

ON BOUNDS OF EIGENVALUES OF MONIC MATRIX POLYNOMIALS

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Abstract. A matrix polynomial is monic if its leading coefficient is $I \in \mathbb{C}^{m \times m}$. In this paper, we derive a family of new explicit and implicit upper and lower bounds on the moduli of eigenvalues of a monic matrix polynomial which generalize the results obtained for scalar polynomials to matrix polynomials. Consequently, different kind of Cauchy-type bounds for matrix polynomials have been obtained as special cases.

Notations

I. Any matrix $A \in \mathbb{C}^{m \times m}$ means A is an $m \times m$ matrix with all its entries belonging to $\mathbb{C}$.

II. If a matrix norm $\|\cdot\|$ is such that $\|A\| = \sup_{v \in \mathbb{C}^m, v \neq 0} \frac{|Av|}{|v|}$, $\forall A \in \mathbb{C}^{m \times m}$, where $|\cdot|$ is a norm over $\mathbb{C}^m$, then $\forall A, B \in \mathbb{C}^{m \times m}$ we have $\|AB\| \leq \|A\| \|B\|$. Such a norm is known as the *subordinate matrix norm*, and we use this notation to mean the same throughout this paper.

III. For two matrices $A, B \in \mathbb{C}^{m \times m}$, we write $A \geq 0$ or $A > 0$, if A is positive semi-definite or positive definite, respectively and $A \geq B \implies A - B \geq 0$.

1. Introduction

Polynomials can be encountered in most scientific fields, and hence a lot of research is done in the interest of finding their zeros. But sometimes it becomes harder, for example, if the degree of the polynomial is very high, to evaluate their zeros with enough accuracy [13]. This problem is often aided by obtaining a region where the zeros lie so that something could be said about the nature of the polynomial. This region can be described by obtaining the lower and upper bounds on their zeros. Gauss may have been among the first mathematicians to localize zeros of polynomials by showing that for any scalar polynomial

$$f(z) := a_n z^n + a_{n-1} z^{n-1} + \dots + a_0,$$

all its zeros lie inside the disc centered at origin and radius $\max_{1 \leq k \leq n} (n\sqrt{2}|a_k|)^{\frac{1}{k}}$. His work was improved by Cauchy in [1] as follows.

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THEOREM 1.1. (Cauchy) Let $p(z) = \sum_{k=0}^n a_k z^k$, $a_n \neq 0$ be a scalar polynomial with complex coefficients. Then all the zeros of $p(z)$ lie in the disc

$$\{z : |z| \leq \eta\} \subset \{z : |z| < 1 + C\},$$

where

$$C := \max_{1 \leq k \leq n} \frac{|a_{n-k}|}{|a_n|},$$

and η is the unique positive real root of the real-coefficient equation

$$|a_n|z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \dots - |a_0| = 0,$$

when $p(z)$ is not a monomial, and as zero otherwise.

Interested readers can find several classical results related to above Theorem 1.1 in [7], [10] and [12].

1.1. Matrix polynomials

Matrix valued functions are constructed in a manner that they have matrices as the coefficients of the complex variable. All these matrices belong to the set of all complex matrices, denoted by $\mathbb{C}^{m \times m}$, which is the set of all $m \times m$ matrices with entries belonging to $\mathbb{C}$. A number $\lambda \in \mathbb{C}$ is known as the eigenvalue of a matrix if there exists some vector $v \in \mathbb{C}^m$ such that $Av = \lambda v$, where $A \in \mathbb{C}^{m \times m}$ is an $m \times m$ complex valued matrix. This is known as the *standard eigenvalue problem*. This problem can be further generalized to ask more meaningful questions, which is the goal of the *polynomial eigenvalue problem*, i.e., determine an eigenvalue $\lambda \in \mathbb{C}$ and a non-zero vector $v \in \mathbb{C}^m$ such that $P(\lambda)v = 0$. Here $P(z)$ denotes a matrix valued function, specifically a matrix polynomial in the relevant literature. The theory of matrix polynomials (see [5] and [6]) has major applications in theory of *vibrating systems*. Eigenvalues are of utmost importance in this study and there are generalized results (see [3], [4], [8] and [9]) obtained for matrix polynomials as well in order to achieve bounds on the moduli values of these eigenvalues.

2. Family of bounds for matrix polynomials

In this section, we derive a family of upper and lower bounds on moduli of eigenvalues of matrix polynomials, which in turn generalizes Theorem 1.1 by Cauchy for matrix polynomials and a result of Higham and Tisseur [4]. Our result provides multiple bounds as we discuss further in the remarks. The upper bound for a matrix polynomial of degree n , obtained by Higham and Tisseur, requires the computation of a separate degree n scalar polynomial. However, through our results, we get equally good bounds, as illustrated in section 3, using scalar polynomials of degrees much less than n . We assume that both A_n and A_0 are invertible throughout the article.

THEOREM 2.1. Every eigenvalue λ of

$$F(z) = Iz^n + A_{n-1}z^{n-1} + \dots + A_0,$$

satisfies

$$\min\{l_1, l_2\} \leq |\lambda| \leq \max\{r_k, 1 + M\},$$

where

$$l_1 = \max\{s_k, 1\} \quad \text{and} \quad l_2 = \max\{s_k, 1 - N\},$$

$$M = \max_{i=k+1, \dots, n-1} \|A_i\| \quad \text{and} \quad N = \max_{i=k+1, \dots, n} \|A_i\|,$$

fixing $A_n = I_m$, and s_k and r_k are the unique positive zeros of

$$L_k(\lambda) = -\lambda^{k+1} - \sum_{i=1}^k \|A_i\| \lambda^i + \|A_0^{-1}\|^{-1} \quad \text{and} \quad G_k(\lambda) = \lambda^{k+1} - \sum_{i=0}^k \|A_i\| \lambda^i,$$

respectively and $k \in \{0, 1, \dots, n-1\}$.

Proof. We will begin by proving the upper bound. Let $\lambda \in \mathbb{C}$ be the eigenvalue of $F(z)$ and $x \in \mathbb{C}^m$ be its associated unit eigenvector. Let us suppose that λ satisfies $|\lambda| > 1 + M$ and $|\lambda| > r_k$. Then we have

$$\begin{aligned} \|F(\lambda)x\| &\geq \left[|\lambda|^n \|x\| - \left\| \sum_{i=0}^{n-1} A_i x \lambda^i \right\| \right] \geq \left[|\lambda|^n - \sum_{i=0}^{n-1} \|A_i\| |\lambda|^i \right] \\ &= |\lambda|^n - |\lambda|^{k+1} + |\lambda|^{k+1} - \sum_{i=0}^k \|A_i\| |\lambda|^i - \sum_{i=k+1}^{n-1} \|A_i\| |\lambda|^i \\ &= |\lambda|^n - |\lambda|^{k+1} + G_k(|\lambda|) - \sum_{i=k+1}^{n-1} \|A_i\| |\lambda|^i \\ &\geq |\lambda|^n - |\lambda|^{k+1} + G_k(|\lambda|) - M \sum_{i=k+1}^{n-1} |\lambda|^i \\ &= (|\lambda|^n - |\lambda|^{k+1}) \left(\frac{|\lambda| - 1 - M}{|\lambda| - 1} \right) + G_k(|\lambda|). \end{aligned}$$

Since $|\lambda| > r_k$ and r_k is the unique positive zero of G_k , we get $G_k(|\lambda|) > 0$ which implies $\|F(z)x\| > 0$, a contradiction. We thus obtain that $|\lambda| \leq \max\{r_k, 1 + M\}$. Now for the lower bound, let us suppose that $|\lambda| < \min\{l_1, l_2\}$, then $|\lambda| < \max\{1, s_k\}$ and $|\lambda| < \max\{s_k, 1 - N\}$. Then we have

$$\begin{aligned} \|F(\lambda)x\| &\geq \left[\|A_0\| - \left\| \sum_{i=1}^n A_i \lambda^i x \right\| \right] \geq \left[\|A_0^{-1}\|^{-1} - \sum_{i=1}^n \|A_i\| |\lambda|^i \right] \\ &= \|A_0^{-1}\|^{-1} - \sum_{i=1}^k \|A_i\| |\lambda|^i - \sum_{i=k+1}^n \|A_i\| |\lambda|^i - |\lambda|^{k+1} + |\lambda|^{k+1} \\ &= L_k(|\lambda|) - \sum_{i=k+1}^n \|A_i\| |\lambda|^i + |\lambda|^{k+1} \geq L_k(|\lambda|) + |\lambda|^{k+1} - N \sum_{i=k+1}^n |\lambda|^i. \end{aligned}$$

Now, for $|\lambda| < 1$ we can write

$$\begin{aligned} &= L_k(|\lambda|) + |\lambda|^{k+1} - N \left(\frac{|\lambda|^{k+1} - |\lambda|^{n+1}}{1 - |\lambda|} \right) \\ &\geq L_k(|\lambda|) + |\lambda|^{k+1} - N \left(\frac{|\lambda|^{k+1}}{1 - |\lambda|} \right) = L_k(|\lambda|) + \frac{|\lambda|^{k+1}(1 - |\lambda| - N)}{1 - |\lambda|}, \end{aligned}$$

which is positive for $1 - N > |\lambda|$ and $s_k > |\lambda|$. We obtain $\|F(\lambda)x\| > 0$ for $|\lambda| < \min\{l_1, l_2\}$ which is a contradiction if λ is an eigenvalue of $F(z)$. Hence, every eigenvalue of $F(z)$ satisfies $|\lambda| \geq \min\{l_1, l_2\}$. $\square$

REMARK 1. For different values of k we get different bounds, such as if we take $k = 0$, Theorem 2.1 yields the following upper bound:

$$|\lambda| \leq \max\{\|A_0\|, 1 + \|A_1\|, \dots, 1 + \|A_{n-1}\|\}, \quad (1)$$

which is the generalization of the result by Cauchy ([7], p. 96–97) to matrix polynomials.

REMARK 2. For $k = n - 1$, Theorem 2.1 gives the result via Higham and Tisseur [4], when applied to monic matrix polynomials, as follows.

THEOREM 2.2. (Higham and Tisseur) *If $F(z) = I z^n + \sum_{i=0}^{n-1} A_i z^i$ is a matrix polynomial of degree n and $A_i \in \mathbb{C}^{m \times m}$, then all the eigenvalues of $F(z)$ satisfy $r \leq |\lambda| \leq R$ where r and R are the unique positive zeros of scalar polynomials given as*

$$z^n + \|A_{n-1}\|z^{n-1} + \|A_{n-2}\|z^{n-2} + \dots - \|A_0^{-1}\|^{-1},$$

and

$$z^n - \|A_{n-1}\|z^{n-1} - \|A_{n-2}\|z^{n-2} - \dots - \|A_0\|,$$

respectively.

REMARK 3. Scalar polynomials can be viewed as matrix polynomials with 1×1 matrix coefficients, and hence Theorem 2.1 can be applied to scalar polynomials as a particular case given in [2].

THEOREM 2.3. (Deutsch) *Every zero of the complex polynomial*

$$f(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0,$$

satisfies

$$|z| \leq \max\{r_k, 1 + |a_{k+1}|, 1 + |a_{k+2}|, \dots, 1 + |a_{n-1}|\},$$

where $k \in \{0, 1, 2, \dots, n-1\}$ and r_k is the unique positive zero of

$$g_k(z) = z^{k+1} - |a_k|z^k - |a_{k-1}|z^{k-1} - \dots - |a_{n-1}|z^{n-1}.$$

REMARK 4. If we denote the upper bound obtained in Theorem 2.1 in terms of a general k , i.e.,

$$M_k = \max\{r_k, 1 + \|A_{k+1}\|, 1 + \|A_{k+2}\|, \dots, 1 + \|A_{n-1}\|\},$$

we can obtain $M_{k+1} \leq M_k$ for any $k = 0, 1, \dots, n-2$. Upon applying Theorem 2.1 to G_{k+1} , we observe that $r_{k+1} \leq \max\{r_k, 1 + \|A_{k+1}\|\}$, implying $M_{k+1} \leq M_k$. Upon increasing the k value, it is possible to improve the upper bound furthermore, but it becomes less efficient and more computationally expensive in the evaluation of corresponding r_{k+1} .

Further we derive yet another result of matrix polynomials.

THEOREM 2.4. *Every eigenvalue $\lambda \in \mathbb{C}$ of a monic matrix polynomial*

$$F(z) = Iz^n + A_{n-1}z^{n-1} + \dots + A_0,$$

satisfies the inequality

$$|\lambda| \leq \max\{1, \|A_{n-1}\| + \|A_{n-2}\| + \dots + \|A_0\|\}.$$

Proof. Let $D = \|A_{n-1}\| + \|A_{n-2}\| + \dots + \|A_0\|$. The result holds true automatically if $|\lambda| \leq 1$ is an eigenvalue of $F(z)$, so will move further under the case when $|\lambda| > 1$. We will now show that any $\lambda \in \mathbb{C}$ which satisfies $|\lambda| > D$ cannot be an eigenvalue of $F(z)$. Let $x \in \mathbb{C}^m$ be the unit eigenvector associated to certain λ . We have

$$\begin{aligned} \|F(\lambda)x\| &\geq |\lambda|^n - \|A_{n-1}\| |\lambda|^{n-1} - \|A_{n-2}\| |\lambda|^{n-2} - \dots - \|A_0\| \\ &= |\lambda|^{n-1} \left(|\lambda| - \|A_{n-1}\| - \frac{\|A_{n-2}\|}{|\lambda|} - \dots - \frac{\|A_0\|}{|\lambda|^{n-1}} \right) \\ &\geq |\lambda|^{n-1} (|\lambda| - \|A_{n-1}\| - \|A_{n-2}\| - \dots - \|A_0\|) \\ &= |\lambda|^{n-1} (|\lambda| - D). \end{aligned}$$

Since $|\lambda| > D$ and $|\lambda| > 1$, we obtain that $\|F(\lambda)x\| > 0$. Hence, any eigenvalue of $F(z)$ is less than D if it is larger than 1. We thus obtain the required upper bound. $\square$

It should be noted that this result gives the following result due to Cauchy ([11] p. 126) when applied to scalar polynomials.

THEOREM 2.5. (Cauchy) *Every zero of the complex monic polynomial*

$$f(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0,$$

satisfies the inequality

$$|z| \leq \max\{1, |a_{n-1}| + |a_{n-2}| + \dots + |a_0|\}.$$

COROLLARY 2.1. *Every eigenvalue of the monic matrix polynomial*

$$F(z) = Iz^n + A_{n-1}z^{n-1} + \dots + A_1z + A_0,$$

satisfies

$$|\lambda| \leq \max\{1, \|A_0\| + \|A_1\| + \dots + \|A_k\|, 1 + \|A_{k+1}\|, \dots, 1 + \|A_{n-1}\|\},$$

where $k = 0, 1, \dots, n-1$.

Proof. If we apply Theorem 2.5 to G_k , where r_k is the unique positive zero of G_k , we obtain

$$r_k \leq \max\{1, \|A_0\| + \|A_1\| + \dots + \|A_k\|\}.$$

As a consequence of this result and Theorem 2.1, we obtain the desired upper bound. $\square$

3. Numerical results

In this section we will illustrate the results of this paper using MATLAB R2024a. It should be noted that all the theorems discussed and derived in this paper pertaining to only monic matrix polynomials can also be applied to general matrix polynomials. We can do so by multiplying the matrix polynomial by the inverse of its leading coefficient.

For our first example, we generated random 30×30 complex-valued matrices and used them as coefficients in a monic matrix polynomial of degree 50. Matrices were generated in MATLAB as follows:

$$A_j = \text{randi}(100, 30, 30) + 1i * \text{randi}(100, 30, 30); j = 0 : 49; A_{50} = \text{eye}(30),$$

where $\text{randi}(100, 30, 30)$ generates a 30×30 matrix with uniformly distributed integer entries ranging from 1 to 100, $1i$ denotes the imaginary number $\sqrt{-1}$ and $\text{eye}(30)$ is the identity matrix of dimension 30. We applied Theorem 2.1 to the polynomial and plotted the ratio of modulus of actual largest eigenvalue of the polynomial and the upper bound obtained for each corresponding value of k . In simpler words, we calculated $\text{ratio} = \left(\frac{\text{Actual largest } |\lambda|}{\text{Upper bound}} \right)$ and the closer this number is to 1, the better the bound obtained is. We used 2-norm as all p -norms are submultiplicative or subordinate.

One can easily notice in Figure 1, via Remark 2, that if $k = n-1$, then Theorem 2.1 gives the same upper bound from Theorem 2.2. As the k values increase, they approach towards better bounds. The figure illustrates that we can achieve equally good bounds even for smaller values of k for each polynomial. Note that for small values of k the associated G_k is of smaller degree and hence computation of its zeros becomes easier and more efficient.

For the next example, we take a 5×5 monic matrix polynomial of degree 9 with coefficients constructed as such:

$$A_i = 10^{i-3} \text{randn}(5); i = 0 : 8; A_9 = \text{eye}(5),$$

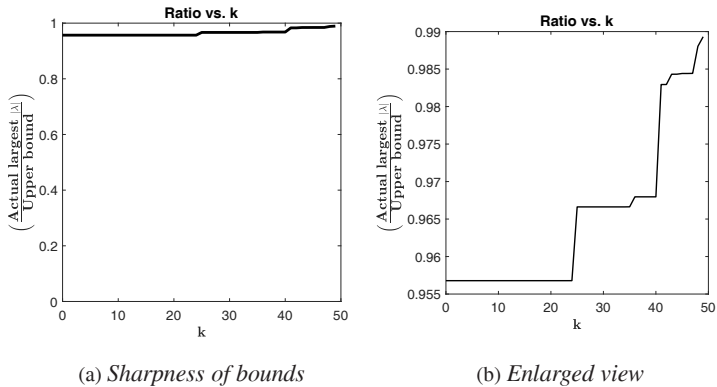


Figure 1: Theorem 2.1 (upper bounds) applied for different values of k .

where $\text{randn}(5)$ is a random 5×5 matrix with entries coming from standard normal distribution and $\text{eye}(5)$ is the identity matrix. We apply Theorem 2.1 to the polynomial and compute the ratio $= \left(\frac{\text{Actual smallest } |\lambda|}{\text{Lower bound}}\right)$ for each k and plot in figure 2.

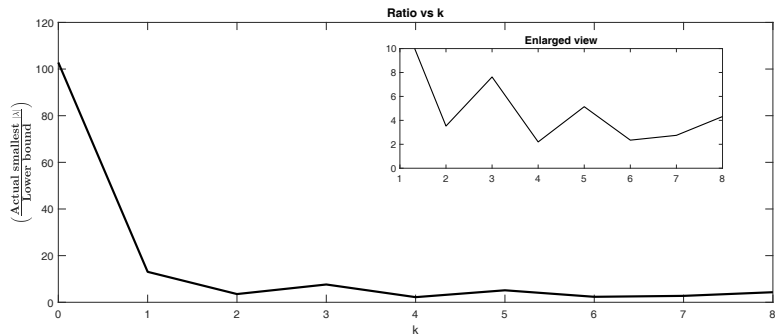


Figure 2: Theorem 2.1 (lower bounds) applied for different values of k .

It should be noted that Figure 2 shows the ratio for a different random matrix polynomial generated for each k and is not the same matrix polynomial throughout. Nevertheless, Theorem 2.1 produces sharper bounds for increasing values of k .

4. Conclusion

We derived a family of upper and lower bounds which generalize some results present in the contemporary literature and also some classical results. We can observe that as the value of the parameter k varies on the extreme ends, we obtain explicit as well as implicit bounds, so our results provide a combination of both types of results

depending upon k . We were able to notice close approximations of the upper bound obtained via Theorem 2.2 in our numerical results. Sharpness of the bounds was noticed to be linked with larger values of k .

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