

A ŠMIGOC GLUING APPROACH FOR CONTROLLING JORDAN STRUCTURES

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Abstract. A list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of complex numbers is said to be realizable if it is the spectrum of an entrywise nonnegative A , which is called a realizing matrix for Λ . Λ is said to be universally realizable if there exists a nonnegative matrix with spectrum Λ for each Jordan structure allowed by Λ . The problems of realizability and universal realizability of a list of complex numbers are solved only for $n \leq 4$, which illustrates their difficulty.

In this work, the universal realizability problem with Jordan control is studied. In particular, a Šmigoc gluing approach for controlling the Jordan structures and constructing the corresponding nonnegative matrices associated to Λ is proposed. The idea is first to construct smaller auxiliary nonnegative matrices realizing the desired Jordan components with relatively small Perron roots, and then these auxiliary blocks are inserted into a real base matrix by means of Šmigoc gluing, preserving nonnegativity, the spectrum and the desired Jordan structure.

1. Introduction

A list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of complex numbers is said to be realizable if it is the spectrum of an entrywise nonnegative matrix A . In this case, A is said to be a realizing matrix for Λ . The problem of the realizability of a list Λ of complex numbers is called Nonnegative Inverse Eigenvalue Problem (*NIEP*). Λ is diagonalizably realizable (*DR*) if a realizing matrix for Λ is diagonalizable and Λ is said to be universally realizable (*UR*) if it is realizable for each Jordan canonical form (*JCF*) allowed by Λ . The problem of the universal realizability of spectra (*URP*) seeks to determine the spectral properties allowed by a nonnegative matrix, not only regarding the eigenvalues themselves, but also from the point of view of the corresponding *JCF*.

The first known results on the *URP*, formerly called nonnegative inverse elementary divisors problem, are due to Minc [7, 8]. In terms of the *URP*, Minc [7] proved that if $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is the spectrum of a diagonalizable positive matrix, then Λ is *UR*. There are spectra, not positively realizable, that are known to be *UR*, as for instance, certain spectra in the left half-plane, that is, $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ with $\lambda_1 > 0$, $\operatorname{Re} \lambda_i \leq 0$, $i = 2, \dots, n$, such as lists of Suleimanova type and Šmigoc type [4, 6, 11, 12].

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Throughout this paper, the first element of a list Λ (usually λ_1 or λ_0 or ρ_i) will be the Perron eigenvalue of a realizing matrix for Λ . If A is a matrix with constant row sums equal to α , we write $A \in \mathcal{CS}_\alpha$. $\sigma(A)$ will denote the spectrum of A and $\mathbf{e}^T = [1 \ 1 \ \cdots \ 1]$ is the vector of all 1. A nonnegative matrix is said to be off-diagonal positive (*ODP*) if all its off-diagonal entries are positive.

Diagonalizability is a necessary condition for a list of complex numbers to be $\mathcal{UR}$. The question of whether Minc's result holds for nonnegative realizations has been open for almost 40 years. Recently, two extensions for Minc's result have been obtained [3, 5]. In particular, in [3], it is shown that if $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ is the spectrum of a diagonalizable nonnegative matrix $A \in \mathcal{CS}_{\lambda_1}$ with a positive column, then Λ is $\mathcal{UR}$. Note that if A is diagonalizable nonnegative with a positive row and A^T has a positive eigenvector, then Λ is also $\mathcal{UR}$. In [5] has been proved that if Λ is realizable by a diagonalizable *ODP* matrix, then Λ is also $\mathcal{UR}$. Since both extensions contain the set of positive matrices, they allow us to significantly increase the set of spectra that can be shown to be $\mathcal{UR}$. Indeed, positive matrices are *ODP*, and since they are also irreducible, they have a positive eigenvector (Perron eigenvector) and then they are similar to matrices with constant row sums. Moreover, from the extension in [5], there are certain trace zero spectra that can be shown to be $\mathcal{UR}$, which is not possible from Minc's result.

The paper is organized as follows: In Section 2, we introduce two known results which will be used to obtain our own results. We also use, in an introductory example, a perturbation result due to Brauer [2], to construct a nonnegative matrix with spectrum Λ of complex numbers for each Jordan matrix allowed by Λ . Brauer perturbations are a powerful tool in the *NIEP*. In particular, they allow one to change a selected eigenvalue of an $n \times n$ matrix without changing any of the remaining $(n - 1)$ eigenvalues. They are especially useful when one starts from a matrix with constant row sums and perturbs it along the Perron eigenvector.

However, when repeated nonreal eigenvalues are involved, the direct Brauer approach may become inefficient if one wants to control the Jordan canonical form. Indeed, one usually starts from a real Jordan block of the desired size and then applies a Brauer type perturbation to make the matrix nonnegative. This often forces the Perron root to be sufficiently large in order to compensate for the negative entries appearing in the real Jordan block. Thus, the direct Brauer construction may require a Perron root larger than what is actually needed. In Section 3, we introduce the main result: a Šmigoc gluing approach for controlling the Jordan structure associated to a list Λ of complex numbers, and for constructing the corresponding nonnegative matrices with spectrum Λ . The idea developed here is different. Instead of applying Brauer directly to the whole list, we first construct smaller auxiliary nonnegative matrices realizing the desired Jordan components with relatively small Perron eigenvalues. These auxiliary blocks are then inserted into a real base matrix by means of Šmigoc gluing, preserving nonnegativity, the constant row sums, the spectrum and the desired Jordan structure.

2. Preliminaries

The following two theorems, due to Brauer [2] and Šmigoc [10, Lemma 5], respectively, have played an important role in the study and development of both problems, the *NIEP* and the *URP*. The Šmigoc result provides a procedure to obtain, from two matrices A and B , a new matrix C , preserving in certain manner, the corresponding Jordan forms of A and B .

THEOREM 2.1. [2] *Let A be an $n \times n$ matrix with spectrum $\Lambda = \{\lambda_1, \dots, \lambda_n\}$. Let $\mathbf{v}^T = [v_1, \dots, v_n]$ be an eigenvector of A associated with the eigenvalue λ_k and let $\mathbf{q}$ be any n -dimensional vector. Then the matrix $A + \mathbf{v}\mathbf{q}^T$ has eigenvalues*

$$\lambda_1, \dots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T \mathbf{q}, \lambda_{k+1}, \dots, \lambda_n.$$

THEOREM 2.2. [10] *Suppose B is an $m \times m$ matrix with a JCF that contain at least one 1×1 Jordan block corresponding to the eigenvalue c ,*

$$J(B) = \begin{bmatrix} c & 0 \\ 0 & I(B) \end{bmatrix}.$$

Let $\mathbf{t}$ and $\mathbf{s}$, respectively, be the left and the right eigenvectors of B associated with the 1×1 Jordan block in the above canonical form. Furthermore, we normalize vectors $\mathbf{t}$ and $\mathbf{s}$ so that $\mathbf{t}^T \mathbf{s} = 1$. Let $J(A)$ be a Jordan matrix for the $n \times n$ matrix

$$A = \begin{bmatrix} A_1 & \mathbf{a} \\ \mathbf{b}^T & c \end{bmatrix},$$

where A_1 is an $(n-1) \times (n-1)$ matrix and $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{C}^n$. Then, the matrix

$$C = \begin{bmatrix} A_1 & \mathbf{a}\mathbf{t}^T \\ \mathbf{s}\mathbf{b}^T & B \end{bmatrix} \quad \text{has} \quad \text{JCF } J(C) = \begin{bmatrix} J(A) & 0 \\ 0 & I(B) \end{bmatrix}.$$

EXAMPLE 2.1. This example illustrates the Šmigoc gluing technique: we may show that the list

$$\Lambda = \{6, -1, -1 \pm 3i, -1 \pm 3i\}$$

is diagonalizably realizable. Indeed, we take the partition $\Lambda = \Lambda_1 \cup \Lambda_2$ with

$$\begin{aligned} \Lambda_1 &= \{6, -1 \pm 3i\}, \quad \Lambda_2 = \{-1, -1 \pm 3i\} \quad \text{and} \\ \Gamma_1 &= \Lambda_1, \quad \Gamma_2 = \{4, -1, -1 \pm 3i\}. \end{aligned}$$

Then, the companion nonnegative matrices

$$A = \begin{bmatrix} 0 & 0 & 60 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & 40 \\ 1 & 0 & 0 & 38 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

have the spectrum Γ_1 and Γ_2 , respectively, and they are diagonalizable since their eigenvalues are distinct. The right and left eigenvectors of B are

$$s^T = [10 \ 12 \ 3 \ 1] \quad \text{and} \quad t^T = [\frac{1}{170} \ \frac{2}{85} \ \frac{8}{85} \ \frac{32}{85}] \quad \text{with} \quad t^T s = 1.$$

Besides,

$$at^T = \begin{bmatrix} 60 \\ 2 \end{bmatrix} [\frac{1}{170} \ \frac{2}{85} \ \frac{8}{85} \ \frac{32}{85}] = \begin{bmatrix} \frac{6}{17} & \frac{24}{17} & \frac{96}{17} & \frac{384}{17} \\ \frac{1}{85} & \frac{4}{85} & \frac{16}{85} & \frac{64}{85} \end{bmatrix}$$

$$sb^T = \begin{bmatrix} 10 \\ 12 \\ 3 \\ 1 \end{bmatrix} [0 \ 1] = \begin{bmatrix} 0 & 10 \\ 0 & 12 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$C = \begin{bmatrix} A_1 & at^T \\ sb^T & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \vdots & \frac{6}{17} & \frac{24}{17} & \frac{96}{17} & \frac{384}{17} \\ 1 & 0 & \vdots & \frac{1}{85} & \frac{4}{85} & \frac{16}{85} & \frac{64}{85} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 10 & \vdots & 0 & 0 & 0 & 40 \\ 0 & 12 & \vdots & 1 & 0 & 0 & 38 \\ 0 & 3 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 & 1 & 1 \end{bmatrix}$$

is nonnegative diagonalizable with spectrum Λ .

REMARK 2.1. Observe that in Theorem 2.2, the auxiliary Perron eigenvalue c in the main diagonal of A ($c = 4$ in the above example) disappears, that is, it is not an eigenvalue of the matrix C .

EXAMPLE 2.2. Consider Brauer's result to construct a nonnegative matrix with spectrum

$$\Lambda = \{23, -1, -1, -1 \pm 2i, -1 \pm 2i, -2 \pm 4i, -2 \pm 4i\},$$

for each Jordan matrix allowed by Λ , that is, for Λ to be $\mathcal{UR}$. Then we apply Brauer's theorem, which is well known, to obtain, with

$$\mathbf{q}^T = [-20, 1, 1, 1, 2, 1, 2, 2, 4, 2, 4],$$

the matrix $A = (B + \mathbf{eq}^T) \in \mathcal{CS}_{23}$, being nonnegative with spectrum Λ and with a diagonal Jordan form. It is also well known that we can obtain all the Jordan structures associated with Λ .

In the next section, we will see that to show that Λ is $\mathcal{UR}$, it is enough to take 18 as the Perron eigenvalue instead of 23. Thus, Brauer procedure will only succeed if the Perron eigenvalue is large enough.

3. A Šmigoc gluing approach for controlling Jordan structures

Consider lists of complex numbers of the form

$$\Lambda = \{\lambda_0, \lambda_1^{(m_1)}, \dots, \lambda_t^{(m_t)}, \underbrace{a_1 \pm b_1 i, \dots, a_1 \pm b_1 i}_{k_1 \text{ times}}, \dots, \underbrace{a_s \pm b_s i, \dots, a_s \pm b_s i}_{k_s \text{ times}}\}, \quad (1)$$

with $\lambda_0 > 0$, $\lambda_i \in \mathbb{R}$, $\lambda_i \neq 0$, $i = 1, \dots, t$, $a_j \in \mathbb{R}$, $b_j > 0$, $j = 1, \dots, s$.

Our strategy is to split Λ into a real part

$$\Lambda_R = \{\lambda_0, \lambda_1^{(m_1)}, \dots, \lambda_t^{(m_t)}\}$$

that controls the repeated real eigenvalues, and a complex part

$$\Lambda_C = \{\underbrace{a_1 \pm b_1 i, \dots, a_1 \pm b_1 i}_{k_1 \text{ times}}, \dots, \underbrace{a_s \pm b_s i, \dots, a_s \pm b_s i}_{k_s \text{ times}}\},$$

that controls the repeated nonreal eigenvalues, and for which we first construct small auxiliary nonnegative matrices realizing the desired Jordan components with relatively small Perron eigenvalues. Next, we apply the Šmigoc glue technique to glue these auxiliary nonnegative matrices with the nonnegative matrices associated to the real part Λ_R .

THEOREM 3.1. *Let*

$$\Lambda = \{\lambda_0, \lambda_1^{(m_1)}, \dots, \lambda_t^{(m_t)}, \underbrace{a_1 \pm b_1 i, \dots, a_1 \pm b_1 i}_{k_1 \text{ times}}, \dots, \underbrace{a_s \pm b_s i, \dots, a_s \pm b_s i}_{k_s \text{ times}}\},$$

with $\lambda_0 > 0$, $\lambda_i \in \mathbb{R}$, $\lambda_i \neq 0$, $i = 1, \dots, t$, $a_j \in \mathbb{R}$, $b_j > 0$, $j = 1, \dots, s$.

For each k_i , $i = 1, \dots, s$, we assume that there exists a nonnegative matrix $D_{k_i} \in CS_{\mu_{k_i}}$ with spectrum

$$\Lambda_{C_{k_i}} = \{\underbrace{\mu_{k_i}, a_i \pm b_i i, \dots, a_i \pm b_i i}_{k_i \text{ times}}\},$$

and with diagonal and nondiagonals Jordan structure, where μ_{k_i} is a simple Perron eigenvalue of D_{k_i} .

We also assume that there is a nonnegative matrix $A_{R_i} \in CS_{\lambda_0}$ with spectrum $\Lambda_{R_i} = \{\lambda_0, \lambda_1^{(m_1)}, \dots, \lambda_t^{(m_t)}\}$ and with diagonal and nondiagonal Jordan structures. Let M_j , $j = 1, \dots, s$, be the nonnegative matrix obtained by applying the Šmigoc glue technique in the form

$$A_R/D_{k_1}, (A_R/D_{k_1})/D_{k_2}, \dots, (((A_R/D_{k_1})/D_{k_2})/D_{k_3})/\dots/D_{k_{s-1}}/D_{k_s}. \quad (2)$$

Then

$$M_s = (((A_R/D_{k_1})/D_{k_2})/D_{k_3})/\dots/D_{k_{s-1}}/D_{k_s} \in CS_{\lambda_0} \quad (3)$$

is nonnegative, with

$$\sigma(M_s) = \sigma(A_R) \cup_{i=1}^s (\sigma(D_{k_i}) - \mu_{k_i}),$$

and

$$J(M_s) = J(A_R) \oplus J(D'_{k_1}) \oplus \cdots \oplus J(D'_{k_s}),$$

where $\sigma(D'_{k_i}) = \sigma(D_{k_i}) - \mu_{k_i}$.

Proof. From the hypotheses, for each k_i , $i = 1, \dots, s$, there exists a nonnegative matrix $D_{k_i} \in CS_{\mu_{k_i}}$ with spectrum

$$\Lambda_{C_{k_i}} = \underbrace{\{\mu_{k_i}, a_i \pm b_i i, \dots, a_i \pm b_i i\}}_{k_i \text{ times}},$$

where μ_{k_i} is a diagonal entry of a matrix $A_{R_i} \in CS_{\lambda_0}$. Since the spectrum $\Lambda_{C_{k_i}}$ is conjugate-even vector, we may always construct a nonnegative circulant matrix $D_{k_i} \in CS_{\mu_{k_i}}$ realizing $\Lambda_{C_{k_i}}$. The reason we need μ_{k_i} to be a diagonal entry in A_{R_i} is that through row and column permutations we can place μ_{k_i} in the last diagonal position of A_{R_i} and thus apply correctly the Šmigoc gluing.

For a no diagonalizable matrix D_{k_i} we may use a nonnegative companion matrix, which is similar to a matrix $\tilde{D}_{k_i} \in CS_{\mu_{k_i}}$. Indeed, if D_{k_i} is nonnegative nonsingular companion, it is irreducible and therefore has a positive eigenvector, say $\mathbf{x} = [x_i]$. Thus, if X is a diagonal matrix with diagonal entries x_i , then $X^{-1}D_{k_i}X = \tilde{D}_{k_i} \in CS_{\mu_{k_i}}$. Now we apply the Šmigoc glue technique in the form (2), where M_s in (3) has spectrum

$$\sigma(M_s) = \sigma(A_R) \cup_{i=1}^s (\sigma(D_{k_i}) - \mu_{k_i}),$$

with

$$J(M_s) = J(A_R) \oplus J(D'_{k_1}) \oplus \cdots \oplus J(D'_{k_s}),$$

where $\sigma(D'_{k_i}) = \sigma(D_{k_i}) - \mu_{k_i}$. $\square$

EXAMPLE 3.1. This example illustrates the improvement we can achieve instead of applying Brauer directly. Consider the complex list

$$\Lambda = \{18, -1, -1, -1 \pm 2i, -1 \pm 2i, -2 \pm 4i, -2 \pm 4i\}.$$

We split Λ into the sublists

$$\Lambda_R = \{18, -1, -1\} \text{ and } \Lambda_C = \{-1 \pm 2i, -1 \pm 2i, -2 \pm 4i, -2 \pm 4i\},$$

and we start by considering the real part $\Lambda_R = \{18, -1, -1\}$, with realizing matrices A_{R_i} with spectrum Λ_R , $i = 1, 2, \dots, 8$.

i) For a diagonal and a nondiagonal structure associated with A_{R_1} and A_{R_2} , respectively, we have from Brauer's result, with $\mathbf{q}^T = [-\frac{64}{5}, \frac{57}{5}, \frac{7}{5}]$,

$$A_{R_1} = \begin{bmatrix} 18 & 0 & 0 \\ 19 & -1 & 0 \\ 19 & 0 & -1 \end{bmatrix} + \mathbf{eq}^T = \begin{bmatrix} \frac{26}{5} & \frac{57}{5} & \frac{7}{5} \\ \frac{31}{5} & \frac{52}{5} & \frac{7}{5} \\ \frac{31}{5} & \frac{57}{5} & \frac{2}{5} \end{bmatrix} \in CS_{18} \quad \text{and}$$

$$A_{R_2} = \begin{bmatrix} 18 & 0 & 0 \\ 20 & -1 & -1 \\ 19 & 0 & -1 \end{bmatrix} + \mathbf{eq}^T = \begin{bmatrix} \frac{26}{5} & \frac{57}{5} & \frac{7}{5} \\ \frac{36}{5} & \frac{52}{5} & \frac{2}{5} \\ \frac{31}{5} & \frac{57}{5} & \frac{2}{5} \end{bmatrix} \in CS_{18},$$

with

$$J(A_{R_1}) = \text{diag}\{J_1(18), J_1(-1), J_1(-1)\},$$

$$J(A_{R_2}) = \text{diag}\{J_1(18), J_2(-1)\}.$$

ii) In the same way, for $\mathbf{q}^T = [-\frac{64}{5}, 9, \frac{19}{5}]$ we have

$$A_{R_3} = \begin{bmatrix} \frac{26}{5} & 9 & \frac{19}{5} \\ \frac{31}{5} & 8 & \frac{19}{5} \\ \frac{31}{5} & 9 & \frac{14}{5} \end{bmatrix} \quad \text{and} \quad A_{R_4} = \begin{bmatrix} \frac{26}{5} & 9 & \frac{19}{5} \\ \frac{30}{5} & 8 & \frac{14}{5} \\ \frac{31}{5} & 9 & \frac{14}{5} \end{bmatrix},$$

with

$$J(A_{R_3}) = \text{diag}\{J_1(18), J_1(-1), J_1(-1)\},$$

$$J(A_{R_4}) = \text{diag}\{J_1(18), J_2(-1)\}.$$

iii) For $\mathbf{q}^T = [-\frac{70}{5}, \frac{57}{5}, \frac{13}{5}]$ we have

$$A_{R_5} = \begin{bmatrix} 4 & \frac{57}{5} & \frac{13}{5} \\ 5 & \frac{52}{5} & \frac{13}{5} \\ 5 & \frac{57}{5} & \frac{8}{5} \end{bmatrix} \quad \text{and} \quad A_{R_6} = \begin{bmatrix} 4 & \frac{57}{5} & \frac{13}{5} \\ 6 & \frac{52}{5} & \frac{8}{5} \\ 5 & \frac{57}{5} & \frac{8}{5} \end{bmatrix},$$

with

$$J(A_{R_5}) = \text{diag}\{J_1(18), J_1(-1), J_1(-1)\},$$

$$J(A_{R_6}) = \text{diag}\{J_1(18), J_2(-1)\}.$$

iv) For $\mathbf{q}^T = [-14, 9, 5]$ we have

$$A_{R_7} = \begin{bmatrix} 4 & 9 & 5 \\ 5 & 8 & 5 \\ 5 & 9 & 4 \end{bmatrix}, \quad \text{and} \quad A_{R_8} = \begin{bmatrix} 4 & 14 & 0 \\ 5 & 8 & 5 \\ 5 & 9 & 4 \end{bmatrix},$$

with

$$\begin{aligned} J(A_{R_7}) &= \text{diag}\{J_1(18), J_1(-1), J_1(-1)\}, \\ J(A_{R_8}) &= \text{diag}\{J_1(18), J_2(-1)\}. \end{aligned}$$

Now, we realize the complex lists $\Lambda_{C1} = \{\rho_1, -1 \pm 2i, -1 \pm 2i\}$ and $\Lambda_{C2} = \{\rho_2, -2 \pm 4i, -2 \pm 4i\}$, with appropriate Perron eigenvalues, ρ_1 and ρ_2 , and with associated diagonal and nondiagonal Jordan matrices, respectively. Moreover, to apply the Šmigoc glue technique, the auxiliary Perron eigenvalues ρ_1 and ρ_2 must be diagonal entries of a realizing matrix A_{R_i} of the real part Λ_R .

For the diagonal Jordan form associated with $\Gamma_{\frac{26}{5}} = \{\frac{26}{5}, -1 \pm 2i, -1 \pm 2i\}$ we look for a realizing circulant matrix $C_{\frac{26}{5}} = \text{circ}\{c_0, c_1, c_2, c_3, c_4\}$ where

$$c_k = \frac{1}{5} \left(\frac{26}{5} + 2 \sum_{j=1}^2 \left(-\cos \frac{2\pi jk}{5} + 2 \sin \frac{2\pi jk}{5} \right) \right), \quad k = 0, 1, \dots, 4.$$

Since $C_{\frac{26}{5}}$ is circulant, its Jordan matrix $J(C_{\frac{26}{5}})$ is diagonal. For the nondiagonal Jordan form associated with $\Gamma_4 = \{4, -1 \pm 2i, -1 \pm 2i\}$, we take the characteristic polynomial

$$p(x) = (x-4)(x+1+2i)^2(x+1-2i)^2,$$

whose companion matrix is

$$D' = \begin{bmatrix} 0 & 0 & 0 & 0 & 100 \\ 1 & 0 & 0 & 0 & 55 \\ 0 & 1 & 0 & 0 & 36 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{with Perron eigenvector } \mathbf{x} = \begin{bmatrix} 25 \\ 20 \\ 14 \\ 4 \\ 1 \end{bmatrix}.$$

Then, for the matrix $X = \text{diag}\{25, 20, 14, 4, 1\}$ we have

$$D = X^{-1}D'X = \begin{bmatrix} 0 & 0 & 0 & 0 & 4 \\ \frac{5}{4} & 0 & 0 & 0 & \frac{11}{4} \\ 0 & \frac{10}{7} & 0 & 0 & \frac{18}{7} \\ 0 & 0 & \frac{7}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} \in CS_4.$$

Since $\Gamma_{\frac{52}{5}} = 2\Gamma_{\frac{26}{5}}$, for the list $\Gamma_{\frac{52}{5}} = \{\frac{52}{5}, -2 \pm 4i, -2 \pm 4i\}$ we take the circulant matrix $2C_{\frac{26}{5}}$, which is nonnegative diagonalizable with spectrum

$$\begin{aligned} \sigma(2C) &= \left\{ \frac{52}{5}, -2 \pm 4i, -2 \pm 4i \right\} \quad \text{and} \\ J(2C) &= \text{diag}\left\{J_1\left(\frac{52}{5}\right), J_1(-2 \pm 4i), J_1(-2 \pm 4i)\right\}. \end{aligned}$$

For the nondiagonal Jordan form associated with $\Gamma_8 = \{8, -2 \pm 4i, -2 \pm 4i\}$, we take the matrix

$$2D = \begin{bmatrix} 0 & 0 & 0 & 0 & 8 \\ \frac{5}{2} & 0 & 0 & 0 & \frac{11}{2} \\ 0 & \frac{20}{7} & 0 & 0 & \frac{36}{7} \\ 0 & 0 & 7 & 0 & 1 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix} \in CS_8,$$

with spectrum Γ_8 and Jordan form

$$J(2D) = \{J_1(8), J_2(-2 \pm 4i)\}.$$

Next, we apply the Šmigoc glue procedure to obtain the eight glued matrices. All of them with spectrum Λ . Let us do this for the nondiagonal case *iv*) $J(A_{R_8}) \oplus J(D) \oplus J(2D)$:

First we glue the matrices

$$A_{R_8} = \begin{bmatrix} 4 & 14 & 0 \\ 5 & 8 & 5 \\ 5 & 9 & 4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 4 \\ \frac{5}{4} & 0 & 0 & 0 & \frac{11}{4} \\ 0 & \frac{10}{7} & 0 & 0 & \frac{18}{7} \\ 0 & 0 & \frac{7}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}.$$

to obtain

$$A_{R_8}/D = \begin{bmatrix} 4 & 14 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & \frac{125}{841} & \frac{400}{841} & \frac{1120}{841} & \frac{1280}{841} & \frac{1280}{841} \\ 5 & 9 & 0 & 0 & 0 & 0 & 4 \\ 5 & 9 & \frac{5}{4} & 0 & 0 & 0 & \frac{11}{4} \\ 5 & 9 & 0 & \frac{10}{7} & 0 & 0 & \frac{18}{7} \\ 5 & 9 & 0 & 0 & \frac{7}{2} & 0 & \frac{1}{2} \\ 5 & 9 & 0 & 0 & 0 & 4 & 0 \end{bmatrix},$$

with

$$J(A_{R_8}/D) = \text{diag}\{J_1(18), J_2(-1), J_2(-1 \pm 2i)\}.$$

Next, we glue the matrices A_{R_8}/D and $2D$ to obtain a matrix M_8 with

$$J(M_8) = \text{diag}\{J_1(18), J_2(-1), J_2(-1 \pm 2i), J_2(-2 \pm 4i)\}.$$

In the same way, we can get the remaining matrices M_i , $i = 1, 2, \dots, 7$, with the desired Jordan forms.

REMARK 3.1. Since Rado's result generalizes Brauer's result, then Rado generally provides better solutions than Brauer, including cases where a list of complex numbers is not Brauer realizable, but is Rado realizable. However, since Theorem

3.1 constructs small auxiliary nonnegative matrices, both diagonalizable and no diagonalizable, realizing real and non-real sublists; and since the real sublists are easy to construct using Brauer, then for our purpose it is enough to apply Brauer. Moreover, to apply Rado's result, it is necessary to have a smaller-size matrix, not only with prescribed eigenvalues but also with prescribed diagonal entries. Unfortunately, as far as we know, there are no results that guarantee this requirement for order $n > 4$, only sufficient conditions (see [1, 9]). Thus for instance, in Example 3.1, the construction of the matrices D and $2D$, with one version being diagonalizable and the other not, suggests using nonnegative circulant matrices (diagonalizable) and nonnegative companion matrices (not diagonalizable unless their eigenvalues are distinct), respectively. Under the hypotheses of Theorem 3.1, this construction is always possible.

The previous example 3.1 is about left half-plane lists of complex numbers. However, Theorem 3.1 also holds for more general lists, as the following example shows:

EXAMPLE 3.2. Consider the list

$$\Lambda = \{10, 8, 8, 4, 4, 1 \pm 2i, 1 \pm 2i\}.$$

We take the decomposition

$$\Gamma_R = \{10, 8, 8, 4, 4\}, \quad \Gamma_C = \{8, 1 \pm 2i, 1 \pm 2i\}.$$

First, from Brauer's theorem, we construct the diagonalizable and nondiagonalizable nonnegative matrices

$$D = \begin{bmatrix} 4 & 0 & 2 & 0 & 2 \\ 5 & 1 & 0 & 0 & 2 \\ 1 & 2 & 3 & 0 & 2 \\ 5 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 2 & 3 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 4 & 0 & 2 & 0 & 2 \\ 4 & 1 & 0 & 1 & 2 \\ 0 & 2 & 3 & 0 & 3 \\ 5 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 2 & 3 \end{bmatrix},$$

both with spectrum $\Lambda_C = \{8, 1 \pm 2i, 1 \pm 2i\}$.

Next, we construct from

$$R_1 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 2 & 8 & 0 & 0 & 0 \\ 2 & 0 & 8 & 0 & 0 \\ 6 & 0 & 0 & 4 & 0 \\ 6 & 0 & 0 & 0 & 4 \end{bmatrix},$$

the matrices

$$R_2 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 1 & 8 & 1 & 0 & 0 \\ 2 & 0 & 8 & 0 & 0 \\ 6 & 0 & 0 & 4 & 0 \\ 6 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 2 & 8 & 0 & 0 & 0 \\ 2 & 0 & 8 & 0 & 0 \\ 5 & 0 & 0 & 4 & 1 \\ 6 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 1 & 8 & 1 & 0 & 0 \\ 2 & 0 & 8 & 0 & 0 \\ 5 & 0 & 0 & 4 & 1 \\ 6 & 0 & 0 & 0 & 4 \end{bmatrix},$$

with spectrum $\Lambda_R = \{10, 8, 8, 4, 4\}$ and with the four Jordan forms allowed by Λ_R , respectively.

Now we apply the Šmigoc glue procedure for the following eight cases:

- 1) R_1/D , 2) R_1/N , 3) R_2/D , 4) R_2/N ,
- 5) R_3/D , 6) R_3/N , 7) R_4/D , 8) R_4/N ,

all with spectrum Λ and with the corresponding associated Jordan matrices.

Conclusion

This paper aims to establish conditions to determine the universal realizability of a list of complex numbers, while at the same time controlling their Jordan structures. Since the problem of universal realizability is still open for order $n > 4$, the problem we study in this work is difficult and, as far as we know, has not been considered in the literature on realizability and universal realizability of spectra. We don't have a general result that allows constructing nonnegative matrices with prescribed eigenvalues and Jordan structure. So indeed, we have proposed a practical procedure that, starting from a given list of complex numbers, which is separated into a real part and a complex part, combines a perturbation result due to Brauer [2] with Šmigoc's gluing technique [10]. This combination allows constructing a nonnegative matrix by controlling its possible Jordan structures in three steps:

1. Realize the real sublists with small auxiliary matrices that control repeated real eigenvalues.
2. Realize the complex sublists with auxiliary matrices that control repeated complex eigenvalues.
3. Apply Šmigoc's gluing technique to glue the auxiliary matrices from both, the real and complex parts.

We believe this procedure could open up opportunities for new research with more general results on the problem proposed here.

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