

MULTIPLICATIVELY PSEUDO SPECTRUM PRESERVING MAPS ON BANACH SPACES

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(Communicated by J. Ball)

Abstract. Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on a complex Banach space X with $\dim X \geq 3$, and $\mathcal{A}, \mathcal{B}$ two subsets of $\mathcal{B}(X)$ containing all operators of rank at most one. For $\varepsilon \in (0, +\infty)$ and $A \in \mathcal{B}(X)$, let $\sigma_\varepsilon(A)$ and $r_\varepsilon(A)$ denote the ε -pseudo spectrum and the ε -pseudo spectral radius of A , respectively. We prove that a pair of surjective maps $\phi, \psi : \mathcal{A} \rightarrow \mathcal{B}$ satisfy

$$\sigma_\varepsilon(\phi(A)\psi(B)) = \sigma_\varepsilon(AB) \quad (A, B \in \mathcal{A})$$

if and only if there exist two bounded invertible linear operators U, V on X with $\|Vx\| = \|x\|$ for each $x \in X$, such that for all $A \in \mathcal{A}$, $\phi(A) = \mu V A U^{-1}$ and $\psi(A) = \nu U A V^{-1}$ for some μ and ν with $\mu\nu = 1$. We also characterize all surjective maps $\phi, \psi : \mathcal{A} \rightarrow \mathcal{B}$ satisfying

$$r_\varepsilon(\phi(A)\psi(B)) = r_\varepsilon(AB) \quad (A, B \in \mathcal{A}).$$

Furthermore, some known results are obtained as immediate consequences of our main results.

1. Introduction and statement of the main results

Throughout this paper, let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on a complex Banach space X . The natural pairing between X and its dual space X^* is defined by $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{C}, (x, f) \mapsto f(x)$. Let $\mathcal{F}_1(X)$ be the set of all operators of rank at most one in $\mathcal{B}(X)$. This means that $\mathcal{F}_1(X) := \{x \otimes f : x \in X \text{ and } f \in X^*\}$, where $(x \otimes f)(z) := \langle z, f \rangle x$ for all $x, z \in X$ and $f \in X^*$. If X has dimension $n < \infty$, we identify X with the vector space $\mathbb{C}^n$ and $\mathcal{B}(X)$ with the algebra $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ -complex matrices. According to the above identifications, we have $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$ and $x \otimes y = x^t y$, where x^t stands for the transpose of x , for all $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{C}^n$. For a ring automorphism τ of $\mathbb{C}$, we set $x^\tau = (\tau(x_1), \tau(x_2), \dots, \tau(x_n))$, and $M^\tau = (\tau(t_{ij}))$ for all $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $M = (t_{ij}) \in \mathcal{M}_n(\mathbb{C})$. The identity operator of $\mathcal{B}(X)$ will be denoted by I , and the adjoint of operator $A \in \mathcal{B}(X)$ is defined by $A^* : X^* \rightarrow X^*; f \mapsto f \circ A$ so that $\langle Ax, f \rangle = \langle x, A^* f \rangle$ for all $(x, f) \in X \times X^*$. Note that if $A : X \rightarrow X$ is a bounded conjugate linear operator on X (i.e., A is additive and $A(\lambda x) = \bar{\lambda} A(x)$ for all $\lambda \in \mathbb{C}$ and $x \in X$), then its adjoint A^* is the bounded conjugate linear operator on X^* defined by

Mathematics subject classification (2020): Primary 47B49; Secondary 47A10, 47A11.

Keywords and phrases: Pseudo spectrum, pseudo spectral radius, nonlinear preserver.

$A^*(f) = \overline{f} \circ A$ for all $f \in X^*$, where $\overline{f}(x) = \overline{f(x)}$ for all $x \in X$. Thus, $\langle Ax, f \rangle = \overline{\langle x, A^*f \rangle}$ for all $(x, f) \in X \times X^*$. For $A \in \mathcal{B}(X)$, we shall denote by $\sigma(A)$ the spectrum and by $r(A)$ its spectral radius. Here $\mathbb{T}$ the unit circle of the complex field $\mathbb{C}$. For $\varepsilon \in (0, +\infty)$, the ε -pseudo spectrum of an operator $A \in \mathcal{B}(X)$ is defined by

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{ \lambda \in \mathbb{C} \setminus \sigma(A) : \|(\lambda I - A)^{-1}\| \geq \frac{1}{\varepsilon} \right\},$$

where the norm used on $\mathcal{B}(X)$ is just the standard operatorial one, that is $\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\}$. It is well known that $\sigma_\varepsilon(A)$ is a compact set containing the spectrum of $A \in \mathcal{B}(X)$. The ε -pseudo spectral radius of an operator A is given by

$$r_\varepsilon(A) := \sup\{|z| : z \in \sigma_\varepsilon(A)\}.$$

Recently, several researchers have been interested in characterizing nonlinear maps on matrices or operators satisfying $F(\phi(A) \bullet \phi(B)) = F(A \bullet B)$. Here, $F(\cdot)$ is a spectral function or a spectral set such as the spectrum, the local spectrum, the numerical range, the numerical radius, the ε -pseudo spectral radius, the ε -pseudo spectrum, the ε -condition spectral radius, and the ε -condition spectrum. On the other hand, $A \bullet B$ stands for different kinds of products such as the usual product AB , the triple product ABA , the Jordan product $AB + BA$, the skew product A^*B , the skew triple product AB^*A , the skew-Jordan product $AB^* + B^*A$, and the Lie product $AB - BA$; see for instance [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18] and the references therein. In [19], Molnár characterized maps preserving the spectrum of operator or matrix products. His result has been extended in several directions. In particular, Abdelali and Aharmim characterized in [5] pair of maps $\phi : \mathcal{M}_{m,n}(\mathbb{C}) \longrightarrow \mathcal{M}_{p,q}(\mathbb{C})$ and $\psi : \mathcal{M}_{n,m}(\mathbb{C}) \longrightarrow \mathcal{M}_{q,p}(\mathbb{C})$ satisfying

$$\sigma(\phi(A)\psi(B)) = \sigma(AB), \quad ((A, B) \in \mathcal{M}_{m,n}(\mathbb{C}) \times \mathcal{M}_{n,m}(\mathbb{C})),$$

where $\sigma(A)$ is the spectrum of A and $\mathcal{M}_{m,n}(\mathbb{C})$ the vector space of matrices of m rows and n columns with complex entries. Recently, Bourhim and Lee studied in [6] surjective maps between algebras of operators $\mathcal{L}(\mathcal{X})$ and $\mathcal{L}(\mathcal{Y})$ on Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively, such that

$$\sigma_{\phi(A)\psi(B)}(x_0) = \sigma_{AB}(y_0), \quad (A, B \in \mathcal{L}(\mathcal{X})),$$

where $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ are fixed nonzero vectors and $\sigma_A(x_0)$ is the local spectrum of A at x_0 . In this paper we characterize all surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ that preserve multiplicatively the ε -pseudo spectrum (ε -pseudo spectral radius) in this way the ε -pseudo spectrum (ε -pseudo spectral radius) of the product $\phi(A)\psi(B)$ coincides with that of the product AB for all $A, B \in \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathcal{B}(X)$ containing all operators of rank at most one. Precisely, we state

THEOREM 1.1. *Let X be an infinite-dimensional complex Banach space and $\mathcal{A}, \mathcal{B}$ two subsets of $\mathcal{B}(X)$ containing $\mathcal{F}_1(X)$. Then, surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfy*

$$r_\varepsilon(\phi(A)\psi(B)) = r_\varepsilon(AB) \quad (A, B \in \mathcal{A}) \quad (1.1)$$

if and only if there exist two scalar-valued functions $h_1, h_2 : \mathcal{A} \longrightarrow \mathbb{C} \setminus \{0\}$ such that $h_1(\mathcal{A})h_2(\mathcal{A}) \subseteq \mathbb{T}$ and two bounded invertible linear or conjugate linear operators U, V on X such that $\|Vx\| = \|x\|$ for each $x \in X$, and

$$\phi(A) = h_1(A)VAU^{-1} \quad \text{and} \quad \psi(A) = h_2(A)UAV^{-1} \quad (A \in \mathcal{A}).$$

THEOREM 1.2. *Let X be an infinite-dimensional complex Banach space and $\mathcal{A}, \mathcal{B}$ two subsets of $\mathcal{B}(X)$ containing $\mathcal{F}_1(X)$. Then, surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfy*

$$\sigma_\varepsilon(\phi(A)\psi(B)) = \sigma_\varepsilon(AB) \quad (A, B \in \mathcal{A}) \quad (1.2)$$

if and only if there exist two nonzero scalars μ and ν such that $\mu\nu = 1$, and two bounded invertible linear operators U, V on X such that $\|Vx\| = \|x\|$ for each $x \in X$, and

$$\phi(A) = \mu VAU^{-1} \quad \text{and} \quad \psi(A) = \nu UAV^{-1} \quad (A \in \mathcal{A}).$$

The corresponding result for the finite-dimensional case is given by the next theorem. In [12, Theorem 3.3] and [9, Theorem 1.2], the norm on $\mathbb{C}^n$ is the Euclidian one (which turns $\mathbb{C}^n$ into a Hilbert space), while in the next statements the norm on $\mathbb{C}^n$ is an arbitrary one.

THEOREM 1.3. *Let $n \geq 3$ and $\mathcal{A}, \mathcal{B}$ two subsets of $\mathcal{M}_n(\mathbb{C})$ containing $\mathcal{F}_1(\mathbb{C}^n)$. Then, surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfy (1.1) if and only if there exist two scalar-valued functions $h_1, h_2 : \mathcal{A} \longrightarrow \mathbb{C} \setminus \{0\}$ such that $h_1(\mathcal{A})h_2(\mathcal{A}) \subseteq \mathbb{T}$ and two nonsingular matrices $U, V \in \mathcal{M}_n(\mathbb{C})$ such that $\|V(x^\tau)\| = \|x\|$ for each $x \in \mathbb{C}^n$, and*

$$\phi(A) = \mu VA^\tau U^{-1} \quad \text{and} \quad \psi(A) = \nu UA^\tau V^{-1} \quad (A \in \mathcal{A}).$$

Here τ is either the identity or the conjugation.

THEOREM 1.4. *Let $n \geq 3$ and $\mathcal{A}, \mathcal{B}$ two subsets of $\mathcal{M}_n(\mathbb{C})$ containing $\mathcal{F}_1(\mathbb{C}^n)$. Then, surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfy (1.2) if and only if there exist two nonzero scalars μ and ν such that $\mu\nu = 1$, and two nonsingular matrices $U, V \in \mathcal{M}_n(\mathbb{C})$ such that $\|Vx\| = \|x\|$ for each $x \in \mathbb{C}^n$, and*

$$\phi(A) = \mu VAU^{-1} \quad \text{and} \quad \psi(A) = \nu UAV^{-1} \quad (A \in \mathcal{A}).$$

We end this section with the following corollaries. First, we characterize all surjective maps preserving the ε -pseudo spectral radius of the product AB . Note that, a surjective map ϕ from $\mathcal{B}(X)$ into itself satisfies (1.1) were described in [15, Theorem 3, Theorem 4]. Such characterization is now a consequence of our main results.

COROLLARY 1.5. *Let X be a complex Banach space with $\dim X \geq 3$, and $\mathcal{A}, \mathcal{B}$ two subsets of $\mathcal{B}(X)$ containing $\mathcal{F}_1(X)$. Then, a surjective map $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (1.1) if and only if there exists a scalar-valued function $h : \mathcal{A} \longrightarrow \mathbb{T}$ and one of the following holds:*

1. If $\dim(X) = \infty$, then there exists a bounded invertible linear or conjugate linear operator U on X such that $\|Ux\| = \|x\|$ for each $x \in X$ and

$$\phi(A) = h(A)UAU^{-1} \quad (A \in \mathcal{A}).$$

2. If $X = \mathbb{C}^n$ with $n \geq 3$, then there exists a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that $\|U(x^\tau)\| = \|x\|$ for each $x \in \mathbb{C}^n$ and

$$\phi(A) = h(A)UA^\tau U^{-1} \quad (A \in \mathcal{A}).$$

Next, let $\mathcal{A}, \mathcal{B}$ be two subsets of $\mathcal{L}(\mathcal{H})$ containing $\mathcal{F}_1(\mathcal{H})$. Surjective maps $\phi : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (1.2) were described in [11, Theorem 4.1] and [12, Theorem 3.3]. Such characterization is now a consequence of our main results.

COROLLARY 1.6. *Let X be a complex Banach space with $\dim X \geq 3$, and $\mathcal{A}, \mathcal{B}$ two subsets of $\mathcal{B}(X)$ containing $\mathcal{F}_1(X)$. Then, a surjective map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ satisfies (1.2) if and only if there exists a nonzero scalar μ such that $\mu^2 = 1$, and one of the following holds:*

1. If $\dim(X) = \infty$, then there exists a bounded invertible linear operator U on X such that $\|Ux\| = \|x\|$ for each $x \in X$ and

$$\phi(A) = \mu UAU^{-1} \quad (A \in \mathcal{A}).$$

2. If $X = \mathbb{C}^n$ with $n \geq 3$, then there exists a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that $\|Ux\| = \|x\|$ for each $x \in \mathbb{C}^n$ and

$$\phi(A) = \mu UAU^{-1} \quad (A \in \mathcal{A}).$$

The rest of this paper is organized as follows. In Section 2, we present some auxiliary results on the numerical range and the numerical radius. Sections 3, 4 and 5 contain proofs of our main results.

2. Preliminaries

In this section, we present some auxiliary results needed for the proof of our main results. The first result summarizes some known basic properties of the ε -pseudo spectrum and the ε -pseudo spectral radius. Formore information about these notions, one can see [15, 20].

PROPOSITION 2.1. *For an operator $A \in \mathcal{B}(X)$, the following statements hold.*

- 1) For every nonzero scalars $\alpha, \beta \in \mathbb{C}$, $\sigma_\varepsilon(\alpha I + \beta A) = \{\alpha\} + \beta \cdot \sigma_{\frac{\varepsilon}{|\beta|}}(A)$.
- 2) For every bijective isometry $U \in \mathcal{B}(X)$, $\sigma_\varepsilon(UAU^{-1}) = \sigma_\varepsilon(A)$.

3) For every conjugate isometry operator U , $\sigma_\varepsilon(UAU^{-1}) = \overline{\sigma_\varepsilon(A)}$.

Consequently, we have:

4) For every nonzero scalar $\beta \in \mathbb{C}$, $r_\varepsilon(\beta A) = |\beta| r_{\frac{\varepsilon}{|\beta|}}(A)$.

5) For every bijective isometry or conjugate isometry $U \in \mathcal{B}(X)$, $r_\varepsilon(UAU^{-1}) = r_\varepsilon(A)$.

The second one we shall need the following observations regarding the ε -pseudo spectral radius of rank one operators. By checking the proof of [15, Lemma 6], one can obtain the following result.

LEMMA 2.2. Let $(x, f) \in X \times X^*$, $(x_n)_n \subseteq X$, $(f_n)_n \subseteq X^*$ and $(\varepsilon_n)_n \subseteq (0, +\infty)$ such that $\lim_{n \rightarrow +\infty} \varepsilon_n = 0$. Then the following statements hold.

1) If $\lim_{n \rightarrow +\infty} |\langle x_n, f \rangle| = +\infty$, then $\lim_{n \rightarrow +\infty} r_{\varepsilon_n}(x_n \otimes f) = +\infty$.

2) If $\lim_{n \rightarrow +\infty} \|x_n - x\| = 0$, then $\lim_{n \rightarrow +\infty} r_{\varepsilon_n}(x_n \otimes f) = |f(x)|$.

3) If $\lim_{n \rightarrow +\infty} |\langle x, f_n \rangle| = +\infty$, then $\lim_{n \rightarrow +\infty} r_{\varepsilon_n}(x \otimes f_n) = +\infty$.

4) If $\lim_{n \rightarrow +\infty} \|f_n - f\| = 0$, then $\lim_{n \rightarrow +\infty} r_{\varepsilon_n}(x \otimes f_n) = |f(x)|$.

Proof. For $A \in \mathcal{B}(X)$, the following equality always holds

$$\sigma_\varepsilon(A) = \bigcup \{ \sigma(A + E) : E \in \mathcal{B}(X), \|E\| \leq \varepsilon \}. \quad (2.1)$$

(See, for example, [14]). Then, (2.1) gives

$$r_\varepsilon(A) = \sup \{ r(A + E) : E \in \mathcal{B}(X), \|E\| \leq \varepsilon \}.$$

In particular,

$$r(A) + \varepsilon \leq r_\varepsilon(A) \leq \|A\| + \varepsilon. \quad (2.2)$$

1) We have $r(x \otimes f) = |f(x)|$, and then for each $n \geq 1$ using the first inequality in (2.2) we have

$$|f(x_n)| + \varepsilon_n = r(x_n \otimes f) + \varepsilon_n \leq r_{\varepsilon_n}(x_n \otimes f).$$

Since

$$\lim_{n \rightarrow +\infty} |f(x_n)| + \varepsilon_n = +\infty,$$

then

$$\lim_{n \rightarrow +\infty} r_{\varepsilon_n}(x_n \otimes f) = +\infty.$$

2) From

$$|f(x_n)| + \varepsilon_n = r(x_n \otimes f) + \varepsilon_n \leq r_{\varepsilon_n}(x_n \otimes f) \quad (n \in \mathbb{N}),$$

since $\lim_{n \rightarrow +\infty} \|x_n - x\| = 0$ and $\lim_{n \rightarrow +\infty} \varepsilon_n = 0$, then,

$$\lim_{n \rightarrow +\infty} |f(x_n)| + \varepsilon_n = |f(x)|.$$

Therefore, $|f(x)| \leq \liminf r_{\varepsilon_n}(x_n \otimes f)$. Now, let $\varepsilon > 0$, for

$$D = \{z \in \mathbb{C} : |z| < |f(x)| + \varepsilon\},$$

using the upper semi-continuity of the spectrum [1, Theorem 3.4.2], we find $\delta > 0$ such that $\|T\| < \delta$ implies

$$\sigma(x \otimes f + T) \subseteq D.$$

Let $p \geq 1$ is a natural number such that $n \geq p$ implies $\varepsilon_n < \frac{\delta}{2}$ and $\|x_n - x\| \|f\| < \frac{\delta}{2}$. Then $n \geq p$ and $\|T\| \leq \varepsilon_n$ implies

$$\begin{aligned} \|x_n \otimes f - x \otimes f + T\| &= \|(x_n - x) \otimes f + T\| \\ &\leq \|(x_n - x) \otimes f\| + \|T\| \\ &\leq \|x_n - x\| \|f\| + \|T\| \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta, \end{aligned}$$

which gives

$$\sigma(x_n \otimes f + T) = \sigma(x \otimes f + x_n \otimes f - x \otimes f + T) \subseteq D.$$

Therefore,

$$r(x_n \otimes f + T) \leq |f(x)| + \varepsilon \quad (n \geq p, \|T\| \leq \varepsilon_n).$$

Taking the supremum over T , then

$$r_{\varepsilon_n}(x_n \otimes f) \leq |f(x)| + \varepsilon \quad (n \geq p).$$

Thus,

$$\limsup r_{\varepsilon_n}(x_n \otimes f) \leq |f(x)| + \varepsilon \quad \text{for all } \varepsilon > 0,$$

hence,

$$\limsup r_{\varepsilon_n}(x_n \otimes f) \leq |f(x)|.$$

Consequently,

$$\lim_{n \rightarrow +\infty} r_{\varepsilon_n}(x_n \otimes f) = |f(x)|.$$

By a similar argument as above, one can obtain that 3) and 4) hold. $\square$

In the case of operators acting on a Hilbert space, the ε -pseudo spectral radius of rank-one operators can be computed explicitly.

REMARK 2.3. ([11, 12]) Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. If x and y are two vectors in $\mathcal{H}$, then

$$r_\varepsilon(x \otimes y) = \frac{1}{2} \left(\sqrt{|\langle x, y \rangle|^2 + 4\varepsilon^2 + 4\varepsilon \|x\| \|y\|} + |\langle x, y \rangle| \right).$$

The third result provides the asymptotic behavior for the ε -pseudo spectral radius of square-zero operators.

LEMMA 2.4. ([15]) *Let $A \in \mathcal{B}(X)$ such that $A^2 = 0$. Then*

$$\lim_{\delta \rightarrow 0} \frac{r_\delta(A)}{\sqrt{\delta}} = \sqrt{\|A\|}.$$

The following lemma, proved in [15], gives a characterization of zero operators in term of the pseudo spectral radius.

LEMMA 2.5. *An operator $A \in \mathcal{B}(X)$ is zero if and only if $r_\varepsilon(A) = \varepsilon$.*

The following lemma is quoted from [3], determines the structure of mappings that preserve the double product divisors of zero.

LEMMA 2.6. *Let X be a complex Banach space with $\dim X \geq 3$, and $\mathcal{A}, \mathcal{B}$ two subsets of $\mathcal{B}(X)$ containing $\mathcal{F}_1(X)$. Suppose that $\phi, \psi : \mathcal{A} \rightarrow \mathcal{B}$ are two surjective maps satisfying*

$$\phi(A)\psi(B) = 0 \iff AB = 0 \quad (2.3)$$

for all $A, B \in \mathcal{A}$. Then the following statements hold.

1. *If $\dim(X) = \infty$, then there exist two maps $k : X \times X^* \rightarrow X$, $(x, f) \mapsto k_f(x)$ and $h : X \times X^* \rightarrow X^*$, $(x, f) \mapsto h_x(f)$ and a bounded invertible linear or conjugate linear operator U on X such that*

$$\phi(x \otimes f) = k_f(x) \otimes (U^{-1})^* f \text{ and } \psi(x \otimes f) = Ux \otimes h_x(f) \quad (2.4)$$

for all $(x, f) \in X \times X^$.*

2. *If $X = \mathbb{C}^n$ with $n \geq 3$, then there exist two maps $k, h : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$, $(x, y) \mapsto k_y(x)$, $(x, y) \mapsto h_x(y)$, a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ such that*

$$\phi(x \otimes y) = k_y(x) \otimes (U^{-1})^* y^\tau \text{ and } \psi(x \otimes y) = Ux^\tau \otimes h_x(y) \quad (2.5)$$

for all $x, y \in \mathbb{C}^n$.

The following is a consequence of the above lemma.

LEMMA 2.7. *Let X be a complex Banach space with $\dim(X) \geq 3$. Suppose that $h : X \rightarrow X$, $k : X^* \rightarrow X^*$ are two surjective maps satisfying*

$$\langle h(x), k(f) \rangle = 0 \iff \langle x, f \rangle = 0 \quad ((x, f) \in X \times X^*). \quad (2.6)$$

Then the following statements hold.

1. If $\dim(X) = \infty$, then there exist two scalar-valued functions $\mu : X \longrightarrow \mathbb{C} \setminus \{0\}$, $\nu : X^* \longrightarrow \mathbb{C} \setminus \{0\}$ and a bounded invertible linear or conjugate linear operator U on X such that

$$h(x) = \mu(x)U(x) \text{ and } k(f) = \nu(f)(U^{-1})^*(f) \quad ((x, f) \in X \times X^*).$$

2. If $X = \mathbb{C}^n$, with $n \geq 3$, then there exist two scalar-valued functions $\mu, \nu : \mathbb{C}^n \longrightarrow \mathbb{C} \setminus \{0\}$, a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ such that

$$h(x) = \mu(x)U(x^\tau) \text{ and } k(x) = \nu(x)(U^{-1})^*(x^\tau) \quad (x \in \mathbb{C}^n).$$

Proof. Applying Lemma 2.6 in the case when $\mathcal{A} = \mathcal{B} = \mathcal{F}_1(X)$ and $\phi(x \otimes f) = x \otimes k(f)$, $\psi(x \otimes f) = h(x) \otimes f$ for all $x \otimes f \in \mathcal{F}_1(X)$. Since

$$\langle h(x), k(f) \rangle = 0 \iff \langle x, f \rangle = 0 \quad ((x, f) \in X \times X^*),$$

it follows that $\phi, \psi : \mathcal{F}_1(X) \longrightarrow \mathcal{F}_1(X)$ are two surjective maps satisfying

$$\phi(A)\psi(B) = 0 \iff AB = 0 \quad (A, B \in \mathcal{F}_1(X)),$$

hence ϕ and ψ satisfy the hypothesis of Lemma 2.6. Therefore, there exist two maps $\alpha : X \times X^* \rightarrow X$ and $\beta : X \times X^* \rightarrow X^*$ such that one of the following results hold:

1. If $\dim(X) = \infty$, then there exists a bounded invertible linear or conjugate linear operator U on X such that

$$\phi(x \otimes f) = \alpha(x, f) \otimes (U^{-1})^*f \text{ and } \psi(x \otimes f) = Ux \otimes \beta(x, f) \quad ((x, f) \in X \times X^*).$$

2. If $X = \mathbb{C}^n$, with $n \geq 3$, then there exist a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ such that

$$\phi(x \otimes f) = \alpha(x, f) \otimes (U^{-1})^*f^\tau \text{ and } \psi(x \otimes f) = Ux^\tau \otimes \beta(x, f) \quad (x, f \in \mathbb{C}^n).$$

On the other hand, since $\phi(x \otimes f) = x \otimes k(f)$, $\psi(x \otimes f) = h(x) \otimes f$ for all $x \otimes f \in \mathcal{F}_1(X)$, we conclude that

1. if $\dim(X) = \infty$, then there exist two scalar-valued functions $\mu, \nu : X \longrightarrow \mathbb{C} \setminus \{0\}$ such that

$$h(x) = \mu(x)U(x) \text{ and } k(x) = \nu(x)(U^{-1})^*(x) \text{ for all } x \in X,$$

2. if $X = \mathbb{C}^n$, with $n \geq 3$, then there exist two scalar-valued functions $\mu, \nu : \mathbb{C}^n \longrightarrow \mathbb{C} \setminus \{0\}$ such that

$$h(x) = \mu(x)U(x^\tau) \text{ and } k(x) = \nu(x)(U^{-1})^*(x^\tau) \text{ for all } x \in \mathbb{C}^n. \quad \square$$

3. Proof of Theorems 1.1 and 1.3

Checking the “if” part is straightforward, so we will only deal with the “only if” part. Then assume that $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ are two surjective maps satisfying (1.1) This, together with Lemma 2.5 imply that

$$\phi(A)\psi(B) = 0 \iff AB = 0$$

for all $A, B \in \mathcal{A}$. Thus we can assume that ϕ and ψ have the forms given in Lemma 2.6. Therefore,

$$\phi(x \otimes f) = k_f(x) \otimes (U^{-1})^* f^{\tau_1} \text{ and } \psi(x \otimes f) = Ux^{\tau_1} \otimes h_x(f) \quad (3.1)$$

for all $x \otimes f \in \mathcal{F}_1(X)$, where

$$x^{\tau_1} = \begin{cases} x^{\tau} & \text{if } X = \mathbb{C}^n \\ x & \text{if } \dim(X) = \infty \end{cases}$$

for all $x \in X$ or $x \in X^*$. We also denote

$$\eta(z) := \begin{cases} \tau(z) & \text{if } X = \mathbb{C}^n \\ \bar{z} & \text{if } \dim(X) = \infty \text{ and } U \text{ is conjugate linear} \\ z & \text{if } \dim(X) = \infty \text{ and } U \text{ is linear} \end{cases}$$

for all $z \in \mathbb{C}$.

We will complete the proof by checking five assertions.

ASSERTION 1. For every vector $(y, f) \in X \times X^*$ such that $\langle y, f \rangle \neq 0$, we have $k_f(\cdot)$ and $h_y(\cdot)$ are surjective maps satisfying

$$|\langle k_f(x), h_y(g) \rangle| = \frac{|\langle y, f \rangle|}{|\eta(\langle y, f \rangle)|} |\langle x, g \rangle| \quad ((x, g) \in X \times X^*). \quad (3.2)$$

Moreover,

$$\langle x, g \rangle = 0 \iff \langle k_f(x), h_y(g) \rangle = 0 \quad ((x, g) \in X \times X^*). \quad (3.3)$$

Proof. Let $(y, f) \in X \times X^*$ such that $\langle y, f \rangle \neq 0$. First, let us show that $k_f(\cdot)$ and $h_y(\cdot)$ are surjective. To do it, for $k_f(\cdot)$, it suffice to show that the following equality holds

$$\phi(\{u \otimes f : u \in X \setminus \{0\}\}) = \{u \otimes (U^{-1})^* f^{\tau_1} : u \in X \setminus \{0\}\}. \quad (3.4)$$

Since $\phi(\{u \otimes f : u \in X \setminus \{0\}\}) \subseteq \{u \otimes (U^{-1})^* f^{\tau_1} : u \in X \setminus \{0\}\}$ is obvious, we only need to establish the other inclusion. Let u be a nonzero vector in X , from [3, Proof of Theorem 2.1, Assertion 3] ϕ preserves rank one operators in both directions, then there exists $v \otimes h \in \mathcal{F}_1(X)$ such that

$$u \otimes (U^{-1})^* f^{\tau_1} = \phi(v \otimes h) = k_h(v) \otimes (U^{-1})^* h^{\tau_1}.$$

It follows that f and h are linearly dependent, and so $h = \alpha f$ for some nonzero scalar α . Thus,

$$\phi(\overline{\alpha}v \otimes f) = \phi(v \otimes h) = u \otimes (U^{-1})^* f^{\tau_1},$$

and the equality (3.4) is proved. The surjectivity of $k_f(\cdot)$ follows immediately. The same arguments prove that $h_y(\cdot)$ is surjective.

Next, let us show that $k_f(\cdot)$ and $h_y(\cdot)$ satisfy the conclusion. Let $(x, g) \in X \times X^*$, pick up a sequence $(z_n)_{n \geq 1} \subseteq X$ such that $\lim_{n \rightarrow +\infty} \|z_n - k_f(x)\| = 0$. Since $k_f(\cdot)$ is surjective, it follows that $k_{nf}(\cdot)$ must be surjective too for each natural number $n \geq 1$. This implies that for each $n \geq 1$, there exists $w_n \in X$ such that $k_{nf}(w_n) = z_n$. Therefore, by Equations (1.1) and (3.1), we have

$$\begin{aligned} r_{\frac{\varepsilon}{p}}(\eta(\langle y, f \rangle) k_f(x) \otimes h_{py}(g)) &= \frac{1}{p} r_{\varepsilon}(p\eta(\langle y, f \rangle) k_f(x) \otimes h_{py}(g)) \\ &= \frac{1}{p} r_{\varepsilon}(k_f(x) \otimes (U^{-1})^* f^{\tau_1} U(py)^{\tau_1} \otimes h_{py}(g)) \\ &= \frac{1}{p} r_{\varepsilon}(\phi(x \otimes f) \psi(py \otimes g)) \\ &= \frac{1}{p} r_{\varepsilon}((x \otimes f)(py \otimes g)) \\ &= \frac{1}{p} r_{\varepsilon}(p \langle y, f \rangle x \otimes g) \\ &= r_{\frac{\varepsilon}{p}}(\langle y, f \rangle x \otimes g) \end{aligned}$$

for all $p \geq 1$. Letting $p \rightarrow +\infty$, by Lemma 2.2 obtain that

$$\lim_{p \rightarrow +\infty} |\langle k_f(x), h_{py}(g) \rangle| = \frac{|\langle y, f \rangle|}{|\eta(\langle y, f \rangle)|} |\langle x, g \rangle| \quad (3.5)$$

On the other hand,

$$\begin{aligned} r_{\frac{\varepsilon}{n}}(\eta(\langle y, f \rangle) z_n \otimes h_y(g)) &= r_{\frac{\varepsilon}{n}}(\eta(\langle y, f \rangle) k_{nf}(w_n) \otimes h_y(g)) \\ &= \frac{1}{n} r_{\varepsilon}(n\eta(\langle y, f \rangle) k_{nf}(w_n) \otimes h_y(g)) \\ &= \frac{1}{n} r_{\varepsilon}(k_{nf}(w_n) \otimes (U^{-1})^* (nf)^{\tau_1} U y^{\tau_1} \otimes h_y(g)) \\ &= \frac{1}{n} r_{\varepsilon}(\phi(w_n \otimes nf) \psi(y \otimes g)) \\ &= \frac{1}{n} r_{\varepsilon}((w_n \otimes nf)(y \otimes g)) \\ &= \frac{1}{n} r_{\varepsilon}(n \langle y, f \rangle w_n \otimes g) \\ &= r_{\frac{\varepsilon}{n}}(\langle y, f \rangle w_n \otimes g). \end{aligned}$$

Letting $n \rightarrow +\infty$, then Lemma 2.2 implies that

$$|\langle k_y(x), h_y(g) \rangle| = \frac{|\langle y, f \rangle|}{|\eta(\langle y, f \rangle)|} \lim_{n \rightarrow +\infty} |\langle w_n, g \rangle|. \quad (3.6)$$

In the same manner, we also have

$$r_{\frac{\varepsilon}{np}}(\eta(\langle y, f \rangle) k_{nf}(w_n) \otimes h_{py}(g)) = r_{\frac{\varepsilon}{np}}(\langle y, f \rangle w_n \otimes g),$$

and therefore, letting $n \rightarrow +\infty$ and $p \rightarrow +\infty$, by Lemma 2.2, we obtain that

$$\lim_{n, p \rightarrow +\infty} |\langle k_{nf}(w_n), h_{py}(g) \rangle| = \frac{|\langle y, f \rangle|}{|\eta(\langle y, f \rangle)|} \lim_{n \rightarrow +\infty} |\langle w_n, g \rangle|.$$

This together with equations (3.5), (3.6) imply that

$$|\langle k_f(x), h_y(g) \rangle| = \frac{|\langle y, f \rangle|}{|\eta(\langle y, f \rangle)|} |\langle x, g \rangle|$$

for all $(x, g) \in X \times X^*$, and hence

$$\langle x, g \rangle = 0 \iff \langle k_f(x), h_y(g) \rangle = 0$$

for all $(x, g) \in X \times X^*$. $\square$

ASSERTION 2. There exist two scalar-valued functions $d, l : X \setminus \{0\} \times X^* \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$, $(x, f) \mapsto d_f(x)$; $(x, f) \mapsto l_x(f)$ such that one of the following results:

1. If $\dim(X) = \infty$, then there exists a bounded invertible linear or conjugate linear operator V such that

$$k_f(x) = d_f(x)V(x) \text{ and } h_x(f) = l_x(f)(V^{-1})^*(f) \quad ((x, f) \in X \setminus \{0\} \times X^* \setminus \{0\}). \quad (3.7)$$

2. If $X = \mathbb{C}^n$, with $n \geq 3$, then there exist a nonsingular matrix $V \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ' of $\mathbb{C}$ such that

$$k_y(x) = d_y(x)V(x^{\tau'}) \text{ and } h_x(y) = l_x(y)(V^{-1})^*(y^{\tau'}) \quad (x, y \in \mathbb{C}^n \setminus \{0\}). \quad (3.8)$$

Proof. Let $(y, f) \in X \times X^*$ such that $\langle y, f \rangle \neq 0$. By Assertion 1, we see that $k_f(\cdot)$ and $h_y(\cdot)$ are surjective maps satisfying (3.3). Then in light of Lemma 2.7, we have the following two possibilities:

1. If $\dim(X) = \infty$, then there exist two scalar-valued functions $\mu_f : X \rightarrow \mathbb{C} \setminus \{0\}$, $\nu_y : X^* \rightarrow \mathbb{C} \setminus \{0\}$ and two bounded invertible linear or conjugate linear operators V_f, V_y on X with V_f and V_y are linearly dependent such that

$$k_f(x) = \mu_f(x)V_f(x) \text{ and } h_y(g) = \nu_y(g)(V_y^{-1})^*(g) \quad ((x, g) \in X \times X^*). \quad (3.9)$$

2. If $X = \mathbb{C}^n$, with $n \geq 3$, then there exist two scalar-valued functions $\mu_f, \nu_y : \mathbb{C}^n \rightarrow \mathbb{C} \setminus \{0\}$ and two nonsingular matrices $V_f, V_y \in \mathcal{M}_n(\mathbb{C})$ with V_f and V_y are linearly dependent and two ring automorphisms τ_f, τ_y of $\mathbb{C}$ such that

$$k_f(x) = \mu_f(x)V_f(x^{\tau_f}) \text{ and } h_y(x) = \nu_y(x)(V_y^{-1})^*(x^{\tau_y}) \quad (x \in \mathbb{C}^n). \quad (3.10)$$

Now, we claim that $\tau_y = \tau_f$ for all $y, f \in \mathbb{C}^n \setminus \{0\}$. Indeed, let $y, f \in \mathbb{C}^n$ with $\langle y, f \rangle \neq 0$ and $x, z \in \mathbb{C}^n$ two unit vectors with $\langle x, z \rangle = 0$. Clearly,

$$\langle \mu x - z, x + \mu z \rangle = 0$$

for all $\mu \in \mathbb{C}$. From (3.3) and (3.10), we have

$$\begin{aligned} 0 &= \langle k_f(\mu x - z), h_y(x + \mu z) \rangle \\ &= \langle V_f((\mu x - z)^{\tau_f}), (V_y^{-1})^*((x + \mu z)^{\tau_y}) \rangle \\ &= \tau_f(\mu) \langle V_f(x^{\tau_f}), (V_y^{-1})^*(x^{\tau_y}) \rangle - \tau_y(\mu) \langle V_f(z^{\tau_f}), (V_y^{-1})^*(z^{\tau_y}) \rangle \\ &\quad + \tau_f(\mu) \tau_y(\mu) \langle V_f(x^{\tau_f}), (V_y^{-1})^*(z^{\tau_y}) \rangle - \langle V_f(z^{\tau_f}), (V_y^{-1})^*(x^{\tau_y}) \rangle \\ &= \tau_f(\mu) \langle V_f(x^{\tau_f}), (V_y^{-1})^*(x^{\tau_y}) \rangle - \tau_y(\mu) \langle V_f(z^{\tau_f}), (V_y^{-1})^*(z^{\tau_y}) \rangle \end{aligned}$$

for all $\mu \in \mathbb{C}$. Therefore,

$$\tau_f(\mu) \langle V_f(x^{\tau_f}), (V_y^{-1})^*(x^{\tau_y}) \rangle - \tau_y(\mu) \langle V_f(z^{\tau_f}), (V_y^{-1})^*(z^{\tau_y}) \rangle = 0 \quad (3.11)$$

for all $\mu \in \mathbb{C}$. Choosing $\mu = 1$ in the above equality, we obtain that

$$\langle V_f(x^{\tau_f}), (V_y^{-1})^*(x^{\tau_y}) \rangle = \langle V_f(z^{\tau_f}), (V_y^{-1})^*(z^{\tau_y}) \rangle. \quad (3.12)$$

This, together with Equation (3.11) imply that

$$(\tau_f(\mu) - \tau_y(\mu)) \langle V_f(x^{\tau_f}), (V_y^{-1})^*(x^{\tau_y}) \rangle = 0$$

for all $\mu \in \mathbb{C}$. Again by (3.3) and the fact that $\langle x, x \rangle \neq 0$, we have $\langle V_f(x^{\tau_f}), (V_y^{-1})^*(x^{\tau_y}) \rangle \neq 0$. Then

$$\tau_f(\mu) = \tau_y(\mu)$$

for all $\mu \in \mathbb{C}$. Thus, $\tau_y = \tau_f$ for all $y, f \in \mathbb{C}^n$ with $\langle y, f \rangle \neq 0$. In the other case, if $y, f \in \mathbb{C}^n \setminus \{0\}$ with $\langle y, f \rangle = 0$, then there exists $z \in \mathbb{C}^n$ such that $\langle z, y \rangle \neq 0$, and $\langle z, f \rangle \neq 0$. Then $\tau_z = \tau_y$ and $\tau_z = \tau_f$. Therefore, $\tau_y = \tau_f$ for all $y, f \in \mathbb{C}^n \setminus \{0\}$. Hence, there exists a ring automorphism τ' of $\mathbb{C}$ with $\tau_x = \tau'$ for all $x \in \mathbb{C}^n \setminus \{0\}$, which proves the claim.

Next, let us show that there exist a nonzero scalar β_x depending only on x , and if $\dim(X) = \infty$ (resp, $X = \mathbb{C}^n$) there exists a bounded invertible linear or conjugate linear operator (resp, nonsingular matrix) V independent of x such that $V_x = \beta_x V$ for all $x \in X \setminus \{0\}$. Indeed, note that V_f and V_y are linearly dependent for all $(y, f) \in X \times X^*$ with $\langle y, f \rangle \neq 0$. In the other case, if $(y, f) \in X \setminus \{0\} \times X^* \setminus \{0\}$ such that $\langle y, f \rangle = 0$, then there exists $(z, h) \in X \times X^*$ such that $\langle z, h \rangle \neq 0$, $\langle y, h \rangle \neq 0$ and $\langle z, f \rangle \neq 0$. It follows that V_z, V_h are linearly dependent, V_y, V_h are linearly dependent and V_z, V_f are also linearly dependent. Therefore, V_y and V_f are linearly dependent for all $(y, f) \in X \setminus \{0\} \times X^* \setminus \{0\}$. So, there exists a scalar-valued function $\beta : X \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$; $x \mapsto \beta_x$ and if $\dim(X) = \infty$ (resp, $X = \mathbb{C}^n$) there exists a bounded invertible linear or conjugate linear operator (resp, nonsingular matrix) V independent of x such that

$$V_x = \beta_x V \text{ for all } x \in X \setminus \{0\}. \quad (3.13)$$

Finally, set $d_f(x) = \beta_x \mu_f(x)$, $l_x(f) = \frac{v_x(f)}{\beta_x}$ for all $(x, f) \in X \setminus \{0\} \times X^* \setminus \{0\}$. This, together with (3.9), (3.10) and (3.13) tell us that

$$k_f(x) = d_f(x)V(x\tau'_1) \quad \text{and} \quad h_x(f) = l_x(f)(V^{-1})^*(f\tau'_1) \quad (3.14)$$

for all $(x, f) \in X \setminus \{0\} \times X^* \setminus \{0\}$, where

$$x\tau'_1 = \begin{cases} x^{\tau'} & \text{if } X = \mathbb{C}^n \\ x & \text{if } \dim(X) = \infty \end{cases}$$

for all $x \in X$ or $x \in X^*$. Thus, the proof of Assertion 2 is complete. $\square$

ASSERTION 3. There exist two scalar-valued functions $\alpha, \beta : \mathcal{A} \mapsto \mathbb{C} \setminus \{0\}$ satisfying $\alpha(\mathcal{A})\beta(\mathcal{A}) \subseteq \mathbb{T}$ and

$$\phi(A) = \alpha(A)VAU^{-1} \quad \text{and} \quad \psi(A) = \beta(A)UAV^{-1} \quad \text{for all } A \in \mathcal{A}.$$

Proof. From equations (2.4) and (2.5) in Lemma 2.7 and Assertion 2, it follows that

$$\phi(x \otimes f) = d_f(x)Vx\tau'_1 \otimes (U^{-1})^*f\tau'_1 \quad \text{and} \quad \psi(x \otimes f) = \overline{l_x(f)}Ux\tau'_1 \otimes (V^{-1})^*f\tau'_1$$

for all $(x, f) \in X \setminus \{0\} \times X^* \setminus \{0\}$. We define the maps $h_1, h_2 : \mathcal{F}_1(X) \mapsto \mathbb{C} \setminus \{0\}$ by

$$h_1(x \otimes f) = d_f(x) \quad \text{if } x \otimes f \neq 0, \quad h_1(0) = 1, \quad (3.15)$$

and

$$h_2(x \otimes f) = \overline{l_x(f)} \quad \text{if } x \otimes f \neq 0, \quad h_2(0) = 1. \quad (3.16)$$

Hence,

$$\phi(x \otimes f) = h_1(x \otimes f)Vx\tau'_1 \otimes (U^{-1})^*f\tau'_1 \quad \text{and} \quad \psi(x \otimes f) = h_2(x \otimes f)Ux\tau'_1 \otimes (V^{-1})^*f\tau'_1 \quad (3.17)$$

for all $x \otimes f \in \mathcal{F}_1(X)$.

To complete the proof of Assertion 3, we will proceed in two steps.

Step 1. There exist nonzero scalars μ and ν satisfying $\mu\nu = 1$ and

$$|h_1(x \otimes f)| = \mu \quad \text{and} \quad |h_2(x \otimes f)| = \nu \quad \text{for all nonzero } x \otimes f \in \mathcal{F}_1(X). \quad (3.18)$$

By equations (3.2), (3.14), (3.15) and (3.16), we have

$$\begin{aligned} \frac{|\langle y, f \rangle|}{|\eta(\langle y, f \rangle)|} |\langle x, g \rangle| &= |\langle k_f(x), h_y(g) \rangle| \\ &= \left| \left\langle d_f(x)Vx\tau'_1, l_y(g)(V^{-1})^*g\tau'_1 \right\rangle \right| \\ &= \left| \left\langle h_1(x \otimes f)Vx\tau'_1, \overline{h_2(y \otimes g)}(V^{-1})^*y\tau'_1 \right\rangle \right| \\ &= |h_1(x \otimes f)h_2(y \otimes g)| |\eta'(\langle x, g \rangle)| \end{aligned}$$

for all $(x, f), (y, g) \in X \times X^*$ with $\langle y, f \rangle \neq 0$, where

$$\eta'(z) := \begin{cases} \tau'(z) & \text{if } X = \mathbb{C}^n \\ \bar{z} & \text{if } \dim(X) = \infty \text{ and } V \text{ is conjugate linear} \\ z & \text{if } \dim(X) = \infty \text{ and } V \text{ is linear} \end{cases}$$

for all $z \in \mathbb{C}$. Therefore,

$$|h_1(x \otimes f)h_2(y \otimes g)| = \frac{|\langle y, f \rangle| |\langle x, g \rangle|}{|\eta(\langle y, f \rangle)| |\eta'(\langle x, g \rangle)|} \quad (3.19)$$

for all $(x, f), (y, g) \in X \times X^*$ with $\langle x, g \rangle \neq 0$ and $\langle y, f \rangle \neq 0$. We have two cases to discuss:

Case 1. If $\dim(X) = \infty$, we have

$$|h_1(x \otimes f)h_2(y \otimes g)| = \frac{|\langle y, f \rangle| |\langle x, g \rangle|}{|\eta(\langle y, f \rangle)| |\eta'(\langle x, g \rangle)|} = 1 \quad (3.20)$$

for all $(x, f), (y, g) \in X \times X^*$ with $\langle y, f \rangle \neq 0$ and $\langle x, g \rangle \neq 0$. Now, let $(x, f), (y, g) \in X \times X^*$ with $\langle x, g \rangle \neq 0$ and $\langle y, f \rangle = 0$. Choose $(w, h) \in X \times X^*$ with $\langle w, h \rangle \neq 0$, $\langle w, f \rangle \neq 0$ and $\langle y, h \rangle \neq 0$. Therefore, the equality (3.20) implies that

$$|h_1(x \otimes f)h_2(w \otimes g)| = 1, |h_1(x \otimes h)h_2(w \otimes g)| = 1 \text{ and } |h_1(x \otimes h)h_2(y \otimes g)| = 1,$$

and thus,

$$\begin{aligned} |h_1(x \otimes f)h_2(y \otimes g)| &= |h_1(x \otimes f)h_1(x \otimes h)h_2(w \otimes g)h_2(y \otimes g)| \\ &= |h_1(x \otimes f)h_2(w \otimes g)h_1(x \otimes h)h_2(y \otimes g)| \\ &= 1. \end{aligned}$$

Next, assume that $\langle x, g \rangle = 0$, choose $(z, k) \in X \times X^*$ with $\langle z, k \rangle \neq 0$, $\langle z, g \rangle \neq 0$ and $\langle x, k \rangle \neq 0$. By (3.20), we have

$$|h_1(x \otimes f)h_2(z \otimes k)| = 1, |h_1(z \otimes f)h_2(z \otimes k)| = 1 \text{ and } |h_1(z \otimes f)h_2(y \otimes g)| = 1.$$

Then,

$$\begin{aligned} |h_1(x \otimes f)h_2(y \otimes g)| &= |h_1(x \otimes f)h_1(z \otimes f)h_2(z \otimes k)h_2(y \otimes g)| \\ &= |h_1(x \otimes f)h_2(z \otimes k)h_1(z \otimes f)h_2(y \otimes g)| \\ &= 1. \end{aligned}$$

Consequently,

$$|h_1(x \otimes f)h_2(y \otimes g)| = 1$$

for all $(x, f), (y, g) \in X \times X^*$.

Case 2. If $X = \mathbb{C}^n$, obviously, from (3.19), we have

$$|h_1(x \otimes u)h_2(y \otimes v)| = \frac{|\langle y, u \rangle| |\langle x, v \rangle|}{|\eta(\langle y, u \rangle)| |\eta'(\langle x, v \rangle)|} = 1 \quad (3.21)$$

for all $x, y, u, v \in \mathbb{C}^n$ with $\langle y, u \rangle$ and $\langle x, v \rangle$ are nonzero integers. Next, assume that $x, y, u, v \in \mathbb{C}^n$ with $\langle y, u \rangle \neq 0$ and $\langle x, v \rangle$ is a nonzero integer. Considering y and u are linearly dependent or independent, we discuss the two cases respectively.

If y and u are linearly independent, then

$$\{z \in \mathbb{C}^n : \langle z, y \rangle = 1\} \text{ and } \{z \in \mathbb{C}^n : \langle z, u \rangle = 1\}$$

are two non-parallel affine hyperplanes of $\mathbb{C}^n$. So

$$\{z \in \mathbb{C}^n : \langle z, y \rangle = 1\} \cap \{z \in \mathbb{C}^n : \langle z, u \rangle = 1\}$$

is a nonempty affine hyperplane, hence there exists $z \in \mathbb{C}^n$ with $\langle z, z \rangle$ is a nonzero integer and

$$\langle z, y \rangle = 1 \text{ and } \langle z, u \rangle = 1.$$

Therefore, the equality (3.21) implies that

$$|h_1(x \otimes u)h_2(z \otimes v)| = 1 \text{ and } |h_1(x \otimes z)h_2(z \otimes v)| = 1 \text{ and } |h_1(x \otimes z)h_2(y \otimes v)| = 1.$$

Hence,

$$\begin{aligned} |h_1(x \otimes u)h_2(y \otimes v)| &= |h_1(x \otimes u)h_1(x \otimes z)h_2(z \otimes v)h_2(y \otimes v)| \\ &= |h_1(x \otimes u)h_2(z \otimes v)h_1(x \otimes z)h_2(y \otimes v)| \\ &= 1. \end{aligned}$$

If y and u are linearly dependent. Choose $z \in \mathbb{C}^n$ such $\langle z, z \rangle$ is positive integer and y, z are linearly independent. By the previous case we conclude that

$$|h_1(x \otimes u)h_2(z \otimes v)| = 1, \quad |h_1(x \otimes z)h_2(z \otimes v)| = 1 \text{ and } |h_1(x \otimes z)h_2(y \otimes v)| = 1,$$

hence

$$\begin{aligned} |h_1(x \otimes u)h_2(y \otimes v)| &= |h_1(x \otimes u)h_1(x \otimes z)h_2(z \otimes v)h_2(y \otimes v)| \\ &= |h_1(x \otimes u)h_2(z \otimes v)h_1(x \otimes z)h_2(y \otimes v)| \\ &= 1. \end{aligned}$$

We conclude that

$$|h_1(x \otimes u)h_2(y \otimes v)| = 1$$

for all $x, y, u, v \in \mathbb{C}^n$ with $\langle y, u \rangle \neq 0$ and $\langle x, v \rangle$ is a nonzero integer.

Finally, if $x, y, u, v \in \mathbb{C}^n$ with $\langle y, u \rangle = 0$ and $\langle x, v \rangle$ is a nonzero integer. Choose a vector $z \in \mathbb{C}^n$ with $\langle z, u \rangle \langle z, y \rangle \neq 0$, then

$$|h_1(x \otimes u)h_2(z \otimes v)| = 1, \quad |h_1(x \otimes z)h_2(z \otimes v)| = 1 \text{ and } |h_1(x \otimes z)h_2(y \otimes v)| = 1,$$

hence

$$\begin{aligned} |h_1(x \otimes u)h_2(y \otimes v)| &= |h_1(x \otimes u)h_1(x \otimes z)h_2(z \otimes v)h_2(y \otimes v)| \\ &= |h_1(x \otimes u)h_2(z \otimes v)h_1(x \otimes z)h_2(y \otimes v)| \\ &= 1. \end{aligned}$$

We conclude that

$$|h_1(x \otimes u)h_2(y \otimes v)| = 1$$

for all $x, y, u, v \in \mathbb{C}^n$ with $\langle x, v \rangle$ is a nonzero integer. Now, in a similar way to the above discussion, we get

$$|h_1(x \otimes u)h_2(y \otimes v)| = 1 \text{ for all } x, y, u, v \in \mathbb{C}^n$$

with $\langle x, v \rangle$ is not nonzero integer. Hence,

$$|h_1(x \otimes u)h_2(y \otimes v)| = 1 \quad (3.22)$$

for all $x, y, u, v \in \mathbb{C}^n$. We conclude that in both cases, that is, $\dim(X)$ is finite or infinite, we have

$$|h_1(x \otimes f)h_2(y \otimes g)| = 1$$

for all $(x, f), (y, g) \in X \times X^*$. Consequently, there exist nonzero scalars μ and ν satisfying $\mu\nu = 1$ and

$$|h_1(x \otimes f)| = \mu \text{ and } |h_2(x \otimes f)| = \nu \quad (3.23)$$

for all $(x, f) \in X \times X^*$.

Step 2. The statement of Assertion 3 holds.

From (3.19), (3.22) we conclude that τ and τ' are continuous, and this implies that τ and τ' are either the identity or the conjugation. So in the rest of this section U denotes the bounded invertible linear or conjugate linear operator on X given in Lemma 2.6, and V is also the bounded invertible linear or conjugate linear operator on X given in the previous assertion. Set $\Phi(A) := V^{-1}\phi(A)U$. Let $A \in \mathcal{A} \setminus \{0\}$, we claim that $\Phi(A)$ and A are linearly dependent for all $A \in \mathcal{A} \setminus \{0\}$. Indeed, from (1.1), (3.17) and (3.23), we have

$$\begin{aligned} r_{\frac{\varepsilon}{n}}(v\phi(A)Ux \otimes (V^{-1})^*f) &= \frac{1}{n}r_{\varepsilon}(\phi(A)h_2(nx \otimes f)U(nx \otimes (V^{-1})^*f)) \\ &= \frac{1}{n}r_{\varepsilon}(\phi(A)\psi(nx \otimes f)) \\ &= \frac{1}{n}r_{\varepsilon}(A(nx \otimes f)) \\ &= r_{\frac{\varepsilon}{n}}(Ax \otimes f) \end{aligned}$$

for all $A \in \mathcal{A} \setminus \{0\}$, $(x, f) \in X \times X^*$ and natural number $n \geq 1$. Therefore,

$$r_{\frac{\varepsilon}{n}}(v\phi(A)Ux \otimes (V^{-1})^*f) = r_{\frac{\varepsilon}{n}}(Ax \otimes f) \quad (3.24)$$

for all $A \in \mathcal{A} \setminus \{0\}$, $(x, f) \in X \times X^*$ and natural number $n \geq 1$. Letting $n \rightarrow +\infty$, then Lemma 2.2 and (3.24) imply that

$$|\langle v\phi(A)Ux, (V^{-1})^*f \rangle| = |\langle Ax, f \rangle|$$

for all $A \in \mathcal{A} \setminus \{0\}$, $(x, f) \in X \times X^*$. Thus,

$$|\langle vV^{-1}\phi(A)Ux, f \rangle| = |\langle Ax, f \rangle| \quad (3.25)$$

for all $A \in \mathcal{A} \setminus \{0\}$, $(x, f) \in X \times X^*$. It then follows that $\ker(\Phi(A)) = \ker(A)$ for all $A \in \mathcal{A} \setminus \{0\}$. Denote $M = \ker(A)$ and for the quotient $\widehat{X} := X/M$, consider the linear bounded operators

$$\widehat{\Phi(A)}, \widehat{A} : \widehat{X} \mapsto \widehat{X} \text{ given by } \widehat{\Phi(A)}(\widehat{x}) = V^{-1}\phi(A)U(x) \text{ and } \widehat{A}(\widehat{x}) = A(x).$$

Let $x \in X/M$ there exists a unique nonzero scalar λ_x such that $\Phi(A)x = \lambda_x A^*x$. Now, we claim that the scalar-valued function $x \mapsto \lambda_x$ is constant on X/M . Indeed, let $x, y \in X/M$, observe first that if x and y are linearly dependent, then have obviously $\lambda_x = \lambda_y$. Thus we may assume that x and y are linearly independent. Then, we have

$$\Phi(A)(x+y) = \Phi(A)x + \Phi(A)y = \lambda_x Ax + \lambda_y Ay,$$

on the other hand,

$$\Phi(A)(x+y) = \lambda_{x+y}A(x+y) = \lambda_{x+y}Ax + \lambda_{x+y}Ay,$$

therefore,

$$(\lambda_{x+y} - \lambda_x)Ax = (\lambda_y - \lambda_{x+y})Ay,$$

thus,

$$A((\lambda_{x+y} - \lambda_x)x - (\lambda_y - \lambda_{x+y})y) = 0.$$

Hence,

$$\widehat{A}((\lambda_{x+y} - \lambda_x)x - (\lambda_y - \lambda_{x+y})y) = 0.$$

Since $\widehat{A}$ is injective operator then, $(\lambda_{x+y} - \lambda_x)x - (\lambda_y - \lambda_{x+y})y = 0$ and thus, $\lambda_x = \lambda_{x+y} = \lambda_y$. Hence, there exists a nonzero scalar λ such that $\Phi(A)x = \lambda Ax$ for all $x \in X/M$, so, $\Phi(A)x = \lambda Ax$ for all $x \in X$. Consequently $\Phi(A)$ and A are linearly dependent, this implies that there exists a scalar-valued function $\alpha : \mathcal{A} \mapsto \mathbb{C} \setminus \{0\}$ such that

$$\phi(A) = \alpha(A)VAU^{-1} \quad (3.26)$$

for all $A \in \mathcal{A}$. Equation (3.25) implies that $|\alpha(A)| = \mu$ for all $A \in \mathcal{A}$. Clearly, we have $\alpha(A) = h_1(A)$ if A is of rank one. The same arguments prove that there exists a scalar-valued function $\beta : \mathcal{A} \mapsto \mathbb{C} \setminus \{0\}$ such that

$$\psi(A) = \beta(A)UAV^{-1} \quad (3.27)$$

for all $A \in \mathcal{A}$, with $\beta(A) = h_2(A)$ if $A \in \mathcal{F}_1(X)$ and $|\beta(A)| = \nu$ for all $A \in \mathcal{A}$. $\square$

ASSERTION 5. There exists $\alpha > 0$ such that $\|Vx\| = \alpha \|x\|$ for all $x \in X$.

Proof. Let $A, B \in \mathcal{A}$ and let $n \geq 1$ be a natural number. From equations (1.1), (3.26), (3.27) and Assertion 4, we have

$$\begin{aligned} r_{\frac{\varepsilon}{n}}(VABV^{-1}) &= \frac{1}{n}r_{\varepsilon}(V(nA)BV^{-1}) = \frac{1}{n}r_{\varepsilon}(\alpha(nA)\beta(B)V(nA)U^{-1}UBV^{-1}) \\ &= \frac{1}{n}r_{\varepsilon}(\phi(nA)\psi(B)) \\ &= \frac{1}{n}r_{\varepsilon}((nA)B) \\ &= r_{\frac{\varepsilon}{n}}(AB). \end{aligned}$$

In particular, for $A = x \otimes f$ and $B = y \otimes g$ with $\langle y, f \rangle = 1$ and $\langle x, g \rangle = 0$, we get

$$r_{\frac{\varepsilon}{n}}(Vx \otimes gV^{-1}) = r_{\frac{\varepsilon}{n}}(x \otimes g).$$

Since $(x \otimes g)^2 = 0$ if and only if $(Vx \otimes gV^{-1})^2 = 0$, it follows from Lemma 2.4, we obtain

$$\begin{aligned} \sqrt{\|Vx\| \|(V^{-1})^*g\|} &= \sqrt{\|Vx \otimes gV^{-1}\|} = \lim_{n \rightarrow +\infty} \frac{r_{\frac{\varepsilon}{n}}(Vx \otimes gV^{-1})}{\sqrt{\frac{\varepsilon}{n}}} \\ &= \lim_{n \rightarrow +\infty} \frac{r_{\frac{\varepsilon}{n}}(x \otimes g)}{\sqrt{\frac{\varepsilon}{n}}} = \sqrt{\|x \otimes g\|} = \sqrt{\|x\| \|g\|}. \end{aligned}$$

Hence,

$$\|Vx\| \|(V^{-1})^*g\| = \|x\| \|g\| \quad (3.28)$$

for all $(x, g) \in X \times X^*$ such that $\langle x, g \rangle = 0$. Now, since $\dim X \geq 3$, then for two arbitrary nonzero vectors $x, y \in X$ by considering $f \in X^* \setminus \{0\}$ such that $\langle x, f \rangle = \langle y, f \rangle = 0$. By formula (3.28) we obtain that $\frac{\|Vx\|}{\|x\|} = \frac{\|Vy\|}{\|y\|} = \frac{\|(V^{-1})^*f\|}{\|f\|}$. This implies that V is a λ multiple of a unitary operator W . So, again by Equations (3.26), (3.27) we get

$$\phi(A) = \alpha(A)\lambda WAU^{-1} \text{ and } \psi(A) = \frac{\beta(A)}{\lambda}UAW^{-1} \quad (3.29)$$

for all $A \in \mathcal{A}$. Since $|\alpha(A)\lambda \frac{\beta(B)}{\lambda}| = |\alpha(A)\beta(B)| = 1$ for all $A, B \in \mathcal{A}$, then the proof of Theorems 1.1 and 1.3 is complete. $\square$

4. Proof of Theorems 1.2, 1.4

Since the “only if” part is trivial, we only have to prove the “if part”; So, assume that the maps ϕ and ψ satisfy (1.2) Clearly,

$$r_{\varepsilon}(\phi(A)\psi(B)) = r_{\varepsilon}(AB) \quad (4.1)$$

for all $A, B \in \mathcal{A}$. Thus we can assume that ϕ and ψ have the forms given in Theorems 1.1 and 1.3, i.e.,

$$\phi(A) = h_1(A)VA^{\tau_1}U^{-1} \text{ and } \psi(A) = h_2(A)UA^{\tau_1}V^{-1} \quad (A \in \mathcal{A}), \quad (4.2)$$

where

$$A^{\tau_1} = \begin{cases} A^{\tau} & \text{if } X = \mathbb{C}^n \\ A & \text{if } \dim(X) = \infty \end{cases}$$

for all $A \in \mathcal{B}(X)$, and τ is either the identity or the conjugation. We also denote

$$\eta(z) := \begin{cases} \tau(z) & \text{if } X = \mathbb{C}^n \\ \bar{z} & \text{if } \dim(X) = \infty \text{ and } U \text{ is conjugate linear} \\ z & \text{if } \dim(X) = \infty \text{ and } U \text{ is linear} \end{cases}$$

for all $z \in \mathbb{C}$.

Now, we claim that U and V are linear and there exist two scalars μ, ν such that $\mu\nu = 1$, $h_1(A) = \mu$ and $h_2(A) = \nu$ for all $A \in \mathcal{A}$. Moreover, if $X = \mathbb{C}^n$ then τ is the identity of $\mathbb{C}$. Indeed, let us first show that $h_1, h_2 : \mathcal{A} \rightarrow \mathbb{C} \setminus \{0\}$ are in fact constants. Let $A \in \mathcal{A} \setminus \{0\}$, $(x, f) \in X \times X^*$ and natural number $n \geq 1$, by equations (1.2) and (4.2) we have

$$\begin{aligned}
 & \left(\sigma_{\frac{\varepsilon}{n}} \left(h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} x^{\tau_1} \otimes f^{\tau_1} \right) \right)^V \\
 &= \left(\sigma_{\frac{\varepsilon}{n}} \left(h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} (x \otimes f)^{\tau_1} \right) \right)^V \\
 &= \frac{1}{n} \sigma_{\varepsilon} \left(nV h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} (x \otimes f)^{\tau_1} V^{-1} \right) \\
 &= \frac{1}{n} \sigma_{\varepsilon} \left(V h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} (nx \otimes f)^{\tau_1} V^{-1} \right) \\
 &= \frac{1}{n} \sigma_{\varepsilon} \left(h_1(A) h_2(nx \otimes f) (VA^{\tau_1} U^{-1}) (U(nx \otimes f)^{\tau_1} V^{-1}) \right) \\
 &= \frac{1}{n} \sigma_{\varepsilon} \left((h_1(A) VA^{\tau_1} U^{-1}) (h_2(nx \otimes f) U(nx \otimes f)^{\tau_1} V^{-1}) \right) \\
 &= \frac{1}{n} \sigma_{\varepsilon} (\phi(A) \psi(nx \otimes f)) \\
 &= \frac{1}{n} \sigma_{\varepsilon} (A(nx \otimes f)) \\
 &= \sigma_{\frac{\varepsilon}{n}} (Ax \otimes f),
 \end{aligned}$$

where $\lambda^V = \lambda$ (resp. $\overline{\lambda}$) if V is linear (resp. conjugate linear) for all scalars λ . Hence

$$\sigma_{\frac{\varepsilon}{n}} \left(h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} x^{\tau_1} \otimes f^{\tau_1} \right)^V = \sigma_{\frac{\varepsilon}{n}} (Ax \otimes f) \quad (4.3)$$

for all $A \in \mathcal{A} \setminus \{0\}$, $(x, f) \in X \times X^*$ and natural number $n \geq 1$.

Now, let $A, B \in \mathcal{A} \setminus \{0\}$, let us fix $(x, f) \in X \times X^*$ such that $\langle Ax, f \rangle = \langle Bx, f \rangle \neq 0$, by equation (4.3), we have

$$\sigma_{\frac{\varepsilon}{n}} \left(h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} x^{\tau_1} \otimes f^{\tau_1} \right)^V = \sigma_{\frac{\varepsilon}{n}} (Ax \otimes f) \quad (n \geq 1). \quad (4.4)$$

This implies that

$$\bigcap_{n \geq 1} \sigma_{\frac{\varepsilon}{n}} \left(h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} x^{\tau_1} \otimes f^{\tau_1} \right)^V = \bigcap_{n \geq 1} \sigma_{\frac{\varepsilon}{n}} (Ax \otimes f) = \sigma(Ax \otimes f) = \{\langle Ax, f \rangle\}.$$

Since $\sigma(A) + D(0, \varepsilon) \subseteq \sigma_{\varepsilon}(A)$ for all $A \in \mathcal{B}(X)$. Then,

$$\begin{aligned}
 & D \left(h_1(A)^V h_2(nx \otimes f)^V \tau_1 (\langle Ax, f \rangle), \frac{\varepsilon}{n} \right)^V \\
 &= \sigma \left(h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} x^{\tau_1} \otimes f^{\tau_1} \right)^V + D \left(0, \frac{\varepsilon}{n} \right)^V \\
 &\subseteq \sigma_{\frac{\varepsilon}{n}} \left(h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1} x^{\tau_1} \otimes f^{\tau_1} \right)^V \quad (n \geq 1).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \bigcap_{n \geq 1} D \left(h_1(A)^V h_2(nx \otimes f)^V \tau_1(\langle Ax, f \rangle), \frac{\varepsilon}{n} \right)^V \\
 & \subseteq \bigcap_{n \geq 1} \sigma_{\frac{\varepsilon}{n}} \left(h_1(A)^V h_2(nx \otimes f)^V A^{\tau_1 x \tau_1} \otimes f^{\tau_1} \right)^V \\
 & \subseteq \bigcap_{n \geq 1} \sigma_{\frac{\varepsilon}{n}} (Ax \otimes f) \\
 & \subseteq \sigma(Ax \otimes f) \\
 & \subseteq \{ \langle Ax, f \rangle \}.
 \end{aligned}$$

In the same manner, we also have

$$\bigcap_{n \geq 1} D \left(h_1(B)^V h_2(nx \otimes f)^V \tau_1(\langle Bx, f \rangle), \frac{\varepsilon}{n} \right)^V \subseteq \{ \langle Bx, f \rangle \}.$$

Thus,

$$\lim_{n \rightarrow +\infty} h_2(nx \otimes f)^V = \frac{\langle Ax, f \rangle}{h_1(A)^V \tau_1(\langle Ax, f \rangle)} = \frac{\langle Bx, f \rangle}{h_1(B)^V \tau_1(\langle Bx, f \rangle)}.$$

This implies that $h_1(A) = h_1(B)$, so $h_1 : \mathcal{A} \rightarrow \mathbb{C} \setminus \{0\}$ is a constant map. By a similar argument as above, we conclude that $h_2 : \mathcal{A} \rightarrow \mathbb{C} \setminus \{0\}$ is a constant map. Consequently, there exist two nonzero scalars μ and ν such that

$$h_1(A) = \mu \quad \text{and} \quad h_2(A) = \nu \tag{4.5}$$

for all $A \in \mathcal{A}$. Second, let us show that the operators U and V are linear and $\mu\nu = 1$. Moreover, if $X = \mathbb{C}^n$ then τ is the identity of $\mathbb{C}$. We claim that V is linear and $\mu\nu = 1$. To do so, let $(x, f) \in X \times X^*$ such that $\langle x, f \rangle = 1$, then by equations (1.2), (4.2) and (4.5), we have

$$\begin{aligned}
 & \sigma_{\frac{\varepsilon}{n}} (\mu^V \nu^V \tau_1(\alpha) x^{\tau_1} \otimes f^{\tau_1})^V \\
 &= \frac{1}{n} \sigma_{\varepsilon} (n \mu^V \nu^V \tau_1(\langle \alpha x, f \rangle) x^{\tau_1} \otimes f^{\tau_1})^V \\
 &= \frac{1}{n} \sigma_{\varepsilon} (\mu^V \nu^V (x^{\tau_1} \otimes f^{\tau_1}) ((n \alpha x)^{\tau_1} \otimes f^{\tau_1}))^V \\
 &= \frac{1}{n} \sigma_{\varepsilon} (\mu^V \nu^V (x \otimes f)^{\tau_1} (n \alpha x \otimes f)^{\tau_1})^V \\
 &= \frac{1}{n} \sigma_{\varepsilon} (V \mu^V \nu^V (x \otimes f)^{\tau_1} (n \alpha x \otimes f)^{\tau_1} V^{-1}) \\
 &= \frac{1}{n} \sigma_{\varepsilon} (\mu \nu (V(x \otimes f)^{\tau_1} U^{-1}) (U(n \alpha x \otimes f)^{\tau_1} V^{-1})) \\
 &= \frac{1}{n} \sigma_{\varepsilon} ((\mu V(x \otimes f)^{\tau_1} U^{-1}) (\nu U(n \alpha x \otimes f)^{\tau_1} V^{-1})) \\
 &= \frac{1}{n} \sigma_{\varepsilon} ((h_1(x \otimes f) V(x \otimes f)^{\tau_1} U^{-1}) (h_2(n \alpha x \otimes f) U(n \alpha x \otimes f)^{\tau_1} V^{-1}))
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n} \sigma_{\varepsilon}(\phi(x \otimes f) \psi(n \alpha x \otimes f)) \\
&= \frac{1}{n} \sigma_{\varepsilon}((x \otimes f)(n \alpha x \otimes f)) \\
&= \frac{1}{n} \sigma_{\varepsilon}(n \alpha \langle x, f \rangle x \otimes f) \\
&= \sigma_{\frac{\varepsilon}{n}}(\alpha x \otimes f)
\end{aligned}$$

for all natural number $n \geq 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$. It follows that

$$\begin{aligned}
\{\mu \nu \tau_1(\alpha)^V\} &= \{(\mu \nu)^V \tau_1(\alpha) \tau_1(\langle x, f \rangle)\}^V \\
&= \sigma((\mu \nu)^V \tau_1(\alpha) x^{\tau_1} \otimes f^{\tau_1})^V \\
&= \bigcap_{n \geq 1} \sigma_{\frac{\varepsilon}{n}}((\mu \nu)^V \tau_1(\alpha) x^{\tau_1} \otimes f^{\tau_1})^V \\
&= \bigcap_{n \geq 1} \sigma_{\frac{\varepsilon}{n}}(\alpha x \otimes f) \\
&= \sigma(\alpha x \otimes f) = \{\alpha \langle x, f \rangle\} = \{\alpha\}
\end{aligned}$$

for all $\alpha \in \mathbb{C} \setminus \{0\}$. This implies that $\mu \nu \tau_1(\alpha)^V = \alpha$ for all $\alpha \in \mathbb{C} \setminus \{0\}$. If $X = \mathbb{C}^n$ then, for $\alpha = 1$ we obtain $\mu \nu = 1$, hence $\tau(\alpha) = \alpha$ for all $\alpha \in \mathbb{C} \setminus \{0\}$. Thus τ is the identity of $\mathbb{C}$. Now, if $\dim(X) = \infty$, then, for $\alpha = 1$ we obtain $\mu \nu = 1$, hence $\alpha^V = \alpha$ for all $\alpha \in \mathbb{C} \setminus \{0\}$. Thus V is linear operator.

Finally, we prove that if $\dim(X) = \infty$, then U is linear. Indeed, assume by the way of contradiction that U is conjugate linear. Then, let $(x, f) \in X \times X^*$ such that $\langle x, f \rangle = 1$, then by equations (1.2), (4.2) and (4.5), we have

$$\begin{aligned}
\sigma_{\frac{\varepsilon}{n}}(\overline{\alpha} x \otimes f) &= \frac{1}{n} \sigma_{\varepsilon} \left((V \mu \nu n \overline{\alpha \langle x, f \rangle} x \otimes f V^{-1}) \right) \\
&= \frac{1}{n} \sigma_{\varepsilon} ((\mu V(n x \otimes f) U^{-1})(\nu U(\alpha x \otimes f) V^{-1})) \\
&= \frac{1}{n} \sigma_{\varepsilon} ((h_1(n x \otimes f) V(n x \otimes f) U^{-1})(h_2(\alpha x \otimes f) U(\alpha x \otimes f) V^{-1})) \\
&= \frac{1}{n} \sigma_{\varepsilon}(\phi(n x \otimes f) \psi(\alpha x \otimes f)) \\
&= \frac{1}{n} \sigma_{\varepsilon}((n x \otimes f)(\alpha x \otimes f)) \\
&= \frac{1}{n} \sigma_{\varepsilon}(n \alpha \langle x, f \rangle x \otimes f) \\
&= \sigma_{\frac{\varepsilon}{n}}(\alpha x \otimes f)
\end{aligned}$$

for all natural number $n \geq 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Thus,

$$\begin{aligned}
\{\overline{\alpha}\} &= \{\overline{\alpha \langle x, f \rangle}\} = \sigma(\overline{\alpha} x \otimes f) = \bigcap_{n \geq 1} \sigma_{\frac{\varepsilon}{n}}(\overline{\alpha} x \otimes f) \\
&= \bigcap_{n \geq 1} \sigma_{\frac{\varepsilon}{n}}(\alpha x \otimes f) = \sigma(\alpha x \otimes f) = \{\alpha \langle x, f \rangle\} = \{\alpha\}
\end{aligned}$$

for all $\alpha \in \mathbb{C} \setminus \{0\}$. This implies that $\bar{\alpha} = \alpha$ for all $\alpha \in \mathbb{C} \setminus \{0\}$, which is a contradiction and so U is linear. Thus, we obtain the desired forms of ϕ and ψ , then the proof of Theorems 1.2 and 1.4 is complete. $\square$

5. Proof of Corollaries 1.5 and 1.6

We start with the proof of Corollary 1.5. The “if” part is straightforward. Then assume that $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ are two surjective maps satisfying (1.1). From Theorems 1.1 and 1.3, there exist two scalar-valued functions $h_1, h_2 : \mathcal{A} \longrightarrow \mathbb{C} \setminus \{0\}$ with $h_1(\mathcal{A})h_2(\mathcal{A}) \subseteq \mathbb{T}$, and two bounded invertible linear or conjugate linear operators U, V on X such that $\|Vx\| = \|x\|$ for each $x \in X$,

$$\phi(A) = h_1(A)VAU^{-1}$$

and

$$\phi(A) = h_2(A)UAV^{-1}$$

for all $A \in \mathcal{A}$. It then follows that

$$h_1(x \otimes f)V(x \otimes f)U^{-1} = \phi(x \otimes f) = h_2(x \otimes f)U(x \otimes f)V^{-1}$$

for all $x \otimes f \in \mathcal{F}_1(X)$. This implies that,

$$h_1(x \otimes f)Vx \otimes (U^{-1}V)^*f = h_1(x \otimes f)V(x \otimes f)U^{-1}V = h_2(x \otimes f)Ux \otimes f$$

for all $x \otimes f \in \mathcal{F}_1(X)$. Hence Ux and Vx are linearly dependent for all $x \in X$. It then follows that there exists $\beta \in \mathbb{C}$ such that $Vx = \beta Ux$ for all $x \in X$, and then, $V = \beta U$. So, $\phi(A) = h(A)UAU^{-1}$, with $h(A) = \beta h_1(A)$. Finally, we claim that $|h(A)| = 1$ for all $A \in \mathcal{A}$. Indeed, let us first show that $|h| : \mathcal{F}_1(X) \longrightarrow \mathbb{C} \setminus \{0\}$, $A \mapsto |h(A)|$ is in fact constant. Let $A \in \mathcal{A}$, $(x, f) \in X \times X^*$ and natural number $n \geq 1$, from equation (1.1), we have

$$\begin{aligned} nr_{\frac{\varepsilon}{n}}(h(A)h(nx \otimes f)Ax \otimes f) &= r_{\varepsilon}(h(A)h(nx \otimes f)A(nx \otimes f)) \\ &= r_{\varepsilon}(h(A)UAU^{-1}h(nx \otimes f)U(nx \otimes f)U^{-1}) \\ &= r_{\varepsilon}(\phi(A)\phi(nx \otimes f)) \\ &= r_{\varepsilon}(A(nx \otimes f)) \\ &= nr_{\frac{\varepsilon}{n}}(Ax \otimes f). \end{aligned}$$

Therefore,

$$r_{\frac{\varepsilon}{n}}(h(A)h(nx \otimes y)Ax \otimes f) = r_{\frac{\varepsilon}{n}}(Ax \otimes f) \quad (5.1)$$

for all $A \in \mathcal{A}$, $(x, f) \in X \times X^*$. Now, let $A, B \in \mathcal{A}$, let us fix $(x, f) \in X \times X^*$ such that $Ax \neq 0$, $Bx \neq 0$ and $\langle Ax, f \rangle = \langle Bx, f \rangle \neq 0$. Letting $n \rightarrow +\infty$, by Lemma 2.2 and the equation (5.1), we get

$$|h(A)| = \frac{1}{\lim_{n \rightarrow +\infty} |h(nx \otimes f)|}. \quad (5.2)$$

Similarly, we have

$$|h(B)| = \frac{1}{\lim_{n \rightarrow +\infty} |h(nx \otimes f)|}.$$

This, together with Equation (5.2) imply that

$$|h(A)| = |h(B)|.$$

Consequently, there exists a complex unit scalar μ such that

$$|h(A)| = \mu$$

for all $A \in \mathcal{A}$. Finally, we claim that $\mu = 1$. Indeed, let us fix $A \in \mathcal{A}$, and $(x, f) \in X \times X^*$ such that $Ax \neq 0$ and $\langle Ax, f \rangle \neq 0$, by applying Lemma 2.2, and the equation (5.1), we conclude that

$$\mu = 1,$$

and this completes the proof. $\square$

Now, we prove Corollary 1.6. Checking the “if” part is straightforward. For the “only if” part, assume that $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ are two surjective maps satisfying (1.2). By Theorems 1.2 and 1.4 there exist two nonzero scalars μ and ν such that $\mu\nu = 1$, and two bounded invertible linear operators U, V on X such that $\|Vx\| = \|x\|$ for each $x \in X$, and

$$\phi(A) = \mu V A U^{-1}$$

and

$$\phi(A) = \nu U A V^{-1}$$

for all $A \in \mathcal{A}$. In particular, we have

$$\mu V x \otimes (U^{-1})^* f = \phi(x \otimes f) = \nu U x \otimes (V^{-1})^* f \quad (5.3)$$

for all $x \otimes f \in \mathcal{F}_1(X)$. Equation (5.3) implies that Ux and Vx are linearly dependent for all $x \in X$. It then follows that there exists $\alpha \in \mathbb{C}$ such that $Vx = \alpha Ux$ for all $x \in X$, and then, $V = \alpha U$. So, $\phi(A) = \lambda U A U^{-1}$, with $\lambda = \alpha\mu$. Finally, we claim that $\lambda^2 = 1$. Indeed, let $A \in \mathcal{A}$, $(x, f) \in X \times X^*$ with $\langle Ax, f \rangle = 1$, by equation (1.2), we have

$$\begin{aligned} n\sigma_n(\lambda^2 Ax \otimes f) &= \sigma_n(\lambda^2 A(nx \otimes f)) \\ &= \sigma_n(\lambda U A U^{-1} \lambda U(nx \otimes f) U^{-1}) \\ &= \sigma_n(\phi(A) \phi(nx \otimes f)) \\ &= \sigma_n(A(nx \otimes f)) \\ &= n\sigma_n(Ax \otimes f) \end{aligned}$$

for all natural number $n \geq 1$. Therefore,

$$\sigma_n(\lambda^2 Ax \otimes f) = \sigma_n(Ax \otimes f)$$

for all natural number $n \geq 1$. It then follows that

$$\begin{aligned}\{\lambda^2\} &= \{\lambda^2 \langle Ax, f \rangle\} = \sigma(\lambda^2 Ax \otimes f) = \bigcap_{n \geq 1} \sigma_{\mathbb{E}}(\lambda^2 Ax \otimes f) = \bigcap_{n \geq 1} \sigma_{\mathbb{E}}(Ax \otimes f) \\ &= \sigma(Ax \otimes f) = \{\langle Ax, f \rangle\} = \{1\}.\end{aligned}$$

Thus, $\lambda^2 = 1$, so, we conclude that $\phi(A) = \lambda UAU^{-1}$ for all $A \in \mathcal{A}$; as desired. The proof is therefore complete. $\square$

QUESTION. Under which conditions the assertion of Theorems 1.1, 1.2, 1.3 and 1.4) are valid in some $\mathbb{C}^*$ -algebras?

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(Received January 23, 2026)

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