

ON ITERATIVE APPROXIMATE SCHAUDER FRAMES IN BANACH SPACES

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Abstract. Let X be a separable Banach space with the dual space X^* . The purpose of the paper is to investigate an approximate Schauder frame $(x_k, f_k)_{k \in \mathbb{Z}} \subset X \times X^*$ of the form $(W^k x_0, T^k f_0)_{k \in \mathbb{Z}}$ where $W \in B(X)$ and $T \in B(X^*)$ are invertible, namely, iterative approximate Schauder frames. A main result determines the operator W in terms of the adjoint of T and the frame operator. As a consequence, we also obtain some relationships between the norms of the operators W and T . Moreover, using the theory of q -Riesz basis ($1 < q < \infty$), we give sufficient conditions for an approximate Schauder frame to be iterative. Finally, we obtain some results on the duality construction of an iterative approximate Schauder frame; in particular, if $(W^k x_0, T^k f_0)_{k \in \mathbb{Z}}$ is shrinking, then its canonical dual also has an iterative structure.

1. Introduction

In 2000, Han and Larson, motivated by [17, Theorem 1.7], introduced the notion of a Schauder frame in the sense of an inner direct summand of a Schauder basis. Similarly, they defined an unconditional Schauder frame as an inner direct summand of an unconditional Schauder basis. Schauder frames extend the concept of frames to Banach spaces; moreover, even when the underlying space is a Hilbert space, Schauder frames are more general than ordinary frames. Casazza et al. followed this approach and presented a concrete form of the abstract definition:

DEFINITION 1.1. ([9]) Let X be a (finite or infinite dimensional) separable Banach space. A sequence $(x_i, f_i)_{i \in I}$, with $\{x_i\}_{i \in I} \subset X$ and $\{f_i\}_{i \in I} \subset X^*$, and $I = \mathbb{N}$ or $I = \{1, 2, \dots, N\}$, for some $N \in \mathbb{N}$, is called a Schauder frame for X if for every $x \in X$

$$x = \sum_{i \in I} f_i(x) x_i. \quad (1)$$

In the case that $I = \mathbb{N}$, we mean that the series in (1) converges in norm, i.e. $x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f_i(x) x_i$.

An unconditional Schauder frame of X is a Schauder frame $(x_i, f_i)_{i \in \mathbb{N}}$ for X for which the convergence in (1) is unconditional.

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In this paper, however, we use the notion of an approximate Schauder frame. This concept was first introduced in [15], where Freeman et al. provided an explicit construction of an approximate Schauder frame of translates for $L^p(\mathbb{R}^d)$ for $d \in \mathbb{N}$ and $2 < p < \infty$. In general, let X be a separable Banach space. A sequence $(x_i, f_i)_{i=1}^\infty \subset X \times X^*$ is called an approximate Schauder frame for X if the operator $S : X \rightarrow X$, defined by $Sx = \sum_{i=1}^\infty f_i(x)x_i$, is (well defined) bounded and bijective. This operator, which plays a key role in the definition of an approximate Schauder frame, is called the frame operator associated with $(x_i, f_i)_{i=1}^\infty$. On the other hand, by extending the frame inequality from the standard definition of a frame in a Hilbert space to Banach spaces, Aldroubi, Sun, and Tang [4] considered two types of frames for Banach spaces: Banach frames and p -frames ($1 < p < \infty$).

Today, the theory of frames has many applications in signal processing, image processing, sampling theory and so on. For example, we refer to [5, 6, 7, 16] for more information. In 2015, Aldroubi et al. introduced the dynamical sampling problem [3]. More precisely, let $F = \{f(i), Tf(i), \dots, T^l f(i) : i \in \Omega\}$ where $\Omega \subset I$ and $T : \ell^2(I) \rightarrow \ell^2(I)$ is a bounded operator. They investigated the conditions on T , Ω and the set $\{l_i : i \in \Omega\}$ to reconstruct any $f \in \ell^2(I)$ from the samples F . Therefore, dynamical sampling deals with the study of systems arising via iterative actions of a bounded operator. Specifically, motivated by the concept of the frame of translates for $L^2(\mathbb{R})$, Christensen and Hasannasab analyzed the properties of frames in a Hilbert space H that have the form $\{T^n \psi\}_{n \in I}$ where either $I = \mathbb{N} \cup \{0\}$ or $I = \mathbb{Z}$ and $T \in B(H)$. In particular, a characterization of frames of the form $\{T^n \psi\}_{n=0}^\infty$, where $T \in B(H)$ was established in [13] by Christensen, Hasannasab, and Rashidi. Subsequently, [12] characterized the frames that admit such a representation with an at least closable operator T , for $I = \mathbb{N} \cup \{0\}$ or $I = \mathbb{Z}$.

These research studies are our starting point to develop the theory of iterative frames from Hilbert spaces to Banach spaces. For the generalization, we first use the structures of p -frames and q -Riesz bases ($1 < p, q < \infty$) in a separable Banach space X . For example, we gave some characterization of p -frames for X having the representation $\{T^n \psi\}_{n \in \mathbb{N}} \subset X^*$ for some $T \in B(X^*)$ and $\psi \in X^*$. Specifically, we prove that any q -Riesz basis for X^* has the form $\{T^n \psi\}_{n=0}^\infty$ where $T : X^* \rightarrow X^*$ is a bounded injective operator with closed range. Furthermore, we analyzed some properties of such frames in the sense of the perturbation theory. In particular, we obtained a sufficient condition to approximate any p -frame via a subset $\{T^{\alpha(i)} \psi\}_{i=1}^\infty \subset \{T^n \psi\}_{n=0}^\infty$ for some bounded operator $T : X^* \rightarrow X^*$, such as hypercyclic operators. In the case $I = \mathbb{Z}$, for a p -frame $\{T^k \psi\}_{k \in \mathbb{Z}}$, we determine the structure of the canonical dual and also characterize the operator of any dual having the same construction. For the details and further results, we refer to the papers [1, 2].

Now, by the concept of an approximate Schauder frame, there is a new method for the generalization of the iterative systems of frames in Banach spaces. We will introduce the definition of an iterative approximate Schauder frame for a separable Banach space, and then we will investigate this new structure. The paper is organized as follows. Section 2 forms the background to obtain our main results, so it contains some preliminaries, notations and basic facts of approximate Schauder frames and q -Riesz bases which will be used later. In Section 3, we consider iterative approximate

Schauder frames and investigate their properties. Indeed, for an approximate Schauder frame of the form $(W^k x_0, T^k f_0)_{k \in \mathbb{Z}} \subset X \times X^*$ where $W \in B(X)$ and $T \in B(X^*)$, we establish a connection between the operators W and T . Moreover, letting S denote the frame operator associated with the given approximate Schauder frame, we prove that $\|T\| = \|S^{-1}W^{-1}S\|$ and $\|T^{-1}\| = \|S^{-1}WS\|$. To be more precise, Section 3 also has two specific parts: the first one is related to q -Riesz bases and the second part deals with the duality theory.

2. Notations and preliminaries

Throughout the paper, X will denote an infinite-dimensional and separable Banach space with dual space X^* and I is a countable index set. We also use the notation $B(X)$ for the set of all bounded linear operators on X .

The concept of an approximate Schauder frame was introduced in [15]. An equivalent definition of a Schauder frame in terms of the frame operator was also provided there.

DEFINITION 2.1. A sequence $(x_i, f_i)_{i \in I} \subset X \times X^*$ is called an approximate Schauder frame for X if the operator $S : X \rightarrow X$ is defined by

$$S(x) = \sum_{i \in I} f_i(x)x_i, \quad (2)$$

is (well defined) bounded and invertible. This operator is called the frame operator of $(x_i, f_i)_{i \in I}$.

If the series in (2) converges unconditionally for all $x \in X$, then an approximate Schauder frame $(x_i, f_i)_{i \in I} \subset X \times X^*$ is called an unconditional approximate Schauder frame.

The sequence $(x_i, f_i)_{i \in I} \subset X \times X^*$ is called a Schauder frame for X if the frame operator is the identity on X .

Motivated by [8, Proposition 2.3], we introduce the concept of shrinking property for approximate Schauder frames.

DEFINITION 2.2. An approximate Schauder frame $(x_i, f_i)_{i \in I} \subset X \times X^*$ is said to be shrinking if for all $g \in X^*$, the series $\sum_{i \in I} g(x_i)f_i$ is convergent in X^* .

The following lemma will play a key role in Section 3.2. Before we state this result, we recall that the Gelfand map $\phi : X \rightarrow X^{**}$ defined by $\phi(x) = \hat{x}$ is an isometry, where $\hat{x}(g) = g(x)$, for any $g \in X^*$. Furthermore, if X is reflexive, then ϕ is invertible.

LEMMA 2.3. Let $(x_i, f_i)_{i \in I} \subset X \times X^*$ be an approximate Schauder frame for X with the frame operator S . Then $(x_i, f_i)_{i \in I}$ is shrinking if and only if $(f_i, \hat{x}_i)_{i \in I} \subset X^* \times X^{**}$ is an approximate Schauder frame for X^* with the frame operator $S^* : X^* \rightarrow X^*$.

Proof. We first assume that $(x_i, f_i)_{i \in I}$ is shrinking. Hence, for any $x \in X$ and $g \in X^*$, we have

$$(S^*g)(x) = g \circ S(x) = \sum_{i \in I} f_i(x)g(x_i) = \left(\sum_{i \in I} \hat{x}_i(g)f_i \right)(x).$$

Also, $\|S^*\| = \|S\|$ and $(S^*)^{-1} = (S^{-1})^*$. Thus, the operator $S^* : X^* \rightarrow X^*$ defined by $S^*g = \sum_{i \in I} \hat{x}_i(g)f_i$ is (well-defined) bounded and invertible. In other words, the sequence $(f_i, \hat{x}_i)_{i \in I}$ is an approximate Schauder frame for X^* with the frame operator S^* .

Conversely, let $(f_i, \hat{x}_i)_{i \in I}$ be an approximate Schauder frame for X^* with the frame operator S^* . Then, it is clear that for all $g \in X^*$, the series $\sum_{i \in I} g(x_i)f_i$ converges to S^*g in X^* . $\square$

In the rest of this section, we state some basic facts about q -Riesz bases and their relationship with p -frames that will be used later. Throughout the paper, we assume that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ whenever the sequence spaces $\ell^p(I)$ and $\ell^q(I)$ are considered. The following definitions and results are mainly from [14]. Recall that a sequence $\{f_i\}_{i \in I} \subset X^*$ is called a p -frame for X ($1 < p < \infty$) if there exist constants $A, B > 0$ such that

$$A\|x\|_X \leq \left(\sum_{i \in I} |f_i(x)|^p \right)^{\frac{1}{p}} \leq B\|x\|_X, \quad (x \in X).$$

For such a p -frame, the associated analysis and synthesis operators are given by

$$\begin{aligned} U : X &\rightarrow \ell^p(I), & Uf &:= \{f_i(x)\}_{i \in I}, \\ V : \ell^q(I) &\rightarrow X^*, & V\{c_i\}_{i \in I} &:= \sum_{i \in I} c_i f_i. \end{aligned}$$

The operator U is bounded with closed range, and X is reflexive.

We will apply the following definition in the dual space X^* , so in order to avoid confusion, we state it in a Banach space Y .

DEFINITION 2.4. Let Y be a Banach space. A family $\{f_i\}_{i \in I} \subset Y$ is a q -Riesz basis ($1 < q < \infty$) for Y if $\overline{\text{span}}\{f_i\}_{i \in I} = Y$ and there exist constants $A, B > 0$ such that for all finite scalar sequences $\{c_i\}_{i \in I}$,

$$A \left(\sum_{i \in I} |c_i|^q \right)^{\frac{1}{q}} \leq \left\| \sum_{i \in I} c_i f_i \right\|_Y \leq B \left(\sum_{i \in I} |c_i|^q \right)^{\frac{1}{q}}.$$

The numbers A, B are called q -Riesz basis bounds.

We will use the following relation between p -frames and q -Riesz bases: every q -Riesz basis $\{f_i\}_{i \in I} \subset X^*$ for X^* is also a p -frame for X [14]. In addition, it follows directly from the definition that every q -Riesz basis for X^* is linearly independent.

The following lemma characterizes the structure of a q -Riesz basis in terms of its synthesis operator.

LEMMA 2.5. *Suppose that $1 < q < \infty$. Then a sequence $\{f_i\}_{i \in I} \subset X^*$ is a q -Riesz basis for X^* if and only if it is of the form $\{Ce_i\}_{i \in I}$ for some bounded and invertible operator $C : \ell^q(I) \rightarrow X^*$, where $\{e_i\}_{i \in I}$ is the canonical Schauder basis of $\ell^q(I)$.*

The following result concerns the biorthogonal sequence associated with a q -Riesz basis and will be used in Proposition 2.7.

THEOREM 2.6. ([14]) *If $\{f_i\}_{i \in I} \subset X^*$ is a q -Riesz basis for X^* with bounds $A, B > 0$, then there exists a unique p -Riesz basis $\{x_i\}_{i \in I} \subset X$ for X that is biorthogonal to $\{f_i\}_{i \in I}$. Moreover,*

$$\begin{aligned} x &= \sum_{i \in I} f_i(x)x_i, \quad (x \in X), \\ f &= \sum_{i \in I} f(x_i)f_i, \quad (f \in X^*). \end{aligned}$$

Let us end this section with one more result that describes the relationship between approximate Schauder frames and q -Riesz bases. Indeed, for a given approximate Schauder frame $(x_i, f_i)_{i \in I} \subset X \times X^*$, the p -Riesz basis property of $\{x_i\}_{i \in I}$ is equivalent to the q -Riesz basis property of $\{f_i\}_{i \in I}$:

PROPOSITION 2.7. *Let $(x_i, f_i)_{i \in I} \subset X \times X^*$ be an approximate Schauder frame for X . Then $\{x_i\}_{i \in I} \subset X$ is a p -Riesz basis for X if and only if $\{f_i\}_{i \in I} \subset X^*$ is a q -Riesz basis for X^* .*

Proof. Suppose that $\{x_i\}_{i \in I}$ is a p -Riesz basis for X , and let S denote the frame operator for $(x_i, f_i)_{i \in I}$. By Theorem 2.6, let $\{g_i\}_{i \in I}$ be the biorthogonal sequence to $\{x_i\}_{i \in I}$. Then, for every $x \in X$,

$$g_i(Sx) = g_i\left(\sum_{j \in I} f_j(x)x_j\right) = f_i(x).$$

Hence $f_i = S^*g_i$, for all $i \in I$. Therefore $\{f_i\}_{i \in I}$ is a q -Riesz basis for X^* . Since $\{g_i\}_{i \in I}$ is a q -Riesz basis for X^* and S^* is bounded and invertible.

Conversely, let $\{y_i\}_{i \in I}$ be the biorthogonal sequence to $\{f_i\}_{i \in I}$. Then

$$Sy_i = \sum_{j \in I} f_j(y_i)x_j = x_i.$$

Thus $x_i = Sy_i$, for all $i \in I$, and hence $\{x_i\}_{i \in I}$ is a p -Riesz basis for X . $\square$

For further background on approximate Schauder frames and q -Riesz bases, we refer the reader to [8, 14, 15, 19].

3. Iterative approximate Schauder frames

This section presents our main results. We study some basic properties of approximate Schauder frames generated by iterative actions of bounded linear operators, which we refer to as iterative approximate Schauder frames. In addition, we will analyze the relation between these frames and q -Riesz bases. The last topic deals with the duality structure, especially the canonical dual of an iterative approximate Schauder frame.

In the sequel, unless otherwise stated, we work with reflexive Banach spaces. In particular, the Gelfand map $\phi : X \rightarrow X^{**}$ is invertible.

Our starting point is the concept of an iterative approximate Schauder frame:

DEFINITION 3.1. Let X be a separable infinite-dimensional Banach space, and let $(x_k, f_k)_{k \in \mathbb{Z}} \subset X \times X^*$ be an approximate Schauder frame for X . If there exist invertible operators $W : X \rightarrow X$ and $T : X^* \rightarrow X^*$ such that $\{x_k\}_{k \in \mathbb{Z}} = \{W^k x_0\}_{k \in \mathbb{Z}}$ and $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$, then

$$(x_k, f_k)_{k \in \mathbb{Z}} = (W^k x_0, T^k f_0)_{k \in \mathbb{Z}} \subset X \times X^*,$$

is called an iterative approximate Schauder frame for X . The linear maps W and T are called the generator operators.

The following example presents some general conditions for an iterative Schauder frame to be approximate; we then discuss a special case based on the (iterative) p -frame theory.

EXAMPLE 3.2.

- (i) Let $(x_k, f_k)_{k \in \mathbb{Z}} = (W^k x_0, T^k f_0)_{k \in \mathbb{Z}} \subset X \times X^*$ be an iterative Schauder frame for X and a bounded invertible operator $S : X \rightarrow X$ be such that $SWx_0 = WSx_0$. Then the sequence $(W^k(Sx_0), T^k f_0)_{k \in \mathbb{Z}} \subset X \times X^*$ is an iterative approximate Schauder frame for X with the frame operator S .
- (ii) Consider a bounded invertible operator $T : X^* \rightarrow X^*$. Let $\{g_k\}_{k \in \mathbb{Z}} = \{T^k g_0\}_{k \in \mathbb{Z}} \subset X^*$ is a p -frame for X with the invertible p -frame map S_p (we refer to [19], for more details on the p -frame map). Then by [2, Theorem 3.3], the sequence $(W^k x_0, T^k g_0)_{k \in \mathbb{Z}} \subset X \times X^*$ is an iterative Schauder frame for X where $x_0 = S_p^{-1} g_0$ and $W : X \rightarrow X$ is defined by $W = \phi^{-1}(T^*)^{-1}\phi$. Indeed, for all $x \in X$,

$$x = \sum_{k \in \mathbb{Z}} (T^k g_0)(x) W^k x_0.$$

Furthermore, by part (i), consider the special case $S = W$, then

$$Wx = \sum_{k \in \mathbb{Z}} (T^k g_0)(x) W^k (Wx_0).$$

That is $(W^k(Wx_0), T^k g_0)_{k \in \mathbb{Z}}$ is an iterative approximate Schauder frame for X with the frame operator W .

Now, we provide a concrete example of an iterative Schauder frame:

EXAMPLE 3.3. Let $X = \ell^1(\mathbb{Z})$, which is not reflexive. Then $X^* = \ell^\infty(\mathbb{Z})$. Define $x_0 = \{c_j\}_{j \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$ and $f_0 = \{d_j\}_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ by

$$\{c_j\}_{j \in \mathbb{Z}} = \begin{cases} 1 & j = 0 \\ -1 & j = 1 \\ 0 & \text{otherwise} \end{cases} \quad \{d_j\}_{j \in \mathbb{Z}} = \begin{cases} 0 & j \geq 1 \\ 1 & j \leq 0 \end{cases}$$

Consider

$$(x_k, f_k)_{k \in \mathbb{Z}} := (W^k x_0, T^k f_0)_{k \in \mathbb{Z}},$$

where W and T are the bilateral right shift operators on $\ell^1(\mathbb{Z})$ and $\ell^\infty(\mathbb{Z})$, respectively.

Then $(x_k, f_k)_{k \in \mathbb{Z}} \subset \ell^1(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$ is an iterative Schauder frame for $\ell^1(\mathbb{Z})$. In fact, for any $\alpha = \{\alpha_j\}_{j \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} f_k(\alpha) x_k &= \sum_{k \in \mathbb{Z}} (T^k f_0) (\{\alpha_j\}_{j \in \mathbb{Z}}) W^k x_0 \\ &= \sum_{k \in \mathbb{Z}} \{d_{j-k}\}_{j \in \mathbb{Z}} (\{\alpha_j\}_{j \in \mathbb{Z}}) W^k x_0 \\ &= \sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} d_{j-k} \alpha_j \right) T_1^k x_0 \\ &= \{\beta_j\}_{j \in \mathbb{Z}}, \end{aligned}$$

where the j -th component of $\{\beta_j\}_{j \in \mathbb{Z}}$ is given by

$$\beta_j = \sum_{i \in \mathbb{Z}} d_{i-j} \alpha_i - \sum_{i \in \mathbb{Z}} d_{i-(j-1)} \alpha_i = \sum_{i \in \mathbb{Z}} (d_{i-j} - d_{i-j+1}) \alpha_i.$$

Now, by the definition of $f_0 = \{d_j\}_{j \in \mathbb{Z}}$ we have $\beta_j = (1 - 0) \alpha_j = \alpha_j$. Hence

$$\alpha = \sum_{k \in \mathbb{Z}} f_k(\alpha) x_k, \quad (\alpha \in \ell^1(\mathbb{Z})).$$

It is worth noticing that the choice of $f_0 = \{d_j\}_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ is not unique, for example, we can also choose the element f_0 as follows:

$$\{d_j\}_{j \in \mathbb{Z}} = \begin{cases} -1 & j \geq 1 \\ 0 & j \leq 0. \end{cases}$$

In the rest of the paper, all approximate Schauder frames considered are unconditional.

The following theorem presents some properties of generator operators of an iterative approximate Schauder frame. We will later use parts of the proof of Theorem 3.4 as a technical tool.

THEOREM 3.4. *Let $(x_k, f_k)_{k \in \mathbb{Z}} = (W^k x_0, T^k f_0)_{k \in \mathbb{Z}} \subset X \times X^*$ be an approximate Schauder frame for X with the frame operator S , for some invertible operators $W \in B(X)$ and $T \in B(X^*)$. Then*

(i) *The equality $W = S\phi^{-1}(T^*)^{-1}\phi S^{-1}$ holds.*

(ii) $\|T\| = \|S^{-1}W^{-1}S\|$ and $\|T^{-1}\| = \|S^{-1}WS\|$.

Proof. Let $x \in X$. Put $y := (\phi^{-1}(T^*)^{-1}\phi)x \in X$. Then, using bracket notation,

$$\langle x, f_k \rangle = \langle x, T^k f_0 \rangle = \langle \phi^{-1}T^*\phi y, T^k f_0 \rangle = \langle y, T^{k+1}f_0 \rangle.$$

Now, using the approximate Schauder frame property and boundedness of W , we get

$$\begin{aligned} WSx &= W \left(\sum_{k \in \mathbb{Z}} \langle x, f_k \rangle x_k \right) = \sum_{k \in \mathbb{Z}} \langle x, f_k \rangle Wx_k \\ &= \sum_{k \in \mathbb{Z}} \langle y, T^{k+1}f_0 \rangle W^{k+1}x_0 = \sum_{k \in \mathbb{Z}} \langle y, T^k f_0 \rangle W^k x_0 \\ &= Sy = S(\phi^{-1}(T^*)^{-1}\phi)x. \end{aligned}$$

Hence $WS = S\phi^{-1}(T^*)^{-1}\phi$, which proves (i).

Furthermore, by part (i), one can see that $T^* = \phi S^{-1}W^{-1}S\phi^{-1}$. Using the properties of the Gelfand map ϕ , we get

$$\begin{aligned} \|T\| &= \|T^*\| \\ &= \|\phi S^{-1}W^{-1}S\phi^{-1}\| \\ &= \sup_{\|\hat{x}\|=1} \|\phi S^{-1}W^{-1}S\phi^{-1}\hat{x}\| \\ &= \sup_{\|\hat{x}\|=1} \|S^{-1}W^{-1}S\phi^{-1}\hat{x}\| \\ &= \sup_{\|x\|=1} \|S^{-1}W^{-1}Sx\| \\ &= \|S^{-1}W^{-1}S\|. \end{aligned}$$

Similarly, one can show that $\|T^{-1}\| = \|S^{-1}WS\|$. This proves (ii). $\square$

According to the fact that a Schauder frame is a special case of an approximate Schauder frame, all the results for an approximate Schauder frame also hold for Schauder frames. For example, in Theorem 3.4, if $(W^k x_0, T^k f_0)_{k \in \mathbb{Z}}$ is an iterative Schauder frame, then $W = \phi^{-1}(T^*)^{-1}\phi$, $\|T\| = \|W^{-1}\|$ and $\|T^{-1}\| = \|W\|$.

3.1. p -Riesz bases and iterative approximate Schauder frames

Riesz Bases play a key role in frame theory, so it is useful to investigate the relation between them. Some preparation is needed before we state our main result.

Motivated by [11], we have the following propositions in the theory of iterative p -frames in the case of the index set $I = \mathbb{Z}$. The proof of Proposition 3.5 is similar to [11, Proposition 2.1].

PROPOSITION 3.5. *Let Y be a vector space. Consider any sequence $\{f_k\}_{k \in \mathbb{Z}}$ in Y , for which $\text{span}\{f_k\}_{k \in \mathbb{Z}}$ is infinite-dimensional. Then the following are equivalent:*

- (i) *The sequence $\{f_k\}_{k \in \mathbb{Z}}$ is linearly independent.*
- (ii) *The mapping $Tf_k = f_{k+1}$ is well-defined, and extend to a linear and invertible operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$.*

In the affirmative case, $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$.

The proof of next proposition is similar to the case $I = \mathbb{N} \cup \{0\}$ in [1, Theorem 3.5].

PROPOSITION 3.6. *Assume that $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ is a p -frame for X with bounds $A, B > 0$, where $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$ is a linear operator. Denoting the kernel of the synthesis operator V by $\mathcal{N}_V$ and let $\mathcal{N}_V$ be complemented in $\ell^q(\mathbb{Z})$, then the following are equivalent:*

- (i) *The operator T is bounded.*
- (ii) *The subspace $\mathcal{N}_V$ is invariant under the bilateral right shift operator $R : \ell^q(\mathbb{Z}) \rightarrow \ell^q(\mathbb{Z})$ defined by $R(\{d_k\}_{k \in \mathbb{Z}}) = \{d_{k-1}\}_{k \in \mathbb{Z}}$.*

As a consequence of Proposition 3.6, we have the following corollary, which plays a key role in the theory of iterative approximate Schauder frames. In particular, Corollary 3.7 gives an iterative construction of q -Riesz bases.

COROLLARY 3.7. *Let $\{f_k\}_{k \in \mathbb{Z}}$ be a q -Riesz basis for X^* . Then there exists a bounded invertible operator $T : X^* \rightarrow X^*$, such that $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$.*

Proof. By the definition of a q -Riesz basis, $\{f_k\}_{k \in \mathbb{Z}}$ is linearly independent. Use Proposition 3.5 to conclude that $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$, for some invertible operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$. On the other hand, one can see that $\mathcal{N}_V = \{0\}$ is invariant under the bilateral right shift operator, so by Proposition 3.6, the operator T is bounded. Hence, it can be extended to a bounded operator $T : X^* \rightarrow X^*$. Furthermore, Theorem 2.6 implies the invertibility of T . $\square$

The following result gives sufficient conditions for an approximate Schauder frame to be iterative.

PROPOSITION 3.8. *Let $(x_k, f_k)_{k \in \mathbb{Z}} \subset X \times X^*$ be an approximate Schauder frame for X . If $\{f_k\}_{k \in \mathbb{Z}}$ is a q -Riesz basis for X^* , then $(x_k, f_k)_{k \in \mathbb{Z}}$ is of the form of an iterative approximate Schauder frame for X .*

Proof. This result follows directly from Proposition 2.7 and Corollary 3.7. Indeed, by Proposition 2.7, one can see that the condition $\{f_k\}_{k \in \mathbb{Z}}$ being a q -Riesz basis for X^* is equivalent with $\{x_k\}_{k \in \mathbb{Z}}$ being a p -Riesz basis for X , since $(x_k, f_k)_{k \in \mathbb{Z}} \subset X \times X^*$ is an approximate Schauder frame for X . Now, for $1 < q < \infty$, we can apply Corollary 3.7

to conclude that any q -Riesz basis $\{f_k\}_{k \in \mathbb{Z}}$ for X^* has the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for some bounded linear operator $T : X^* \rightarrow X^*$ and similarly there exists $W \in B(X)$ such that $\{x_k\}_{k \in \mathbb{Z}} = \{W^k x_0\}_{k \in \mathbb{Z}}$. Hence,

$$(x_k, f_k)_{k \in \mathbb{Z}} = \left(W^k x_0, T^k f_0 \right)_{k \in \mathbb{Z}},$$

that is the approximate Schauder frame $(x_k, f_k)_{k \in \mathbb{Z}}$ is iterative. $\square$

REMARK 1. In Proposition 3.8, we can use Proposition 2.7 to replace the q -Riesz basis condition of $\{f_k\}_{k \in \mathbb{Z}}$ by the p -Riesz basis condition of $\{x_k\}_{k \in \mathbb{Z}}$.

We also note that under the assumptions in Proposition 3.8, by Theorem 3.4, the approximate Schauder frame $(x_k, f_k)_{k \in \mathbb{Z}}$ indeed has the form

$$(x_k, f_k)_{k \in \mathbb{Z}} = \left((S\phi^{-1}(T^*)^{-1}\phi S^{-1})^k x_0, T^k f_0 \right)_{k \in \mathbb{Z}},$$

for some $T \in B(X^*)$.

3.2. Duality structure

In the rest of the paper, we will present some results of duality theory for an iterative approximate Schauder frame arising via a bounded operator. We begin with the definition of a dual of an approximate Schauder frame. The concept is motivated by [18, Definition 2.10].

DEFINITION 3.9. Let $(x_k, f_k)_{k \in \mathbb{Z}} \subset X \times X^*$ be an approximate Schauder frame for X . An approximate Schauder frame $(y_k, g_k)_{k \in \mathbb{Z}}$ for X is called a dual of $(x_k, f_k)_{k \in \mathbb{Z}}$ if for any $x \in X$,

$$x = \sum_{k \in \mathbb{Z}} g_k(x) x_k = \sum_{k \in \mathbb{Z}} f_k(x) y_k.$$

REMARK 2. Consider an approximate Schauder frame $(x_k, f_k)_{k \in \mathbb{Z}}$ for X with the frame operator S . Then $(S^{-1}x_k, f_k S^{-1})_{k \in \mathbb{Z}}$ is an approximate Schauder frame for X with the frame operator $S^{-1} : X \rightarrow X$. Indeed, for any $x \in X$,

$$\sum_{k \in \mathbb{Z}} (f_k S^{-1})(x) S^{-1} x_k = S^{-1} \left(\sum_{k \in \mathbb{Z}} f_k(S^{-1}x) x_k \right) = S^{-1}(S(S^{-1}x)) = S^{-1}x.$$

Furthermore

$$x = S^{-1}Sx = S^{-1} \left(\sum_{k \in \mathbb{Z}} f_k(x) x_k \right) = \sum_{k \in \mathbb{Z}} f_k(x) S^{-1} x_k$$

and

$$x = SS^{-1}x = \sum_{k \in \mathbb{Z}} f_k(S^{-1}x) x_k = \sum_{k \in \mathbb{Z}} (f_k S^{-1})(x) x_k.$$

For an approximate Schauder frame $(x_k, f_k)_{k \in \mathbb{Z}} \subset X \times X^*$ with the frame operator S , the approximate Schauder frame $(S^{-1}x_k, f_k S^{-1})_{k \in \mathbb{Z}}$ is called the canonical dual of $(x_k, f_k)_{k \in \mathbb{Z}}$. We refer to the paper [18] for more information.

The following result gives a sufficient condition for the canonical dual of an iterative approximate Schauder frame being iterative. More precisely, consider bounded invertible operators $W : X \rightarrow X$ and $T : X^* \rightarrow X^*$, the next theorem shows that the canonical dual of a shrinking approximate Schauder frame $(W^k x_0, T^k f_0)_{k \in \mathbb{Z}}$ is iterative.

THEOREM 3.10. *Assume that operators $W \in B(X)$ and $T \in B(X^*)$ are invertible and let $(x_k, f_k)_{k \in \mathbb{Z}} = (W^k x_0, T^k f_0)_{k \in \mathbb{Z}}$ be an approximate Schauder frame for X with the frame operator S . If $(x_k, f_k)_{k \in \mathbb{Z}}$ is shrinking, then the canonical dual $(S^{-1}x_k, f_k S^{-1})_{k \in \mathbb{Z}}$ has the form $(W^k (S^{-1}x_0), T^k ((S^*)^{-1}f_0))_{k \in \mathbb{Z}}$, where $\mathcal{W} = \phi^{-1}(T^*)^{-1}\phi$ and $\mathcal{T} = (S^*)^{-1}TS^*$.*

Proof. By Theorem 3.4, we have $W = S\phi^{-1}(T^*)^{-1}\phi S^{-1}$. Hence $S^{-1}WS = \phi^{-1}(T^*)^{-1}\phi$, which implies that $S^{-1}W^k S = \phi^{-1}(T^*)^{-k}\phi$, for all $k \in \mathbb{Z}$. Thus, we can obtain that

$$S^{-1}x_k = S^{-1}W^k x_0 = \phi^{-1}(T^*)^{-k}\phi(S^{-1}x_0) = \mathcal{W}^k(S^{-1}x_0), \quad (k \in \mathbb{Z}).$$

On the other hand, use Lemma 2.3 to conclude that $(f_k, \hat{x}_k)_{k \in \mathbb{Z}} \subset X^* \times X^{**}$ is an approximate Schauder frame for X^* with the frame operator S^* . Hence

$$\begin{aligned} f_k S^{-1} &= (S^{-1})^* (T^k f_0) = (S^{-1})^* (T^k (S^* (S^*)^{-1} f_0)) \\ &= (S^*)^{-1} T^k S^* ((S^*)^{-1} f_0) = T^k ((S^*)^{-1} f_0), \end{aligned}$$

for all $k \in \mathbb{Z}$. $\square$

In Theorem 3.10, it is clear that for the canonical dual

$$(S^{-1}x_k, f_k S^{-1})_{k \in \mathbb{Z}} = (\mathcal{W}^k (S^{-1}x_0), T^k ((S^*)^{-1}f_0))_{k \in \mathbb{Z}},$$

we have $\mathcal{W} = S^{-1}WS$. That is, the operators $\mathcal{W}$ and $\mathcal{T}$ can be determined directly in terms of the operators W, S and T, S^* , respectively.

Let us end this paper with one more result on duality construction; for a given iterative approximate Schauder frame, the following theorem determines the operators of any dual, which is also iterative.

THEOREM 3.11. *Consider bounded invertible operators $W, \mathcal{W} : X \rightarrow X$ and $T, \mathcal{T} : X^* \rightarrow X^*$. Let $(x_k, f_k)_{k \in \mathbb{Z}} = (W^k x_0, T^k f_0)_{k \in \mathbb{Z}}$ be an approximate Schauder frame for X with dual $(y_k, g_k)_{k \in \mathbb{Z}} = (\mathcal{W}^k y_0, \mathcal{T}^k g_0)_{k \in \mathbb{Z}}$. Then $\mathcal{W} = S^{-1}WS$ and $\mathcal{T}^* = \phi S^{-1}\phi^{-1}T^*\phi S\phi^{-1}$ where S and $\mathcal{S}$ are the frame operators of approximate Schauder frames $(x_k, f_k)_{k \in \mathbb{Z}}$ and $(y_k, g_k)_{k \in \mathbb{Z}}$, respectively.*

Proof. For approximate Schauder frame $(W^k x_0, T^k f_0)_{k \in \mathbb{Z}}$, by Theorem 3.4, one can see that $S^{-1}WS = \phi^{-1}(T^*)^{-1}\phi$. Similarly, we have $S^{-1}\mathcal{W}S = \phi^{-1}(T^*)^{-1}\phi$, corresponding to $(\mathcal{W}^k y_0, T^k g_0)_{k \in \mathbb{Z}}$. Let $x \in X$ and put $y := (\phi^{-1}(T^*)^{-1}\phi)x$. Then by the same argument as the proof of Theorem 3.4, we have $\langle x, f_k \rangle = \langle y, T^{k+1}f_0 \rangle$, for all $k \in \mathbb{Z}$. Now, via the duality property, we get

$$\begin{aligned} \mathcal{W}x &= \mathcal{W} \left(\sum_{k \in \mathbb{Z}} \langle x, f_k \rangle y_k \right) = \sum_{k \in \mathbb{Z}} \langle y, T^{k+1}f_0 \rangle \mathcal{W}^{k+1}y_0 = \sum_{k \in \mathbb{Z}} \langle y, T^k f_0 \rangle \mathcal{W}^k y_0 \\ &= \sum_{k \in \mathbb{Z}} \langle y, f_k \rangle y_k = y = (\phi^{-1}(T^*)^{-1}\phi)x \\ &= S^{-1}WSx. \end{aligned}$$

That is $\mathcal{W} = S^{-1}WS$. On the other hand

$$\phi^{-1}(T^*)^{-1}\phi = S^{-1}\mathcal{W}S = S^{-1}(S^{-1}WS)S = S^{-1}(\phi^{-1}(T^*)^{-1}\phi)S,$$

so $(T^*)^{-1} = \phi S^{-1}\phi^{-1}(T^*)^{-1}\phi S\phi^{-1}$, which leads to the desired result. $\square$

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