

INJECTIVITY OF CANONICAL MAPS BETWEEN OPERATOR SPACE TENSOR PRODUCTS

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Abstract. We establish the injectivity of the canonical inclusion from the Haagerup tensor product to the injective tensor product for operator spaces. The injectivity of the canonical maps from the operator space projective tensor product to the Haagerup, Banach space injective, and operator space injective tensor products of TROs is also proved.

1. Introduction

The systematic study of operator spaces, initiated by Effros and Ruan [4, 5] and developed further by Blecher, Paulsen, [2, 3] and others, provides a natural framework for analyzing matricially normed spaces and their tensor products. A central theme in this theory is the comparison of various operator space tensor norms, such as the Haagerup norm, the operator space projective norm, and the operator space injective norm. These tensor products arise naturally in the study of completely bounded maps, operator algebras, and ternary rings of operators (TROs).

For C^* -algebras, several equivalence and embedding results among different tensor norms have been established, often through Grothendieck-type inequalities, lifting arguments, and second dual techniques. In particular, when two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ satisfy $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$, the identity map on the algebraic tensor product extends to a contractive map between the corresponding completions. The question of injectivity of such canonical maps is subtle, and has been resolved only in specific settings, such as the case of Banach space tensor norms on the tensor products of C^* -algebras or TROs [1, 2, 6, 7, 9, 10, 11, 12, 13, 16].

In this paper, We first show that the canonical inclusion map from the Haagerup tensor product of operator spaces to the operator space injective tensor product is injective. We then extend this analysis to TROs, proving that the canonical maps from the projective tensor product to the Haagerup, Banach space injective, and operator space injective tensor products are injective. Finally, we show that every jointly completely bounded bilinear form on TROs extends uniquely to their biduals, yielding a canonical bi-continuous embedding of the corresponding bidual tensor products.

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2. Preliminaries

We first recall some of the norms on the tensor product of operator spaces.

DEFINITION 1. Given Banach spaces X and Y , the *Banach space injective tensor norm* of an element $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ is defined by

$$\|u\|_\lambda = \sup \left\{ \left| \sum_{i=1}^n f(x_i)g(y_i) \right| : f \in B_{X^*}, g \in B_{Y^*} \right\},$$

where B_{X^*} and B_{Y^*} denote the unit balls of X^* and Y^* . The completion of $X \otimes Y$ with respect to this norm is denoted by $X \otimes^\lambda Y$.

DEFINITION 2. The *Haagerup tensor norm* $\|u\|_h$ of an element $u \in E \otimes F$, where E and F are operator spaces, is defined as

$$\|u\|_h = \inf \left\{ \|a\| \|b\| : a = (a_{1j})_{1 \times n}, b = (b_{j1})_{n \times 1} \text{ and } u = \sum_{j=1}^n a_{1j} \otimes b_{j1} \right\}$$

where the infimum is taken over all such representations of u . The completion of $E \otimes F$ with respect to this norm is denoted by $E \otimes^h F$.

DEFINITION 3. The *operator space projective tensor norm* on $E \otimes F$ is given by

$$\|u\|_\wedge = \inf \{ \|\alpha\| \|x\| \|y\| \|\beta\| : u = \alpha(x \otimes y)\beta \},$$

where the infimum is taken over arbitrary decompositions with $x \in M_p(E)$, $y \in M_q(F)$, and $\alpha \in M_{1,pq}$, $\beta \in M_{pq,1}$, for some $p, q \in \mathbb{N}$. The completion is denoted $E \hat{\otimes} F$.

DEFINITION 4. Suppose that $E \subseteq B(\mathcal{H})$ and $F \subseteq B(\mathcal{K})$ be operator spaces. The norm induced on $E \otimes F$ via the inclusion $E \otimes F \hookrightarrow B(\mathcal{H} \otimes \mathcal{K})$ is known as the *operator space injective tensor norm*. The completion is denoted $E \otimes^\vee F$.

These tensor norms on $E \otimes F$ satisfies $\|\cdot\|_\lambda \leq \|\cdot\|_\vee \leq \|\cdot\|_h \leq \|\cdot\|_\wedge$. Moreover, the identity map extends to complete contractions

$$E \hat{\otimes} F \rightarrow E \otimes^h F \rightarrow E \otimes^\vee F.$$

Next we recall the notion of ternary rings of operators (TRO) [8, 14, 15, 17].

DEFINITION 5. Let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the Banach space of all bounded linear operators from a Hilbert space $\mathcal{H}$ into another Hilbert space $\mathcal{K}$. A Banach subspace V of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is called a TRO if V is closed under the ternary operation $(x, y, z) \mapsto xy^*z$ for all $x, y, z \in V$. We note that $\mathcal{B}(\mathcal{H}, \mathcal{K})$, the matrix space $M_{m,n}$ of all $m \times n$ complex matrices, and any C^* -algebra are TROs.

Given a TRO $V \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$, its *linking C^* -algebra* is the C^* -subalgebra of $\mathcal{B}(\mathcal{K} \oplus \mathcal{H})$ defined by

$$A(V) = \begin{bmatrix} C(V) & V \\ V^* & D(V) \end{bmatrix},$$

where $V^* = \{x^* : x \in V\}$, $C(V) \subseteq \mathcal{B}(\mathcal{K})$ is the C^* -subalgebra generated by elements of the form xy^* with $x, y \in V$, and $D(V) \subseteq \mathcal{B}(\mathcal{H})$ is the C^* -subalgebra generated by elements of the form y^*x with $x, y \in V$. The linking C^* -algebra $A(V)$ is uniquely determined (up to $*$ -isomorphism) by the TRO V , and is independent of the particular Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ used to represent V .

There is a natural linear map

$$i_V : V \longrightarrow A(V), \quad i_V(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix},$$

whose range is the $(1, 2)$ -corner of $A(V)$. The map i_V is an injective TRO homomorphism and hence a complete isometry. Moreover, if

$$e = \begin{bmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{H}} \end{bmatrix},$$

then i_V induces a TRO isomorphism from V onto $eA(V)f$, so that V can be identified with the $(1, 2)$ -corner of its linking C^* -algebra.

3. Injectivity of the canonical map $E \otimes^h F \rightarrow E \otimes^\vee F$

Let E and F be operator spaces, and let $T : E \times F \rightarrow \mathbb{C}$ be a bounded bilinear map. Then there exists a unique bounded linear operator $\tilde{T} : E \rightarrow F^*$ satisfying

$$T(a, b) = \tilde{T}(a)(b), \quad a \in E, b \in F.$$

Effros and Ruan [4] established that if T is completely bounded, then the associated operator $\tilde{T}$ factors through a Hilbert space L in the form

$$E \xrightarrow{v} L \xrightarrow{w} F^*,$$

so that $\tilde{T} = wv$ and

$$\|T\|_{cb} = \|w\|_{cb} \|v\|_{cb}.$$

For convenience, we denote

$$K_T := \|w\|_{cb} \|v\|_{cb},$$

and note that this definition is independent of the particular choice of factorization, since $\|T\|_{cb} = K_T$.

PROPOSITION 1. *Let E and F be operator spaces, and let $T : E \times F \rightarrow \mathbb{C}$ be a completely bounded bilinear map with associated linearization $\tilde{T} : E \rightarrow F^*$. Then there exists a net (T_α) of finite-rank completely bounded maps from E into F^* such that*

$$\|T_\alpha\|_{cb} \leq K_T \quad \text{and} \quad T_\alpha(x) \rightarrow \tilde{T}(x) \text{ for all } x \in E.$$

Proof. Let $\tilde{T} = wv$ where $v : E \rightarrow L$ and $w : L \rightarrow F^*$ are completely bounded maps and L is a Hilbert space. Let $\{p_\alpha\}$ be the net of finite-rank orthogonal projections on L associated to finite-dimensional subspaces spanned by subsets of an orthonormal basis. Then $p_\alpha \rightarrow I_L$ strongly, each p_α has cb-norm 1, and hence the maps

$$T_\alpha := w p_\alpha v : E \rightarrow F^*$$

are finite-rank and satisfy

$$\|T_\alpha\|_{cb} \leq \|w\|_{cb} \|p_\alpha\|_{cb} \|v\|_{cb} \leq K_T.$$

Moreover, for each $x \in E$, we have $T_\alpha(x) \rightarrow w(v(x)) = \tilde{T}(x)$, as $p_\alpha \rightarrow I_L$ strongly. $\square$

THEOREM 1. *Let E and F be operator spaces. Then the canonical map*

$$\tilde{i} : E \otimes^h F \longrightarrow E \otimes^\lambda F$$

is injective.

Proof. Let $z \in E \otimes^h F$ be nonzero. By the Hahn–Banach theorem, there exists a functional $\phi \in (E \otimes^h F)^*$ such that $\phi(z) \neq 0$. Using the identification

$$(E \otimes^h F)^* \cong \text{CB}(E \times F, \mathbb{C})$$

([5], Corollary 7.1.5), ϕ corresponds to a completely bounded bilinear form $\Phi : E \times F \rightarrow \mathbb{C}$. By above proposition, the associated map $\Phi : E \rightarrow F^*$ can be approximated pointwise by a net (Φ_α) of finite-rank completely bounded maps with $\|\Phi_\alpha\|_{cb} \leq K_\Phi$. Since Φ_α has finite rank, its range lies in a finite-dimensional subspace $V \subseteq F^*$. Choosing an Auerbach basis $\{\psi_1, \dots, \psi_n\}$ for V with dual functionals $\{\tau_1, \dots, \tau_n\} \subset V^*$, we may write

$$\Phi_\alpha(x) = \sum_{j=1}^n \lambda_j(x) \psi_j, \quad \lambda_j(x) := \tau_j(\Phi_\alpha(x)).$$

Thus, for $x \in E$ and $y \in F$,

$$\phi_\alpha(x \otimes y) = \sum_{j=1}^n \lambda_j(x) \psi_j(y),$$

so that $\phi_\alpha = \sum_{j=1}^n f_j \otimes g_j \in E^* \otimes F^* \hookrightarrow (E \otimes^h F)^*$ ([5], Theorem 9.4.7). Thus, each Φ_α induces a functional $\phi_\alpha \in (E \otimes^h F)^*$ satisfying

$$\phi_\alpha(x \otimes y) := \Phi_\alpha(x)(y),$$

We claim that $\lim_{\alpha} \phi_{\alpha}(z) = \phi(z)$. First of all, note that for $w = \sum_{i=1}^n a_i \otimes b_i \in E \otimes F$, we have

$$|\phi_{\alpha}(w) - \phi(w)| = \left| \sum_{i=1}^n \Phi_{\alpha}(a_i) b_i - \sum_{i=1}^n \Phi(a_i, b_i) \right| \leq \sum_{i=1}^n \|\Phi_{\alpha}(a_i) - \tilde{\Phi}(a_i)\| \|b_i\|,$$

which converges to 0, since $\lim_{\alpha} \|\Phi_{\alpha}(a_i) - \tilde{\Phi}(a_i)\| = 0$ for each $i = 1, 2, \dots, n$. So, $\lim_{\alpha} \phi_{\alpha}(w) = \phi(w)$ for all $w \in E \otimes F$.

Now, given an arbitrary $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$, $z_n \in E \otimes F$ such that $\|z_n - z\|_h < \varepsilon$ for all $n \geq n_0$. For this fixed n_0 , we have $\lim_{\alpha} \phi_{\alpha}(z_{n_0}) = \phi(z_{n_0})$, so there exists α_0 such that

$$|\phi_{\alpha}(z_{n_0}) - \phi(z_{n_0})| < \varepsilon \quad \text{for all } \alpha \geq \alpha_0.$$

Thus,

$$\begin{aligned} |\phi_{\alpha}(z) - \phi(z)| &\leq |\phi_{\alpha}(z) - \phi_{\alpha}(z_{n_0})| + |\phi_{\alpha}(z_{n_0}) - \phi(z_{n_0})| + |\phi(z_{n_0}) - \phi(z)| \\ &\leq (\|\phi_{\alpha}\| + \|\phi\|) \|z_{n_0} - z\|_h + \varepsilon < (1 + K_{\phi} + \|\phi\|) \varepsilon \rightarrow 0, \end{aligned}$$

and this holds for all $\alpha \geq \alpha_0$. Thus, $\lim_{\alpha} \phi_{\alpha}(z) = \phi(z)$. In particular, $\phi_{\alpha}(z) \neq 0$ for some α . Fix such an α .

Since $\phi_{\alpha}(z) \neq 0$, at least one term $(f_j \otimes g_j)(z)$ is nonzero. As $f_j \otimes g_j$ is continuous, we have

$$\lim_n (f_j \otimes g_j)(z_n) = (f_j \otimes g_j)(z) \neq 0.$$

Hence for sufficiently large n ,

$$|(f_j \otimes g_j)(z_n)| \geq \delta > 0,$$

for some $\delta > 0$. After normalizing f_j to have norm one (rescaling g_j accordingly), this gives

$$\|z_n\|_{\lambda} \geq |(f_j \otimes g_j)(z_n)| \geq \delta.$$

Thus $\|\tilde{i}(z)\|_{\lambda} > 0$, proving that $\tilde{i}(z) \neq 0$. Therefore the canonical map $\tilde{i}$ is injective. $\square$

COROLLARY 1. *Let E and F be operator spaces. Then the canonical map*

$$i : E \otimes^h F \longrightarrow E \otimes^{\vee} F$$

is injective.

Proof. Let $z \in E \otimes^h F$ be nonzero, and let $(z_n) \subset E \otimes F$ be a sequence converging to z in the Haagerup norm. By Theorem 1, there exists $K > 0$ and $n_0 \in \mathbb{N}$ such that $\|z_n\|_{\lambda} > K$ for all $n \geq n_0$. Since the injective norm dominates the λ -norm on $E \otimes F$, we have

$$\|z_n\|_{\vee} \geq \|z_n\|_{\lambda} > K, \quad \forall n \geq n_0.$$

The map i is the identity on the algebraic tensor product and continuous with respect to the Haagerup norm. Hence, $i(z_n) \rightarrow i(z)$ in $E \otimes^{\vee} F$. Because $\|i(z_n)\|_{\vee}$ stays bounded away from zero, it follows that $\|i(z)\|_{\vee} \geq K > 0$. Thus, $i(z) \neq 0$, proving the injectivity of the canonical map. $\square$

4. Embedding operator space projective tensor product into second duals

Let V and W be TROs, and let

$$i_V : V \longrightarrow A(V) \quad \text{and} \quad i_W : W \longrightarrow A(W)$$

denote their respective natural injective TRO homomorphisms into the corresponding linking C^* -algebras. Consider the canonical map

$$j = i_V \widehat{\otimes} i_W : V \widehat{\otimes} W \longrightarrow A(V) \widehat{\otimes} A(W).$$

Although the operator space projective tensor norm is not injective in general, the following holds.

PROPOSITION 2. *The map $j : V \widehat{\otimes} W \rightarrow A(V) \widehat{\otimes} A(W)$ is a complete isometry.*

Proof. By symmetry of the operator-space projective tensor product ([5], Proposition 7.1.4), it is enough to show that the canonical map

$$\tilde{j} = i_V \widehat{\otimes} \text{id}_W : V \widehat{\otimes} W \longrightarrow A(V) \widehat{\otimes} W$$

is a complete isometry. Let $V = eA(V)f$ for some projections $e, f \in A(V)$. Define

$$P : A(V) \rightarrow V, \quad P(x) = exf.$$

For any a in a C^* -algebra, the left and right multiplication maps

$$L_a(x) = ax, \quad R_a(x) = xa$$

satisfy

$$\|L_a\|_{cb}, \|R_a\|_{cb} \leq \|a\|,$$

since for every $[x_{ij}] \in M_n(\mathbb{C})$ we have $[ax_{ij}] = (a \otimes I_n)[x_{ij}]$, giving the matrix-level estimate. In particular, as e and f are projections (so $\|e\| = \|f\| = 1$), the maps L_e and R_f are complete contractions on $A(V)$. Therefore,

$$P = L_e \circ R_f$$

is a completely contractive idempotent, i.e., a completely contractive projection onto V .

By functoriality of the operator-space projective tensor product, the map

$$P \otimes \text{id}_W : A(V) \widehat{\otimes} W \longrightarrow V \widehat{\otimes} W$$

extends and satisfies $\|P \otimes \text{id}_W\|_{cb} \leq 1$.

Now let $v \in V$ and $b \in W$. Identifying $v \otimes b$ with its image under the canonical inclusion j , we compute

$$(P \otimes \text{id}_W)(\tilde{j}(v \otimes b)) = P(v) \otimes b = evf \otimes b = v \otimes b,$$

since $v = evf$. By linearity and continuity, this equality extends to all of $V \widehat{\otimes} W$. Hence,

$$(P \otimes \text{id}_W) \circ \tilde{j} = \text{id}_{V \widehat{\otimes} W}.$$

Consequently, $\tilde{j}$ admits a completely contractive left inverse, and therefore $\tilde{j}$ is a *complete isometry*. $\square$

COROLLARY 2. *Let V and W be TROs. Then the canonical maps*

$$i' : V \widehat{\otimes} W \rightarrow V \otimes^h W, \quad i'' : V \widehat{\otimes} W \rightarrow V \otimes^\lambda W, \quad \text{and} \quad i''' : V \widehat{\otimes} W \rightarrow V \otimes^\vee W$$

are all injective.

Proof. The canonical map $A(V) \widehat{\otimes} A(W) \rightarrow A(V) \otimes^h A(W)$ is injective ([11], Corollary 1) and the Haagerup tensor product is injective on operator spaces ([5], Proposition 9.2.5). The commutative diagram

$$\begin{array}{ccc} V \widehat{\otimes} W & \xrightarrow{i'} & V \otimes^h W \\ \downarrow j & & \downarrow \\ A(V) \widehat{\otimes} A(W) & \longrightarrow & A(V) \otimes^h A(W) \end{array}$$

shows that i' is injective. Injectivity of i'' follows similarly. Finally, since the canonical map

$$i : V \otimes^h W \rightarrow V \otimes^\vee W$$

is injective (Corollary 1), the composition

$$i''' = i \circ i' : V \widehat{\otimes} W \rightarrow V \otimes^\vee W$$

is injective. $\square$

Recall that a bilinear map $u : E \times F \rightarrow \mathbb{C}$ is *jointly completely bounded* (jcb) if

$$\|u\|_{\text{jcb}} := \sup_{n \geq 1} \|u_n\| < \infty,$$

where $u_n : M_n(E) \times M_n(F) \rightarrow M_n(\mathbb{C})$ is defined entrywise by

$$u_n([v_{ij}], [w_{kl}]) = [u(v_{ij}, w_{kl})].$$

PROPOSITION 3. *Let V and W be TROs. Then every jcb bilinear map*

$$u : V \times W \rightarrow \mathbb{C}$$

admits a unique extension to a jcb bilinear map

$$\tilde{u} : V^{**} \times W^{**} \rightarrow \mathbb{C}$$

such that

$$\|\tilde{u}\|_{\text{jcb}} = \|u\|_{\text{jcb}}.$$

Proof. Define a bilinear form $U : A(V) \times A(W) \rightarrow \mathbb{C}$ by setting

$$U\left(\begin{pmatrix} a & v \\ v^* & b \end{pmatrix}, \begin{pmatrix} c & w \\ w^* & d \end{pmatrix}\right) := u(v, w).$$

Fix $n \in \mathbb{N}$ and let

$$X = (X_{ij}) \in M_n(A(V)), \quad Y = (Y_{ij}) \in M_n(A(W)).$$

Write each X_{ij} and Y_{ij} in 2×2 block form according to the corner decomposition:

$$X_{ij} = \begin{pmatrix} a_{ij} & v_{ij} \\ v_{ij}^* & b_{ij} \end{pmatrix}, \quad Y_{ij} = \begin{pmatrix} c_{ij} & w_{ij} \\ w_{ij}^* & d_{ij} \end{pmatrix}.$$

Then the (i, j) -entry of the matrix valued amplification $U_n(X, Y) \in M_n(\mathbb{C})$ is

$$(U_n(X, Y))_{ij} = U(X_{ij}, Y_{ij}) = u(v_{ij}, w_{ij}).$$

Consequently the full matrix $U_n(X, Y)$ coincides with the amplification $u_n(V, W)$ formed from the $n \times n$ matrices $V = (v_{ij}) \in M_n(V)$ and $W = (w_{ij}) \in M_n(W)$:

$$U_n(X, Y) = u_n((v_{ij}), (w_{ij})).$$

Observe that the operator space norms satisfy

$$\|V\|_{M_n(V)} \leq \|X\|_{M_n(A(V))}, \quad \|W\|_{M_n(W)} \leq \|Y\|_{M_n(A(W))},$$

because the inclusion of the $(1, 2)$ -corner into the linking C^* -algebra is a complete isometry. From these inequalities it follows immediately that

$$\|U_n(X, Y)\| = \|u_n(V, W)\| \leq \|u\|_{\text{jcb}} \|X\|_{M_n(A(V))} \|Y\|_{M_n(A(W))}.$$

Taking the supremum over all n and over all X, Y with unit norms yields

$$\|U\|_{\text{jcb}} \leq \|u\|_{\text{jcb}}.$$

For the converse inequality, given n and arbitrary $V \in M_n(V), W \in M_n(W)$ choose $X \in M_n(A(V)), Y \in M_n(A(W))$ whose $(1, 2)$ -corners are exactly V and W and whose other corners are zero (i.e. take $X = \begin{pmatrix} 0 & v_{ij} \\ 0 & 0 \end{pmatrix}$ and similarly for Y). Then $\|X\|_{M_n(A(V))} = \|V\|_{M_n(V)}$ and $\|Y\|_{M_n(A(W))} = \|W\|_{M_n(W)}$, and the computation above gives $U_n(X, Y) = u_n(V, W)$. Hence

$$\|u_n(V, W)\| \leq \|U\|_{\text{jcb}} \|V\|_{M_n(V)} \|W\|_{M_n(W)},$$

and taking supremum over n, V, W yields $\|u\|_{\text{jcb}} \leq \|U\|_{\text{jcb}}$. Combining the two inequalities we obtain the desired equality

$$\|U\|_{\text{jcb}} = \|u\|_{\text{jcb}}.$$

By ([10], Proposition 2.5), U admits a unique jcb bilinear extension

$$\tilde{U} : A(V)^{**} \times A(W)^{**} \rightarrow \mathbb{C}$$

with $\|\tilde{U}\|_{\text{jcb}} = \|U\|_{\text{jcb}}$. Finally, we obtain $\tilde{u}$ by restriction. Let $i_V^{**} : V^{**} \rightarrow A(V)^{**}$ and $i_W^{**} : W^{**} \rightarrow A(W)^{**}$ be the second dual maps of i_V and i_W respectively. Define

$$\tilde{u} : V^{**} \times W^{**} \longrightarrow \mathbb{C}, \quad \tilde{u}(x, y) := \tilde{U}(i_V^{**}(x), i_W^{**}(y)).$$

Since $\tilde{U}$ extends U , the map $\tilde{u}$ extends u . Moreover restriction does not increase the j.c.b. norm, so

$$\|\tilde{u}\|_{\text{jcb}} \leq \|\tilde{U}\|_{\text{jcb}} = \|U\|_{\text{jcb}} = \|u\|_{\text{jcb}}.$$

Conversely, as $\tilde{u}$ restricts to u on $V \times W$ we have $\|\tilde{u}\|_{\text{jcb}} \geq \|u\|_{\text{jcb}}$, hence equality of norms follows. Uniqueness of $\tilde{u}$ is immediate from uniqueness of $\tilde{U}$. $\square$

Let $u : V \times W \rightarrow \mathbb{C}$ be a jcb bilinear map. From the preceding proposition, we know that u admits a unique extension

$$\tilde{u} : V^{**} \times W^{**} \longrightarrow \mathbb{C}$$

which is norm-preserving. Through the identification ([3], Proposition 5.4)

$$\text{JCB}(V \times W, \mathbb{C}) \cong (V \hat{\otimes} W)^*,$$

this gives rise to a bounded linear operator

$$\phi : (V \hat{\otimes} W)^* \longrightarrow (V^{**} \hat{\otimes} W^{**})^*, \quad \phi(u) = \tilde{u},$$

which is isometric. Taking the Banach adjoint of ϕ , and composing with the canonical embedding

$$J : V^{**} \hat{\otimes} W^{**} \hookrightarrow (V^{**} \hat{\otimes} W^{**})^{**},$$

we obtain a linear map

$$\Phi := \phi^* \circ J : V^{**} \hat{\otimes} W^{**} \longrightarrow (V \hat{\otimes} W)^{**}.$$

The next result shows that the inverse of the map Φ is bounded.

THEOREM 2. *For TROs V and W , there is a canonical bi-continuous embedding*

$$\Phi : V^{**} \hat{\otimes} W^{**} \longrightarrow (V \hat{\otimes} W)^{**},$$

satisfying

$$\frac{1}{2} \|z\| \leq \|\Phi(z)\| \leq \|z\|, \quad \forall z \in V^{**} \hat{\otimes} W^{**}.$$

Proof. The canonical inclusions $j_V : V \hookrightarrow V^{**}$ and $j_W : W \hookrightarrow W^{**}$ together induce a map

$$\kappa = j_V^{**} \otimes j_W^{**} : V^{**} \hat{\otimes} W^{**} \longrightarrow A(V)^{**} \hat{\otimes} A(W)^{**},$$

We now show that the canonical map κ is a complete isometry. Indeed, let $P_V : A(V)^{**} \rightarrow V^{**}$ and $P_W : A(W)^{**} \rightarrow W^{**}$ be the canonical completely contractive projections onto the TROs V^{**} and W^{**} , respectively. Consider the composition

$$V^{**} \hat{\otimes} W^{**} \xrightarrow{\kappa} A(V)^{**} \hat{\otimes} A(W)^{**} \xrightarrow{P_V \otimes P_W} V^{**} \hat{\otimes} W^{**}.$$

Since $P_V \otimes P_W$ is completely contractive and the composition is the identity on $V^{**} \widehat{\otimes} W^{**}$, it follows that κ must be a complete isometry.

On the other hand, the natural inclusion $j : V \widehat{\otimes} W \hookrightarrow A(V) \widehat{\otimes} A(W)$ induces, by taking biduals, a canonical embedding

$$j^{**} : (V \widehat{\otimes} W)^{**} \longrightarrow (A(V) \widehat{\otimes} A(W))^{**}.$$

Finally, $\widetilde{\Phi} : A(V)^{**} \widehat{\otimes} A(W)^{**} \rightarrow (A(V) \widehat{\otimes} A(W))^{**}$ is defined analogously to Φ . With these maps, we obtain the diagram

$$\begin{array}{ccc} V^{**} \widehat{\otimes} W^{**} & \xrightarrow{\Phi} & (V \widehat{\otimes} W)^{**} \\ \downarrow \kappa & & \downarrow j^{**} \\ A(V)^{**} \widehat{\otimes} A(W)^{**} & \xrightarrow{\widetilde{\Phi}} & (A(V) \widehat{\otimes} A(W))^{**} \end{array}$$

We now show the diagram is commutative. Note that for $z \in V^{**} \widehat{\otimes} W^{**}$ and $u \in (V \widehat{\otimes} W)^*$ we have the following duality relation for Φ

$$\langle \Phi(z), u \rangle = \langle (\phi^* \circ J)(z), u \rangle = \langle J(z), \phi(u) \rangle = \langle z, \phi u \rangle = \langle z, \widetilde{u} \rangle.$$

Similarly, viewing u as a functional on the subspace $V \widehat{\otimes} W \subseteq A(V) \widehat{\otimes} A(W)$ we define $\overline{u} \in (A(V) \widehat{\otimes} A(W))^*$ to be the functional which vanishes on the complementary corners and agrees with u on the $(1, 2)$ -corner; $\overline{u}$ admits a unique norm-preserving extension

$$\widetilde{\overline{u}} \in (A(V)^{**} \widehat{\otimes} A(W)^{**})^* = \text{JCB}(A(V)^{**} \times A(W)^{**}, \mathbb{C}).$$

By the analogous construction we obtain $\widetilde{\Phi} : A(V)^{**} \widehat{\otimes} A(W)^{**} \rightarrow (A(V) \widehat{\otimes} A(W))^{**}$ satisfying the duality relation

$$\langle \widetilde{\Phi}(\xi), \overline{u} \rangle = \langle \xi, \widetilde{\overline{u}} \rangle \quad (\xi \in A(V)^{**} \widehat{\otimes} A(W)^{**}).$$

Fix

$$z \in V^{**} \widehat{\otimes} W^{**} \quad \text{and} \quad u \in (V \widehat{\otimes} W)^*.$$

Note that

$$\langle j^{**}(\Phi(z)), u \rangle = \langle \Phi(z), u \rangle = \langle z, \widetilde{u} \rangle.$$

On the other hand, by the duality for $\widetilde{\Phi}$,

$$\langle \widetilde{\Phi}(\kappa(z)), \overline{u} \rangle = \langle \kappa(z), \widetilde{\overline{u}} \rangle,$$

But the restriction of $\widetilde{\overline{u}}$ to the $(1, 2)$ -corner of $A(V)^{**} \times A(W)^{**}$ is an extension of u on $V^{**} \times W^{**}$. By uniqueness of such extensions this restriction coincides with $\widetilde{u}$, hence

$$\langle \kappa(z), \widetilde{\overline{u}} \rangle = \langle z, \widetilde{u} \rangle.$$

Combining the displayed identities we obtain

$$\langle j^{**}(\Phi(z)), u \rangle = \langle \tilde{\Phi}(\kappa(z)), \bar{u} \rangle.$$

To conclude that $j^{**}(\Phi(z)) = \tilde{\Phi}(\kappa(z))$ in $(A(V) \hat{\otimes} A(W))^{**}$ we observe that both elements lie in the subspace of $(A(V) \hat{\otimes} A(W))^{**}$ corresponding to the $(1, 2)$ -corner (indeed each is obtained by extending an element supported on that corner). The map

$$u \longmapsto \bar{u}$$

identifies $(V \hat{\otimes} W)^*$ isometrically with the subspace of $(A(V) \hat{\otimes} A(W))^*$ consisting of functionals vanishing on the complementary corners. Consequently the equalities of pairings

$$\langle j^{**}(\Phi(z)), u \rangle = \langle \tilde{\Phi}(\kappa(z)), \bar{u} \rangle \quad (\forall u \in (V \hat{\otimes} W)^*)$$

are sufficient to deduce

$$j^{**}(\Phi(z)) = \tilde{\Phi}(\kappa(z)).$$

Since z was arbitrary this proves the operator identity

$$\tilde{\Phi} \circ \kappa = j_{V \otimes W}^{**} \circ \Phi,$$

i.e. the diagram commutes. Finally, for any $z \in V^{**} \hat{\otimes} W^{**}$, we have

$$\|\Phi(z)\| = \|\tilde{\Phi}(\kappa(z))\| \leq \|\kappa(z)\| = \|z\|,$$

where the first equality follows from the commutativity of the diagram, the second from ([10], Theorem 2.9) and the third from the isometry of κ . Similarly,

$$\|\Phi(z)\| = \|\tilde{\Phi}(\kappa(z))\| \geq \frac{1}{2} \|\kappa(z)\| = \frac{1}{2} \|z\|. \quad \square$$

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