

A SHORT PROOF OF FAGUNDES'S THEOREM

LAN LU AND YU WANG*

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Abstract. In 2019, Fagundes used a tedious argument to prove that every multilinear polynomial on strictly upper triangular matrix algebras over any field is a vector space. The goal of the paper is to give a short proof of the result.

1. Introduction

Let F be a field. By $U_n(F)$ we denote the set of all $n \times n$ upper triangular matrices over F . By $U_n(F)^{(i)}$ we denote the set of all upper triangular matrices over F whose entries (j, k) are zero, for $k - j \leq i$, where $i \geq 0$ is an integer. Note that $U_n(F)^{(0)}$ is the set of all $n \times n$ strictly upper triangular matrices over F .

In 2019, Fagundes [3] used some tedious arguments to prove that the image of every multilinear polynomial on $U_n(F)^{(0)}$ is a vector space.

THEOREM 1.1. [3, Theorem 3] *Let F be any field, let $n \geq 2$ and $m \geq 1$ be integers. Let $f(x_1, \dots, x_m)$ be a nonzero multilinear polynomial over F . If $m \geq n$, then $f(U_n(F)^{(0)}) = \{0\}$. If $m < n$, then $f(U_n(F)^{(0)}) = U_n(F)^{(m-1)}$.*

We remark that [3, Theorem 3] is a profound result because it holds true for any field. However, the proof of Theorem 1.1 is very complex and not easy to understand for readers. It is necessary to give a short proof of Theorem 1.1.

In 2019, Fagundes and Mello [5] discussed the images of multilinear polynomials of lower degrees on $U_n(F)$. For the case of multilinear polynomials of arbitrary degrees, they presented the following:

CONJECTURE 1.1. [5] *The image of a multilinear polynomial over F on $U_n(F)$ is always a vector space.*

Since 2019, Conjecture 1.1 has been investigated extensively in [1, 4, 6, 8–10]. We remark that Conjecture 1.1 remains open for $|F| < n$ (see [1] for details).

Recently, Chen and Wang [2] gave a description of the images of generalized multilinear polynomials on $U_n(F)$, which is a generalization of the results on Conjecture 1.1.

In the present paper, by using some arguments from [1] we shall give a short proof of Theorem 1.1.

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* Corresponding author.

2. The proof of Theorem 1.1

Let $f(x_1, \dots, x_m)$ be a nonzero multilinear polynomial over F . By W_m we denote the set of all permutations of $\{1, \dots, m\}$. Without loss of generality we can write

$$f(x_1, \dots, x_m) = \sum_{(i_1, \dots, i_m) \in W_m} \alpha_{i_1 \dots i_m} x_{i_1} \cdots x_{i_m}, \quad (1)$$

where $\alpha_{i_1 \dots i_m} \in F$ with $\alpha_{1 \dots m} \neq 0$.

The following result gives an expression of the image of a multilinear polynomial on $U_n(F)^{(0)}$.

LEMMA 2.1. *Let $m < n$. For $A_i = (a_{jk}^{(i)})_{n \times n} \in U_n(F)^{(0)}$, $i = 1, \dots, m$, we have*

$$f(A_1, \dots, A_m) = (h_{s, s+m+u})_{n \times n} \in U_n(F)^{(m-1)},$$

where

$$h_{s, s+m+u} = \sum_{\substack{s=j_1 < j_2 < \dots < j_{m+1} = s+m+u \\ (i_1, \dots, i_m) \in W_m}} \alpha_{i_1 \dots i_m} a_{j_1 j_2}^{(i_1)} \cdots a_{j_m j_{m+1}}^{(i_m)}$$

is the $(s, s+m+u)$ -th entry of $f(A_1, \dots, A_m)$, where $1 \leq s < s+m+u \leq n$.

Proof. It follows from (1) that

$$\begin{aligned} f(A_1, \dots, A_m) &= \sum_{(i_1, \dots, i_m) \in W_m} \alpha_{i_1 \dots i_m} A_{i_1} \cdots A_{i_m} \\ &= \sum_{(i_1, \dots, i_m) \in W_m} \alpha_{i_1 \dots i_m} \left(\sum_{s=j_1 < j_2 < \dots < j_{m+1} = s+m+u} a_{j_1 j_2}^{(i_1)} \cdots a_{j_m j_{m+1}}^{(i_m)} \right)_{n \times n} \\ &= (h_{s, s+m+u})_{n \times n}. \end{aligned}$$

This proves the result. $\square$

The following result is crucial for the proof of Theorem 1.1.

LEMMA 2.2. *Let $m < n$. Let*

$$g_{s, s+m} = \sum_{(i_1, \dots, i_{m-1}, m) \in W_m} \alpha_{i_1 \dots i_{m-1} m} y_{s, s+1}^{(i_1)} \cdots y_{s+m-2, s+m-1}^{(i_{m-1})}$$

be a polynomial in commutative variables over F , where $1 \leq s < s+m \leq n$. Then there exist $b_{j, j+1}^{(i)} \in F$, where $j = 1, \dots, n-m$ and $i = 1, \dots, m$ with $i \neq m$, such that

$$g_{s, s+m}(b_{j, j+1}^{(i)}) = \sum_{(i_1, \dots, i_{m-1}, m) \in W_m} \alpha_{i_1 \dots i_{m-1} m} b_{s, s+1}^{(i_1)} \cdots b_{s+m-2, s+m-1}^{(i_{m-1})} \neq 0$$

for all $1 \leq s < s+m \leq n$.

Proof. Set

$$g = \prod_{1 \leq s < s+m \leq n} g_{s,s+m}.$$

It is clear that g is a nonzero homogenous polynomial. We claim that $g(F) \neq \{0\}$. If F is an infinite field, we get from Theorem 2.19 [7] that $g(F) \neq \{0\}$. We now assume that F is a finite field. We assume on the contrary that $g(F) = \{0\}$. Set

$$M_{s,s+m} = \alpha_1 \cdots \alpha_{s,s+1}^{(1)} \cdots \alpha_{s+m-2,s+m-1}^{(m-1)}$$

for all $1 \leq s < s+m \leq n$, and

$$M = \prod_{1 \leq s < s+m \leq n} M_{s,s+m}.$$

Since $\alpha_1 \cdots \alpha_m \neq 0$, we get that M is a nonzero monomial of g such that the exponent of every variable in M is 1. By [1, Proposition 4.2] we have that M must contain a variable whose exponent is not less than $|F|$. Since $|F| \geq 2$ we get a contradiction. Therefore, $g(F) \neq \{0\}$. Thus there exist $b_{j,j+1}^{(i)} \in F$, where $j = 1, \dots, n-m$ and $i = 1, \dots, m$ with $i \neq m$, such that

$$g(b_{j,j+1}^{(i)}) = \prod_{1 \leq s < s+m \leq n} g_{s,s+m}(b_{j,j+1}^{(i)}) \neq 0.$$

This implies that $g_{s,s+m}(b_{j,j+1}^{(i)}) \neq 0$ for all $1 \leq s < s+m \leq n$. $\square$

We are ready to give

The proof of Theorem 1.1. If $m \geq n$, we note that $f(U_n(F)^{(0)}) = \{0\}$. We now assume that $m < n$. It is clear that $f(U_n(F)^{(0)}) \subseteq U_n(F)^{(m-1)}$. It suffices to prove that $U_n(F)^{(m-1)} \subseteq f(U_n(F)^{(0)})$.

Take $A_i = (a_{jk}^{(i)}) \in U_n(F)^{(0)}$, $i = 1, \dots, m$. It follows from Lemma 2.1 that

$$f(A_1, \dots, A_m) = (h_{s,s+m+u})_{n \times n},$$

where

$$h_{s,s+m+u} = \sum_{\substack{(i_1, \dots, i_m) \in W_m \\ s=j_1 < j_2 < \dots < j_{m+1}=s+m+u}} \alpha_{i_1 \cdots i_m} a_{j_1 j_2}^{(i_1)} \cdots a_{j_m j_{m+1}}^{(i_m)}$$

for all $1 \leq s < s+m+u \leq n$. Moreover, $\alpha_1 \cdots \alpha_m \neq 0$. We have

$$\begin{aligned} h_{s,s+m+u} = & \left(\sum_{(i_1, \dots, i_{m-1}, m) \in W_m} \alpha_{i_1 \cdots i_{m-1} m} a_{s,s+1}^{(i_1)} \cdots a_{s+m-2,s+m-1}^{(i_{m-1})} \right) a_{s+m-1,s+m+u}^{(m)} \\ & + \left(\sum_{\substack{(i_1, \dots, i_m) \in W_m \\ i_m \neq m}} \alpha_{i_1 \cdots i_m} a_{s,s+1}^{(i_1)} \cdots a_{s+m-2,s+m-1}^{(i_{m-1})} \right) a_{s+m-1,s+m+u}^{(i_m)} \\ & + \sum_{\substack{s \leq j_1 < \dots < j_{m+1} \leq s+m+u \\ (j_m, j_{m+1}) \neq (s+m-1, s+m+u) \\ (i_1, \dots, i_m) \in W_m}} \alpha_{i_1 \cdots i_m} a_{j_1 j_2}^{(i_1)} \cdots a_{j_m j_{m+1}}^{(i_m)} \end{aligned} \quad (2)$$

for all $1 \leq s < s+m+u \leq n$. We set

$$g_{s,s+m} = \sum_{(i_1, \dots, i_{m-1}, m) \in W_m} \alpha_{i_1 \dots i_{m-1} m} y_{s,s+1}^{(i_1)} \cdots y_{s+m-2, s+m-1}^{(i_{m-1})}$$

for all $1 \leq s < s+m \leq n$. By Lemma 2.2 we get that there exist $b_{j,j+1}^{(i)} \in F$, where $j = 1, \dots, n-m$ and $i = 1, \dots, m$ with $i \neq m$, such that

$$g_{s,s+m}(b_{j,j+1}^{(i)}) = \sum_{(i_1, \dots, i_{m-1}, m) \in W_m} \alpha_{i_1 \dots i_{m-1} m} b_{s,s+1}^{(i_1)} \cdots b_{s+m-2, s+m-1}^{(i_{m-1})} \neq 0$$

for all $1 \leq s < s+m \leq n$. We define an order on the following set of the variables

$$\{x_{s+m-1, s+m+u}^{(m)} \mid 1 \leq s < s+m+u \leq n\}$$

as follows:

- (i) $x_{s+m-1, s+m+u}^{(m)} < x_{s'+m-1, s'+m+u'}^{(m)}$ if $u < u'$;
- (ii) $x_{s+m-1, s+m+u}^{(m)} < x_{s'+m-1, s'+m+u'}^{(m)}$ if $u = u'$ and $s < s'$.

By taking

$$\begin{cases} a_{j,j+1}^{(i)} = b_{j,j+1}^{(i)}, & \text{where } j = 1, \dots, n-m \text{ and } i = 1, \dots, m \text{ with } i \neq m; \\ a_{s+m-1, s+m+u}^{(m)} = x_{s+m-1, s+m+u}^{(m)}, & \text{where } 1 \leq s < s+m+u \leq n; \\ a_{jk}^{(i)} = 0, & \text{otherwise} \end{cases}$$

in (2), we get that

$$h_{s,s+m+u} = g_{s,s+m}(b_{j,j+1}^{(i)})x_{s+m-1, s+m+u}^{(m)} + \gamma_{s,s+m+u} \quad (3)$$

for all $1 \leq s \leq s+m+u \leq n$, where $\gamma_{s,s+m+u}$ is a polynomial in some commutative variables that rank in front of $x_{s+m-1, s+m+u}^{(m)}$. Moreover, $\gamma_{1,1+m} = 0$.

For any $A' = (a'_{s,s+m+u})_{n \times n} \in U_n(F)^{(m-1)}$, we consider the following the group of equations:

$$g_{s,s+m}(b_{j,j+1}^{(i)})x_{s+m-1, s+m+u}^{(m)} + \gamma_{s,s+m+u} = a'_{s,s+m+u} \quad (4)$$

for all $1 \leq s < s+m+u \leq n$. It is easy to check that (4) has a solution $b_{s+m-1, s+m+u}^{(m)} \in F$, where $1 \leq s < s+m+u \leq n$. Taking $x_{s+m-1, s+m+u}^{(m)} = b_{s+m-1, s+m+u}^{(m)}$ for all $1 \leq s < s+m+u \leq n$ in (3), we get from (4) that

$$f(A_1, \dots, A_m) = \left(h_{s,s+m+u}(b_{s+m-1, s+m+u}^{(m)}) \right)_{n \times n} = (a'_{s,s+m+u})_{n \times n} = A'.$$

This implies that $U_n(F)^{(m-1)} \subseteq f(U_n(F))$, as desired. The proof of the result is complete. $\square$

Conflicts of interest. The authors declare that they have no competing interests.

Availability of data and material. Not applicable.

Authors' contributions. All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

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Lan Lu
Department of Mathematics and Physics
Changchun Guanghua University
Changchun 130033, China
e-mail: lulan@ghu.edu.cn

Yu Wang
Department of Mathematics
Shanghai Normal University
Shanghai 200234, China
e-mail: ywang2004@126.com