

ANALYTIC α -MORREY SPACES AND VOLTERRA INTEGRAL OPERATORS

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Abstract. A family of analytic α -Morrey spaces $\mathcal{L}^{2,\lambda,\alpha}$ in the unit disc is introduced. Subsequently, some basic properties of $\mathcal{L}^{2,\lambda,\alpha}$ are established. Moreover, the boundedness, compactness, norm, and essential norm of the Volterra integral operators T_g and S_g acting on analytic α -Morrey spaces $\mathcal{L}^{2,\lambda,\alpha}$ are investigated comprehensively.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. The Möbius group, denoted by $\text{Aut}(\mathbb{D})$, is the collection of all bi-holomorphic analytic mappings on $\mathbb{D}$ of the form $\phi = e^{i\theta} \sigma_b$ with $\theta \in [0, 2\pi)$ and $b \in \mathbb{D}$. Here $\sigma_b(z) = \frac{b-z}{1-\bar{b}z}$, which plays a pivotal role in various aspects of complex function theory.

For a positive real-valued parameter α , let $f \in H(\mathbb{D})$, and $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic mapping. The weighted composition of f and ϕ , which was first introduced by Zhao [33], is denoted by $f \circ_\alpha \phi$ and is defined by

$$f \circ_\alpha \phi(z) = \int_0^z f'(\phi(w))(\phi'(w))^\alpha dw + f(\phi(0))(\phi'(0))^{\alpha-1}.$$

Here $(\phi'(0))^{\alpha-1}$ is taken to be its principal value. This construction generalizes the standard composition and provides a powerful tool in the investigation of various problems in complex analysis. In particular, when $\alpha = 1$, $f \circ_\alpha \phi$ will be denoted by $f \circ \phi$.

For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ is defined as the collection of all $f \in H(\mathbb{D})$ satisfying (cf. [4]):

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let H^∞ denote the space of all bounded analytic functions with the supremum norm $\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)|$.

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In 1961, F. John and L. Nirenberg [11] introduced the space of functions of bounded mean oscillation, BMO , during their investigation of differential equations. The $BMOA$ space is a fundamental object in complex analysis, defined as the collection of all analytic functions within the Hardy space H^2 , where the boundary values of these functions lie in BMO . Formally,

$$BMOA = \left\{ f \in H^2(\mathbb{D}) : \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|} \int_I |f(e^{i\theta}) - f_I|^2 d\theta < \infty \right\},$$

where $f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{d\zeta}{2\pi}$ represents the mean value of f over the arc I of the unit circle $\partial\mathbb{D}$. A well-established and highly useful characterization of $BMOA$ states that a function f belongs to $BMOA$ if and only if

$$\sup_{b \in \mathbb{D}} \|f \circ \sigma_b - f \circ \sigma_b(0)\|_{H^2} < \infty.$$

The $BMOA$ space plays a crucial role in numerous problems. C. Fefferman's [5, 6] celebrated duality theorem stands as a cornerstone result. This theorem elegantly establishes that the dual space of the Hardy space H^1 is precisely $BMOA$, thereby providing a deep connection between these two important function spaces and facilitating further progress in harmonic and complex analysis. If $f \in H(\mathbb{D})$ such that $\lim_{|b| \rightarrow 1} \|f \circ \sigma_b - f(b)\|_{H^2} = 0$, then we say that f belongs to $VMOA$.

Let λ satisfy $0 \leq \lambda \leq 1$. In their work [28], Wu and Xie introduced the analytic Morrey space $\mathcal{L}^{2,\lambda}$. A function f in the Hardy space H^2 belongs to $\mathcal{L}^{2,\lambda}$ if and only if the norm $\|f\|_{\mathcal{L}^{2,\lambda}}$ is finite. The norm $\|f\|_{\mathcal{L}^{2,\lambda}}$ is defined as follows:

$$\|f\|_{\mathcal{L}^{2,\lambda}} = |f(0)| + \sup_{b \in \mathbb{D}} (1 - |b|^2)^{\frac{1-\lambda}{2}} \|f \circ \sigma_b - f \circ \sigma_b(0)\|_{H^2}.$$

This space exhibits interesting special cases. When $\lambda = 0$, $\mathcal{L}^{2,0}$ coincides with the Hardy space H^2 , and when $\lambda = 1$, it is identical to the $BMOA$ space. For a comprehensive understanding of the most recent research findings regarding various analytic Morrey spaces, such as Bergman-Morrey, Dirichlet-Morrey, and Besov-Morrey spaces, one may consult [7, 10, 13, 14, 22, 24, 26, 28, 29, 30, 31, 32] and the references therein.

Inspired by the concepts of the analytic Morrey space and weighted composition, we define a new function space $\mathcal{L}^{2,\lambda,\alpha}$, namely the analytic α -Morrey space, in this work. For a function $f \in H(\mathbb{D})$, with parameters $0 < \alpha < \infty$ and $0 \leq \lambda \leq 1$, we say that f belongs to the analytic α -Morrey space $\mathcal{L}^{2,\lambda,\alpha}$ if the norm $\|f\|_{\mathcal{L}^{2,\lambda,\alpha}}$ is finite. Here,

$$\|f\|_{\mathcal{L}^{2,\lambda,\alpha}} = |f(0)| + \sup_{b \in \mathbb{D}} (1 - |b|^2)^{\frac{1-\lambda}{2}} \|f \circ_\alpha \sigma_b - f \circ_\alpha \sigma_b(0)\|_{H^2}.$$

It is easy to check that analytic α -Morrey spaces are indeed Banach spaces. Furthermore, we introduce the little analytic α -Morrey space, denoted by $\mathcal{L}_0^{2,\lambda,\alpha}$. A function $f \in H(\mathbb{D})$ is an element of $\mathcal{L}_0^{2,\lambda,\alpha}$ if

$$\lim_{|b| \rightarrow 1} (1 - |b|^2)^{\frac{1-\lambda}{2}} \|f \circ_\alpha \sigma_b - f \circ_\alpha \sigma_b(0)\|_{H^2} = 0.$$

This new function space framework exhibits interesting special cases. When $\alpha = 1$, the space $\mathcal{L}^{2,\lambda,\alpha}$ coincides with the well-known $\mathcal{L}^{2,\lambda}$ space. When $\lambda = 0$, $\mathcal{L}^{2,\lambda,\alpha}$ reduces to the Hardy space H^2 . In the case where $\lambda = 1$, we give the spaces more specialized names: $\mathcal{L}^{2,\lambda,\alpha}$ is referred to as the α -BMOA space, denoted by $BMOA^\alpha$, and $\mathcal{L}_0^{2,\lambda,\alpha}$ is called the α -VMOA space, denoted by $VMOA^\alpha$, respectively.

Let $g \in H(\mathbb{D})$. We recall the definitions of the Volterra integral operators T_g and S_g , which are induced by the function g . These operators are defined as follows:

$$T_g f(z) = \int_0^z f(w)g'(w)dw \quad \text{and} \quad S_g f(z) = \int_0^z f'(w)g(w)dw,$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. The operator T_g generalizes the well-known Cesàro operator. The operator T_g was initially introduced by Pommerenke in [21]. In this work, Pommerenke established a fundamental result: the operator T_g maps the Hardy space H^2 to itself boundedly if and only if $g \in BMOA$. Li, Liu, and Lou in [13] delved into the properties of the $\mathcal{L}^{2,\lambda}$ space and explored the boundedness, compactness, norm, and essential norm of the Volterra integral operators T_g and S_g acting on $\mathcal{L}^{2,\lambda}$. Over the past nearly four decades, the mathematical community has conducted extensive research on the boundedness, compactness, norm, and essential norm of T_g and S_g when considered between various function spaces. For a comprehensive overview of results regarding these operators, we refer the reader to [1, 7, 9, 10, 12, 13, 14, 15, 16, 17, 19, 22, 23, 24, 25, 26, 28, 29, 30, 31] and the references therein.

The paper is organized as follows. In section 2, we establish some basic properties of the analytic α -Morrey space $\mathcal{L}^{2,\lambda,\alpha}$ and the little analytic α -Morrey space $\mathcal{L}_0^{2,\lambda,\alpha}$. In section 3, we fully characterize the boundedness, compactness, norm, and essential norm of the Volterra integral operators T_g and S_g on the analytic α -Morrey space $\mathcal{L}^{2,\lambda,\alpha}$.

In this paper, we say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$. The symbol $f \asymp g$ means that $f \lesssim g \lesssim f$.

2. Basic properties of analytic α -Morrey spaces

In this section, we consider some basic properties of analytic α -Morrey spaces $\mathcal{L}^{2,\lambda,\alpha}$. First, let us state some definitions and lemmas, which will be used in the proof of the main results.

Let μ denote a positive Borel measure defined on $\mathbb{D}$, and let $\beta > 0$ be a fixed positive real number. We say that the measure μ is a β -Carleson measure on $\mathbb{D}$ (cf. [3]) if the quantity

$$\|\mu\|_\beta = \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^\beta}$$

is finite. Here, for an arc I on the unit circle $\partial\mathbb{D}$, the Carleson box $S(I)$ is defined as the set

$$S(I) = \left\{ w = te^{i\theta} \in \mathbb{D} : 1 - |I| \leq t < 1, e^{i\theta} \in I \right\},$$

where $|I|$ represents the normalized arc-length measure of the arc I on $\partial\mathbb{D}$. The measure μ is said to be a vanishing β -Carleson measure if

$$\lim_{|I| \rightarrow 0} \frac{\mu(S(I))}{|I|^\beta} = 0.$$

In the special case when $\beta = 1$, the β -Carleson measure reduces to the well-known classical Carleson measure, as studied in [2, 3].

As established in [20], the measure μ on $\mathbb{D}$ is characterized in the context of β -Carleson measures. Specifically, μ is a β -Carleson measure if and only if

$$\sup_{b \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |b|^2)^t}{|1 - \bar{b}w|^{t+\beta}} d\mu(w) < \infty, \quad (1)$$

for all $t \in (-\beta, \infty)$. This integral condition serves as an alternative, yet equivalent, definition to the more traditional one based on Carleson boxes. Moreover, the measure μ is a vanishing β -Carleson measure if and only if

$$\lim_{|b| \rightarrow 1} \int_{\mathbb{D}} \frac{(1 - |b|^2)^t}{|1 - \bar{b}w|^{t+\beta}} d\mu(w) = 0, \quad (2)$$

for every $t \in (-\beta, \infty)$. These integral characterizations provide powerful tools for analyzing and working with β -Carleson and vanishing β -Carleson measures, allowing mathematicians to explore their properties from a different analytical perspective compared to the Carleson box-based definitions.

PROPOSITION 1. *Let $f \in H(\mathbb{D})$, $0 < \lambda \leq 1$ and $0 < \alpha < 1 + \frac{\lambda}{2}$. Then the following statements are equivalent.*

(i) $f \in \mathcal{L}^{2,\lambda,\alpha}$.

(ii)

$$M_1 = \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2) |\sigma'_b(w)|^{3-2\alpha} dA(w) < \infty.$$

(iii)

$$M_2 = \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^{2\alpha-2} (1 - |\sigma_b(w)|^2)^{3-2\alpha} dA(w) < \infty.$$

(iv)

$$M_3 = \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f'(w)|^2 (1 - |w|^2) dA(w) < \infty.$$

(v)

$$M_4 = \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^{2\alpha-\lambda-1} (1 - |\sigma_b(w)|^2)^{2-2\alpha+\lambda} dA(w) < \infty.$$

Moreover,

$$\|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 - |f(0)|^2 \asymp M_1 \asymp M_2 \asymp M_3 \asymp M_4.$$

Proof. Let us first prove that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). Making the change of variable $w = \sigma_b(z)$ and using the following identity

$$\|f\|_{H^2}^2 \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) dA(z),$$

it follows that

$$\begin{aligned} & \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 - |f(0)|^2 \asymp \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \|f \circ \sigma_b - f \circ \sigma_b(0)\|_{H^2}^2 \\ & \asymp \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(\sigma_b(z))|^2 |\sigma'_b(z)|^{2\alpha} (1 - |z|^2) dA(z) \\ & = \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(\sigma_b(z))|^2 (1 - |z|^2)^{1-2\alpha} (1 - |\sigma_b(z)|^2)^{2\alpha} dA(z) \\ & = \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |\sigma_b(w)|^2)^{1-2\alpha} (1 - |w|^2)^{2\alpha} |\sigma'_b(w)|^2 dA(w) \\ & = \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2) |\sigma'_b(w)|^{3-2\alpha} dA(w) \\ & = \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^{2\alpha-2} (1 - |\sigma_b(w)|^2)^{3-2\alpha} dA(w). \end{aligned}$$

(ii) $\Rightarrow$ (iv) Given any arc $I \subset \partial\mathbb{D}$, let $b = (1 - |I|)\xi$, where ξ is the center of I . We have $|1 - \bar{b}w| \asymp 1 - |b|^2 \asymp |I|$, $w \in S(I)$. Hence

$$\begin{aligned} & \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f'(w)|^2 (1 - |w|^2) dA(w) \\ & \lesssim (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2) \frac{(1 - |b|^2)^{3-2\alpha}}{|1 - \bar{b}w|^{2(3-2\alpha)}} dA(w) \\ & \lesssim \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2) |\sigma'_b(w)|^{3-2\alpha} dA(w) \asymp \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 - |f(0)|^2, \end{aligned}$$

as desired.

(iv) $\Rightarrow$ (ii) Let $d\mu_f(w) = |f'(w)|^2 (1 - |w|^2) dA(w)$. We obtain

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu_f(S(I))}{|I|^{2-2\alpha+\lambda}} = \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f'(w)|^2 (1 - |w|^2) dA(w) < \infty,$$

which implies that μ_f is $a(2 - 2\alpha + \lambda)$ -Carleson measure. Here we used the fact that

$\alpha < 1 + \frac{\lambda}{2}$. Then by (1) we have

$$\begin{aligned} & \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2) |\sigma'_b(w)|^{3-2\alpha} dA(w) \\ &= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} \frac{(1 - |b|^2)^{3-2\alpha}}{|1 - \bar{b}w|^{2(3-2\alpha)}} d\mu_f(w) \\ &= \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |b|^2)^{-2\alpha+4-\lambda}}{|1 - \bar{b}w|^{2-2\alpha+\lambda+(4-2\alpha-\lambda)}} d\mu_f(w) \\ &< \infty. \end{aligned}$$

(iv) $\Leftrightarrow$ (v) Let $d\mu_f(z) = |f'(w)|^2 (1 - |w|^2) dA(w)$. By (1), we obtain that μ_f is a $(2 - 2\alpha + \lambda)$ -Carleson measure if and only if

$$\sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^{2\alpha-\lambda-1} (1 - |\sigma_b(w)|^2)^{2-2\alpha+\lambda} dA(w) < \infty.$$

The proof is complete. $\square$

REMARK 1. From the proof of Proposition 1, we see that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) for any $\alpha \in (0, \infty)$. Moreover, $f \in \mathcal{L}^{2,\lambda,\alpha}$ if and only if

$$\sup_{b \in \mathbb{D}} (1 - |b|^2)^{\frac{1-\lambda}{2}} \|f \circ_{\alpha} \sigma_b\|_{H^2} < \infty.$$

REMARK 2. When $\alpha = \frac{1+\lambda}{2}$, $f \in \mathcal{L}^{2,\lambda,\alpha}$ if and only if

$$\sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2 (1 - |\sigma_b(w)|^2) dA(w) < \infty.$$

This implies that the space $\mathcal{L}^{2,\lambda,\frac{1+\lambda}{2}}$ is the *BMOA* space (see [33]). When $\alpha = 1$, the space $\mathcal{L}^{2,\lambda,\alpha}$ is the analytic Morrey space $\mathcal{L}^{2,\lambda}$. Hence, the space $\mathcal{L}^{2,\lambda,\alpha}$ not only generalizes the analytic Morrey space but also include the classical *BMOA* space.

From the proof of Proposition 1 and (2), we have the following characterization of the space $\mathcal{L}_0^{2,\lambda,\alpha}$. We omit the details of the proof.

PROPOSITION 2. Let $f \in H(\mathbb{D})$, $0 < \lambda \leq 1$ and $0 < \alpha < 1 + \frac{\lambda}{2}$. Then the following statements are equivalent.

(i) $f \in \mathcal{L}_0^{2,\lambda,\alpha}$.

(ii)

$$\lim_{|b| \rightarrow 1} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2) |\sigma'_b(w)|^{3-2\alpha} dA(w) = 0.$$

(iii)

$$\lim_{|b| \rightarrow 1} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^{2\alpha-2} (1 - |\sigma_b(w)|^2)^{3-2\alpha} dA(w) = 0.$$

(iv)

$$\lim_{|I| \rightarrow 0} \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f'(w)|^2 (1 - |w|^2) dA(w) = 0.$$

(v)

$$\lim_{|b| \rightarrow 1} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^{2\alpha-\lambda-1} (1 - |\sigma_b(w)|^2)^{2-2\alpha+\lambda} dA(w) = 0.$$

PROPOSITION 3. Let $0 < \alpha < \infty$ and $0 < \lambda \leq 1$. Then $\mathcal{L}^{2,\lambda,\alpha} \subseteq H^2$.

Proof. Let $f \in \mathcal{L}^{2,\lambda,\alpha}$. By Proposition 1, we have

$$\begin{aligned} \infty &> \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^{2\alpha-2} (1 - |\sigma_b(w)|^2)^{3-2\alpha} dA(w) \\ &\geq \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2) dA(w). \end{aligned}$$

The last inequality follows by taking $b = 0$. So $f \in H^2$. Hence, $\mathcal{L}^{2,\lambda,\alpha} \subseteq H^2$. $\square$

LEMMA 1. [18, Lemma 2.5] Let $s > -1$, $r, t > 0$. If $r < s + 2 < t$, then we have

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^s}{|1 - \bar{a}z|^r |1 - \bar{b}z|^t} dA(z) \lesssim \frac{1}{|1 - \bar{a}b|^r (1 - |b|^2)^{t-s-2}}$$

where $a, b \in \mathbb{D}$.

PROPOSITION 4. Let $a \in \mathbb{D}$, $0 < \lambda \leq 1$ and $\frac{\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. Then the function

$$f_a(z) = \frac{(1 - |a|^2)}{(1 - \bar{a}z)^{\frac{1-\lambda}{2} + \alpha}}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{L}^{2,\lambda,\alpha}$. Moreover,

$$\|f_a\|_{\mathcal{L}^{2,\lambda,\alpha}} \lesssim 1.$$

Proof. Using Proposition 1 and Lemma 1, we get

$$\begin{aligned} \|f_a\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 &\asymp \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'_a(w)|^2 (1 - |w|^2)^{2\alpha-2} (1 - |\sigma_b(w)|^2)^{3-2\alpha} dA(w) \\ &\lesssim \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} \frac{(1 - |a|^2)^2 (1 - |w|^2) (1 - |b|^2)^{3-2\alpha}}{|1 - \bar{a}w|^{2\alpha+3-\lambda} |1 - \bar{b}w|^{6-4\alpha}} dA(w) \\ &\lesssim \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} (1 - |a|^2)^2 \int_{\mathbb{D}} \frac{(1 - |w|^2)}{|1 - \bar{b}w|^{3-2\alpha} |1 - \bar{a}w|^{2\alpha+3-\lambda}} dA(w) \\ &\lesssim 1. \end{aligned}$$

Here we used the assumption that $\frac{\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. The proof is complete. $\square$

PROPOSITION 5. Let $0 < \lambda \leq 1$ and $0 < \alpha < 1 + \frac{\lambda}{2}$. If $f \in \mathcal{L}^{2,\lambda,\alpha}$, then for $z \in \mathbb{D}$,

$$|f(z)| \lesssim \begin{cases} \frac{\|f\|_{\mathcal{L}^{2,\lambda,\alpha}}}{(1-|z|^2)^{\frac{1-\lambda}{2}+\alpha-1}}, & \alpha > \frac{1+\lambda}{2}, \\ \log \frac{2}{1-|z|^2} \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}, & \alpha = \frac{1+\lambda}{2}, \\ \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}, & \alpha < \frac{1+\lambda}{2}. \end{cases}$$

Proof. For $z \in \mathbb{D}$ and $r > 0$, set $\mathbb{D}(z, r) = \{w \in \mathbb{D} : \beta(z, w) < r\}$. Here $\beta(\cdot, \cdot)$ denotes the Bergman metric. From [34] we see that

$$\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} \asymp \frac{1}{(1-|w|^2)^2} \asymp \frac{1}{(1-|z|^2)^2},$$

when $w \in \mathbb{D}(z, r)$. Hence,

$$\begin{aligned} |f'(z)|^2 &\lesssim \frac{1}{(1-|z|^2)^{2\alpha}} \int_{\mathbb{D}(z,r)} |f'(w)|^2 (1-|w|^2)^{2\alpha-2} dA(w) \\ &\lesssim \frac{1}{(1-|z|^2)^{2\alpha}} \int_{\mathbb{D}(z,r)} |f'(w)|^2 (1-|w|^2)^{2\alpha-2} (1-|\sigma_z(w)|^2)^{3-2\alpha} dA(w) \\ &\lesssim \frac{1}{(1-|z|^2)^{2\alpha+1-\lambda}} \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2. \end{aligned}$$

Therefore,

$$|f'(z)| \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda,\alpha}}}{(1-|z|^2)^{\frac{1-\lambda}{2}+\alpha}}.$$

If $\alpha > \frac{1+\lambda}{2}$, then for $b \in \mathbb{D}$ and using the last inequality,

$$\begin{aligned} |f(b) - f(0)| &= \left| b \int_0^1 f'(bz) dz \right| \lesssim |b| \int_0^1 |f'(bz)| dz \\ &\lesssim \|f\|_{\mathcal{L}^{2,\lambda,\alpha}} \int_0^1 \frac{|b| dz}{(1-|bz|^2)^{\frac{1-\lambda}{2}+\alpha}} \\ &\lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda,\alpha}}}{(1-|b|^2)^{\frac{1-\lambda}{2}+\alpha-1}}. \end{aligned}$$

Because the point b is arbitrary, we obtain

$$|f(b)| \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda,\alpha}}}{(1-|b|^2)^{\frac{1-\lambda}{2}+\alpha-1}}, \quad b \in \mathbb{D}.$$

Other cases can be proven similarly. The proof is complete. $\square$

The following integral estimate is of great importance in our proof.

LEMMA 2. [34, Lemma 3.10] Suppose $z \in \mathbb{D}$, c is real, $t > -1$, and

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - z\bar{w}|^{2+t+c}} dA(w).$$

(a) If $c < 0$, then as a function of z , $I_{c,t}(z)$ is bounded from above and bounded from below on $\mathbb{D}$.

(b) If $c > 0$, then

$$I_{c,t}(z) \asymp \frac{1}{(1 - |z|^2)^c}, \quad |z| \rightarrow 1^-.$$

(c) If $c = 0$, then

$$I_{0,t}(z) \asymp \log \frac{1}{1 - |z|^2}, \quad |z| \rightarrow 1^-.$$

The following two propositions are devoted to illustrating the relationship between polynomials and $\mathcal{L}_0^{2,\lambda,\alpha}$. We show that $\mathcal{L}_0^{2,\lambda,\alpha}$ is the closure of all polynomials in $\mathcal{L}^{2,\lambda,\alpha}$.

PROPOSITION 6. Let $0 < \lambda \leq 1$ and $0 < \alpha < 1 + \frac{\lambda}{2}$. Then $\mathcal{L}_0^{2,\lambda,\alpha}$ (and so $\mathcal{L}^{2,\lambda,\alpha}$) contains all polynomials.

Proof. If f is a polynomial, then

$$\sup_{w \in \mathbb{D}} |f'(w)|^2 \leq M < \infty.$$

By Lemma 2, we get

$$\begin{aligned} & \lim_{|b| \rightarrow 1} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^{2\alpha-2} (1 - |\sigma_b(w)|^2)^{3-2\alpha} dA(w) \\ & \leq M \lim_{|b| \rightarrow 1} (1 - |b|^2)^{4-\lambda-2\alpha} \int_{\mathbb{D}} \frac{(1 - |w|^2)}{|1 - \bar{b}w|^{2(3-2\alpha)}} dA(w) = 0. \end{aligned}$$

Thus $f \in \mathcal{L}_0^{2,\lambda,\alpha}$ by Proposition 2. $\square$

Similarly to the proof of Proposition 2.15 in [33], we get the following proposition.

PROPOSITION 7. Let $g \in \mathcal{L}_0^{2,\lambda,\alpha}$, $0 < \lambda \leq 1$ and $0 < \alpha < 1 + \frac{\lambda}{2}$. Then

$$\lim_{r \rightarrow 1} \|g_r - g\|_{\mathcal{L}^{2,\lambda,\alpha}} = 0,$$

where $g_r(z) = g(rz)$. In particular, $\mathcal{L}_0^{2,\lambda,\alpha}$ is the closure of all polynomials in $\mathcal{L}^{2,\lambda,\alpha}$.

3. Integral operators T_g and S_g

In this section, we precisely determine the boundedness, compactness, norm and essential norm properties of the operators T_g and S_g acting on the analytic α -Morrey space $\mathcal{L}^{2,\lambda,\alpha}$.

To prove our main results in this section, we need the following lemmas.

LEMMA 3. [8] *Suppose that μ is a non-negative measure on $\mathbb{D}$. Then μ is a Carleson measure if and only if the following inequality*

$$\int_{\mathbb{D}} |f(z)|^2 d\mu \lesssim \|f\|_{H^2}^2$$

holds for all $f \in H^2$. Moreover,

$$\sup_{\|f\|_{H^2}=1} \int_{\mathbb{D}} |f(z)|^2 d\mu \asymp \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|}.$$

LEMMA 4. [8] *Suppose that $f \in H(\mathbb{D})$. Then $f \in BMOA$ if and only if the measure $d\mu_f = |f'(z)|^2(1-|z|^2)dA(z)$ is a Carleson measure. Moreover,*

$$\|f\|_{BMOA}^2 \asymp |f(0)|^2 + \sup_{I \subset \partial\mathbb{D}} \frac{\mu_f(S(I))}{|I|}.$$

From the proof of Lemma 1 in [14], we have the following lemma.

LEMMA 5. *Let $f \in H(\mathbb{D})$ and $b, z \in \mathbb{D}$. Then*

$$|f \circ \sigma_b(z) - f \circ \sigma_b(0)|^2 \lesssim (1-|z|^2)^{-\frac{1}{2}} \int_{\mathbb{D}} |(f \circ \sigma_b)'(u)|^2 \frac{(1-|u|^2)^{\frac{7}{2}}}{|1-\bar{u}z|^3} dA(u).$$

Now we are in a position to state and prove the main result in this section.

THEOREM 1. *Let $g \in H(\mathbb{D})$, $0 < \lambda \leq 1 \leq \alpha < \infty$ such that $\frac{1+\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. Then $T_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is bounded if and only if $g \in BMOA$. Moreover,*

$$\|T_g\| \asymp \|g\|_{BMOA}.$$

Proof. Suppose that $g \in BMOA$. By Lemma 4, μ_g is a Carleson measure where $d\mu_g = |g'(z)|^2(1-|z|^2)dA(z)$. For any $I \subset \partial\mathbb{D}$, let ξ be the center of I and $b = (1-|I|)\xi$. For any $f \in \mathcal{L}^{2,\lambda,\alpha}$,

$$\begin{aligned} & \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f(z)|^2 d\mu_g(z) \\ & \lesssim \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f(b)|^2 d\mu_g(z) + \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f(z) - f(b)|^2 d\mu_g(z) \\ & = G_1 + G_2. \end{aligned}$$

By Proposition 5, Lemma 4 and the assumption that $\frac{1+\lambda}{2} < \alpha$, we have

$$\begin{aligned}
 G_1 &= \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f(b)|^2 d\mu_g(z) \\
 &\lesssim \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} \left(\frac{\|f\|_{\mathcal{L}^{2,\lambda,\alpha}}}{(1-|b|^2)^{\frac{1-\lambda}{2}+\alpha-1}} \right)^2 d\mu_g(z) \\
 &\lesssim \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 \frac{\mu_g(S(I))}{|I|} \\
 &\lesssim \sup_{I \subseteq \partial \mathbb{D}} \frac{\mu_g(S(I))}{|I|} \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 \\
 &\lesssim \|g\|_{BMOA}^2 \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2.
 \end{aligned}$$

Since μ_g is a Carleson measure, by Lemma 3 we see that $I_d : H^2 \rightarrow L^2(\mu_g)$ is bounded. Using Lemmas 3 and 4, and the facts that

$$\left| \frac{d}{dz} \left(\frac{f(z) - f(w)}{1 - \bar{w}z} \right) \right| \leq \frac{|f'(z)|}{|1 - \bar{w}z|} + \frac{|f(z) - f(w)|}{|1 - \bar{w}z|^2}$$

and

$$\|f\|_{H^2}^2 \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) dA(z),$$

we have

$$\begin{aligned}
 G_2 &= \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f(z) - f(b)|^2 d\mu_g(z) \\
 &\lesssim (1 - |b|^2)^{2\alpha-\lambda} \int_{S(I)} \left| \frac{f(z) - f(b)}{1 - \bar{b}z} \right|^2 d\mu_g(z) \\
 &\lesssim (1 - |b|^2)^{2\alpha-\lambda} \int_{\mathbb{D}} \left| \frac{f(z) - f(b)}{1 - \bar{b}z} \right|^2 d\mu_g(z) \\
 &\lesssim (1 - |b|^2)^{2\alpha-\lambda} \int_{\mathbb{D}} \left| \frac{d}{dz} \left(\frac{f(z) - f(b)}{1 - \bar{b}z} \right) \right|^2 (1 - |z|^2) dA(z) \sup_{I \subseteq \partial \mathbb{D}} \frac{\mu_g(S(I))}{|I|} \\
 &\lesssim \|g\|_{BMOA}^2 (1 - |b|^2)^{2\alpha-\lambda} \int_{\mathbb{D}} \frac{|f'(z)|^2 (1 - |z|^2)}{|1 - \bar{b}z|^2} dA(z) \\
 &\quad + \|g\|_{BMOA}^2 (1 - |b|^2)^{2\alpha-\lambda} \int_{\mathbb{D}} \frac{|f(z) - f(b)|^2 (1 - |z|^2)}{|1 - \bar{b}z|^4} dA(z) \\
 &\lesssim \|g\|_{BMOA}^2 \Omega_1 + \|g\|_{BMOA}^2 \Omega_2.
 \end{aligned}$$

Here

$$\Omega_1 := (1 - |b|^2)^{2\alpha-\lambda} \int_{\mathbb{D}} \frac{|f'(z)|^2 (1 - |z|^2)}{|1 - \bar{b}z|^2} dA(z)$$

and

$$\Omega_2 := (1 - |b|^2)^{2\alpha-\lambda} \int_{\mathbb{D}} \frac{|f(z) - f(b)|^2 (1 - |z|^2)}{|1 - \bar{b}z|^4} dA(z).$$

Using the assumption that $1 \leq \alpha$, we have $\left(\frac{1-|b|^2}{|1-\bar{b}z|}\right)^{4-4\alpha} \geq 1$, which implies that

$$\begin{aligned}\Omega_1 &\leq (1-|b|^2)^{2\alpha-\lambda} \int_{\mathbb{D}} \frac{|f'(z)|^2(1-|z|^2)}{|1-\bar{b}z|^2} \left(\frac{1-|b|^2}{|1-\bar{b}z|}\right)^{4-4\alpha} dA(z) \\ &\leq (1-|b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2\alpha-2} (1-|\sigma_b(z)|^2)^{3-2\alpha} dA(z) \\ &\lesssim \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2.\end{aligned}$$

In addition, making the change of variables twice, by Lemma 5, Lemma 1 and the assumption that $\alpha \geq 1$ we obtain

$$\begin{aligned}\Omega_2 &= (1-|b|^2)^{2\alpha-\lambda-1} \int_{\mathbb{D}} \frac{|f(z)-f(b)|^2}{|1-\bar{b}z|^2} (1-|\sigma_b(z)|^2) dA(z) \\ &= (1-|b|^2)^{2\alpha-\lambda-1} \int_{\mathbb{D}} |f \circ \sigma_b(z) - f \circ \sigma_b(0)|^2 \frac{1-|z|^2}{|1-\bar{b}z|^2} dA(z) \\ &\lesssim (1-|b|^2)^{2\alpha-1-\lambda} \int_{\mathbb{D}} \int_{\mathbb{D}} |(f \circ \sigma_b)'(u)|^2 \frac{(1-|u|^2)^{\frac{1}{2}}}{|1-\bar{u}z|^3} dA(u) \frac{(1-|z|^2)^{\frac{1}{2}}}{|1-\bar{b}z|^2} dA(z) \\ &\lesssim (1-|b|^2)^{2\alpha-1-\lambda} \int_{\mathbb{D}} |(f \circ \sigma_b)'(u)|^2 (1-|u|^2)^{\frac{1}{2}} \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{\frac{1}{2}}}{|1-\bar{b}z|^2 |1-\bar{u}z|^3} dA(z) \right) dA(u) \\ &\lesssim (1-|b|^2)^{1-\lambda} \int_{\mathbb{D}} |(f \circ \sigma_b)'(u)|^2 \frac{(1-|u|^2)^3 (1-|b|^2)^{2\alpha-2}}{|1-\bar{b}u|^2} dA(u) \\ &\lesssim (1-|b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(\sigma_b(u))|^2 (1-|\sigma_b(u)|^2)^{2\alpha} (1-|u|^2)^{3-2\alpha} \frac{(1-|u|^2)^2}{|1-\bar{b}u|^2} \\ &\quad \times |1-\bar{b}u|^{4\alpha-4} \frac{dA(u)}{(1-|u|^2)^2} \\ &\lesssim (1-|b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(\sigma_b(u))|^2 (1-|\sigma_b(u)|^2)^{2\alpha} (1-|u|^2)^{3-2\alpha} \frac{dA(u)}{(1-|u|^2)^2} \\ &= (1-|b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2\alpha-2} (1-|\sigma_b(z)|^2)^{3-2\alpha} dA(z) \\ &\lesssim \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2.\end{aligned}$$

Therefore, for any $f \in \mathcal{L}^{2,\lambda,\alpha}$,

$$\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f(w)|^2 d\mu_g(w) \lesssim \sup_{I \subset \partial \mathbb{D}} (G_1 + G_2) \lesssim \|g\|_{BMOA}^2 \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2.$$

Let $f \in \mathcal{L}^{2,\lambda,\alpha}$. We get

$$\begin{aligned} \|T_g f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 &\asymp \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |(T_g f)'(w)|^2 (1-|w|^2) dA(w) \\ &\asymp \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f(w)|^2 d\mu_g(w) \\ &\lesssim \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 \|g\|_{BMOA}^2 < \infty. \end{aligned}$$

Therefore $T_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is bounded and $\|T_g\| \lesssim \|g\|_{BMOA}$.

Conversely, suppose that $T_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is bounded. For any $I \subseteq \partial \mathbb{D}$, let $a = (1 - |I|)\zeta \in \mathbb{D}$, where ζ is the center of I . Set

$$f_a(w) = \frac{(1 - |a|^2)}{(1 - \bar{a}w)^{\frac{1-\lambda}{2} + \alpha}}, \quad w \in \mathbb{D}.$$

By Proposition 4, $f_a \in \mathcal{L}^{2,\lambda,\alpha}$ and $\|f_a\|_{\mathcal{L}^{2,\lambda,\alpha}} \lesssim 1$. Thus,

$$\|T_g f_a\|_{\mathcal{L}^{2,\lambda,\alpha}} \lesssim \|T_g\| \|f_a\|_{\mathcal{L}^{2,\lambda,\alpha}} \lesssim \|T_g\|.$$

By the fact that $|I| = 1 - |a| \asymp |1 - \bar{a}w|$, $w \in S(I)$, we have

$$\begin{aligned} \frac{1}{|I|} \int_{S(I)} |g'(w)|^2 (1 - |w|^2) dA(w) &\lesssim \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |f_a(w)|^2 |g'(w)|^2 (1 - |w|^2) dA(w) \\ &= \frac{1}{|I|^{2-2\alpha+\lambda}} \int_{S(I)} |(T_g f_a)'(w)|^2 (1 - |w|^2) dA(w) \\ &\lesssim \|T_g f_a\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 \lesssim \|T_g\|^2 < \infty. \end{aligned}$$

Since I is arbitrary, by Lemma 4 we see that $g \in BMOA$, moreover $\|g\|_{BMOA} \lesssim \|T_g\|$. The proof is complete. $\square$

THEOREM 2. Let $g \in H(\mathbb{D})$, $0 < \lambda \leq 1$ and $\frac{\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. Then $S_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is bounded if and only if $g \in H^\infty$. Moreover,

$$\|S_g\| \asymp \|g\|_\infty.$$

Proof. Assume that $S_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is bounded. For any $a \in \mathbb{D}$, take f_a in Proposition 4. Let $r > 0$. From [34] we see that

$$\frac{(1 - |a|^2)^2}{|1 - \bar{a}w|^4} \asymp \frac{1}{(1 - |w|^2)^2} \asymp \frac{1}{(1 - |a|^2)^2},$$

when $w \in \mathbb{D}(a, r)$. By Proposition 4.13 in [34] and the fact that $S_g f_a(0) = 0$ we have,

$$\begin{aligned}
 \|S_g\|^2 &\gtrsim \|S_g f_a\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 \\
 &\asymp \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'_a(w)|^2 |g(w)|^2 (1 - |w|^2) |\sigma'_b(w)|^{3-2\alpha} dA(w) \\
 &\gtrsim \int_{\mathbb{D}} \frac{(1 - |a|^2)^{3-\lambda}}{|1 - \bar{a}w|^{2\alpha+3-\lambda}} |g(w)|^2 (1 - |w|^2) |\sigma'_a(w)|^{3-2\alpha} dA(w) \\
 &\gtrsim \int_{\mathbb{D}(a,r)} |g(w)|^2 (1 - |w|^2)^{-2} dA(w) \\
 &\gtrsim \frac{1}{(1 - |a|^2)^2} \int_{\mathbb{D}(a,r)} |g(w)|^2 dA(w) \\
 &\gtrsim |g(a)|^2,
 \end{aligned}$$

which implies that

$$\|S_g\| \gtrsim \|g\|_{\infty}.$$

Conversely, let $g \in H^{\infty}$. For any $f \in \mathcal{L}^{2,\lambda,\alpha}$,

$$\begin{aligned}
 \|S_g f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 &\asymp \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |(S_g f)'(w)|^2 (1 - |w|^2) |\sigma'_b(w)|^{3-2\alpha} dA(w) \\
 &= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(w)|^2 |g(w)|^2 (1 - |w|^2) |\sigma'_b(w)|^{3-2\alpha} dA(w) \\
 &\lesssim \|f\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 \|g\|_{\infty}^2,
 \end{aligned}$$

which implies that $S_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is bounded and

$$\|S_g\| \lesssim \|g\|_{\infty}.$$

The proof is complete. $\square$

Finally, we conduct a comprehensive study of the essential norm of T_g as well as S_g on the space $\mathcal{L}^{2,\lambda,\alpha}$. Let X be a Banach space and T be a bounded linear operator on X . Recall that the essential norm of T , denoted as $\|T\|_e$, is defined as follows:

$$\|T\|_e = \inf\{\|T - K\| : K \text{ are compact operators on } X\}.$$

LEMMA 6. [12, Lemma 3] *Let $g \in BMOA$. Then*

$$\limsup_{r \rightarrow 1^-} \|g - g_r\|_{BMOA} \asymp \limsup_{|b| \rightarrow 1} \|g \circ \sigma_b - g(b)\|_{H^2}.$$

Here $g_r(w) = g(rw)$, $0 < r < 1$, $w \in \mathbb{D}$.

Similar to the proof of Lemma 4.3 in [13], we have the following result.

LEMMA 7. *Let $g \in BMOA$, $0 < \lambda \leq 1 \leq \alpha < \infty$ such that $\frac{1+\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. If $0 < r < 1$, then $T_{g_r} : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is compact.*

THEOREM 3. *Let $g \in BMOA$, $0 < \lambda \leq 1 \leq \alpha < \infty$ such that $\frac{1+\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. Then $T_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ satisfies*

$$\|T_g\|_{e,\mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} \asymp \limsup_{|b| \rightarrow 1} \|g \circ \sigma_b - g(b)\|_{H^2}.$$

Proof. Let $\{I_j\}$ be a sequence of arcs with $|I_j| \rightarrow 0$ as $j \rightarrow \infty$. Set $d_j = (1 - |I_j|)\xi_j$, where ξ_j is the center of I_j . Take

$$f_j(w) = \frac{(1 - |d_j|^2)}{(1 - \overline{d_j}w)^{\frac{1-\lambda}{2} + \alpha}}, \quad w \in \mathbb{D}.$$

It is easy to check that $f_j \in \mathcal{L}^{2,\lambda,\alpha}$ and $\{f_j\}$ converges to zero uniformly on every compact subsets of $\mathbb{D}$ as $j \rightarrow \infty$. Let $K : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ be a compact operator. Using Lemma 2.10 in [27], we have

$$\lim_{j \rightarrow \infty} \|Kf_j\|_{\mathcal{L}^{2,\lambda,\alpha}} = 0.$$

Therefore,

$$\begin{aligned} \|T_g - K\| &\gtrsim \limsup_{j \rightarrow \infty} \|(T_g - K)f_j\|_{\mathcal{L}^{2,\lambda,\alpha}} \\ &\gtrsim \limsup_{j \rightarrow \infty} (\|T_g f_j\|_{\mathcal{L}^{2,\lambda,\alpha}} - \|Kf_j\|_{\mathcal{L}^{2,\lambda,\alpha}}) = \limsup_{j \rightarrow \infty} \|T_g f_j\|_{\mathcal{L}^{2,\lambda,\alpha}} \\ &\gtrsim \limsup_{j \rightarrow \infty} \left(\int_{\mathbb{D}} \frac{(1 - |d_j|^2)^2}{|1 - \overline{d_j}w|^{2\alpha+1-\lambda}} |g'(w)|^2 (1 - |w|^2)^{2\alpha-\lambda-1} (1 - |\sigma_{d_j}(w)|^2)^{2-2\alpha+\lambda} dA(w) \right)^{\frac{1}{2}} \\ &\gtrsim \limsup_{j \rightarrow \infty} \left(\frac{1}{|I_j|} \int_{S(I_j)} |g'(w)|^2 (1 - |w|^2) dA(w) \right)^{\frac{1}{2}}, \end{aligned}$$

which implies that

$$\|T_g\|_{e,\mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} \gtrsim \limsup_{|I| \rightarrow 0} \left(\frac{1}{|I|} \int_{S(I)} |g'(w)|^2 (1 - |w|^2) dA(w) \right)^{\frac{1}{2}}.$$

From the proof of [25, Lemma 3.4], we see that, for $g \in BMOA$,

$$\limsup_{|I| \rightarrow 0} \left(\frac{1}{|I|} \int_{S(I)} |g'(w)|^2 (1 - |w|^2) dA(w) \right)^{\frac{1}{2}} \asymp \limsup_{|b| \rightarrow 1} \|g \circ \sigma_b - g(b)\|_{H^2},$$

which implies that

$$\|T_g\|_{e,\mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} \gtrsim \limsup_{|b| \rightarrow 1} \|g \circ \sigma_b - g(b)\|_{H^2}.$$

On the other hand, for $0 < r < 1$, by Lemma 7 we see that $T_{g_r} : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is compact. By Theorem 1,

$$\|T_g\|_{e,\mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} \leq \|T_g - T_{g_r}\| = \|T_{g-g_r}\| \asymp \|g - g_r\|_{BMOA}.$$

Using Lemma 6, we get

$$\|T_g\|_{e,\mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} \lesssim \limsup_{r \rightarrow 1^-} \|g - g_r\|_{BMOA} \asymp \limsup_{|b| \rightarrow 1} \|g \circ \sigma_b - g(b)\|_{H^2}.$$

The proof is complete. $\square$

It is clear that $T : X \rightarrow Y$ is compact if and only if $\|T\|_{e,X \rightarrow Y} = 0$. The following result can be directly obtained by Theorem 3.

COROLLARY 1. *Let $g \in BMOA$, $0 < \lambda \leq 1 \leq \alpha < \infty$ such that $\frac{1+\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. Then $T_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is compact if and only if $g \in VMOA$.*

THEOREM 4. *Let $g \in H(\mathbb{D})$, $0 < \lambda \leq 1$ and $\frac{\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. If $S_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is bounded, then*

$$\|S_g\|_{e,\mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} \asymp \|g\|_{\infty}.$$

Proof. By Theorem 2,

$$\|S_g\|_{e,\mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} = \inf_K \|S_g - K\| \leq \|S_g\| \lesssim \|g\|_{\infty}.$$

Next we prove that

$$\|S_g\|_{e,\mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} \gtrsim \|g\|_{\infty}.$$

Let $\{d_j\} \subset \mathbb{D}$ such that $|d_j| \rightarrow 1$ as $j \rightarrow \infty$. Set

$$f_j(w) = \frac{(1 - |d_j|^2)}{(1 - \overline{d_j}w)^{\frac{1-\lambda}{2} + \alpha}} \quad w \in \mathbb{D}.$$

From the proof of Theorem 3 we see that $f_j \in \mathcal{L}^{2,\lambda,\alpha}$ and $\{f_j\}$ converges to zero uniformly on every compact subsets of $\mathbb{D}$. By the proof of Theorem 2,

$$\begin{aligned} \|S_g f_j\|_{\mathcal{L}^{2,\lambda,\alpha}}^2 &\asymp \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f_j'(w)|^2 |g(w)|^2 (1 - |w|^2) (1 - |\sigma_b(w)|^2)^{3-2\alpha} dA(w) \\ &\gtrsim |g(d_j)|^2. \end{aligned}$$

Let $K : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ be a compact operator. By Lemma 2.10 in [27], we have

$$\lim_{j \rightarrow \infty} \|K f_j\|_{\mathcal{L}^{2,\lambda,\alpha}} = 0.$$

Hence,

$$\begin{aligned}\|S_g - K\| &\gtrsim \limsup_{j \rightarrow \infty} \|(S_g - K)f_j\|_{\mathcal{L}^{2,\lambda,\alpha}} \\ &\geq \limsup_{j \rightarrow \infty} (\|S_g f_j\|_{\mathcal{L}^{2,\lambda,\alpha}} - \|K f_j\|_{\mathcal{L}^{2,\lambda,\alpha}}) \\ &\geq \limsup_{j \rightarrow \infty} \|S_g f_j\|_{\mathcal{L}^{2,\lambda,\alpha}}.\end{aligned}$$

Therefore,

$$\|S_g\|_{e, \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}} \gtrsim \limsup_{j \rightarrow \infty} |g(d_j)|.$$

Since $\{d_j\}$ is arbitrary, we get the desired result. The proof is complete. $\square$

The following result can be directly derived by Theorem 4.

COROLLARY 2. *Let $g \in H(\mathbb{D})$, $0 < \lambda \leq 1$ and $\frac{\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. Then $S_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is compact if and only if $g = 0$.*

REMARK 3. The operators T_g and S_g exhibit a close connection with the multiplication operator M_g . Specifically, we have the relation

$$T_g f + S_g f = M_g f - f(0)g(0),$$

where the multiplication operator M_g is defined by $M_g f(z) = f(z)g(z)$ for all $f \in H(\mathbb{D})$. From Corollaries 1 and 2, it can be deduced that the operator $M_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is bounded if and only if $g \in H^\infty$ when $0 < \lambda \leq 1 \leq \alpha < \infty$ and $\frac{1+\lambda}{2} < \alpha < 1 + \frac{\lambda}{2}$. Moreover, the operator $M_g : \mathcal{L}^{2,\lambda,\alpha} \rightarrow \mathcal{L}^{2,\lambda,\alpha}$ is compact if and only if the function g is identically zero.

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