

GRAPHS WITH DISTINCT RECIPROCAL DISTANCE LAPLACIAN EIGENVALUES

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This article is dedicated to the cherished memory of Prof. R. B. Bapat.

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Abstract. In this article, we consider the following problem: *Characterize connected graphs G such that G has $k(\in \mathbb{N})$ distinct reciprocal distance Laplacian eigenvalues.* For $k = 2$, we provide the complete answer. For $k = 3$, graphs with three distinct eigenvalues are identified from a class of connected graphs. For $k = 4$, we prove that $K_{n,n} - e$ has four distinct reciprocal distance Laplacian eigenvalues for $n \geq 2$, where $K_{n,n}$ is complete bipartite and e is an edge in $K_{n,n}$. We also establish a relation between the reciprocal distance Laplacian spectrum and the Laplacian spectrum of a connected graph.

1. Introduction

We consider simple, connected graphs G on the vertex set $V(G) = \{1, 2, \dots, n\}$ and edge set $E(G) \subseteq \binom{V(G)}{2}$. We use $[i, j]$ to denote the edge between the vertices i and j . The complement of a graph G is denoted by $\bar{G}$. The star graph with n vertices, the complete graph with n vertices, and regular complete bipartite graph with $n = 2r$ vertices are denoted by S_n , K_n , and $K_{r,r}$, respectively. The *adjacency matrix* of a graph G with n vertices $A(G) = [(a_{ij})]_{n \times n}$ is defined as $a_{ij} = 1$ if and only if $[i, j] \in E(G)$ and 0 otherwise. The *Laplacian matrix* $L(G)$ of G with n vertices is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees in G . Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$ be the Laplacian spectrum of G . It is known that the Laplacian eigenvalues of $\bar{G}$ are $n - \mu_{n-1} \geq n - \mu_{n-2} \geq \dots \geq n - \mu_1 \geq 0$. Let λ be an eigenvalue of a matrix M . The algebraic multiplicity (or simply, the multiplicity) of λ is denoted by

$m(\lambda)$. The notation $\left(\overbrace{\lambda_1, \dots, \lambda_1}^{(m_1)}, \overbrace{\lambda_2, \dots, \lambda_2}^{(m_2)}, \dots, \overbrace{\lambda_l, \dots, \lambda_l}^{(m_l)} \right)$ is used to denote the distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_l$ with multiplicities $m_1, m_2, \dots, m_l$, respectively. The *distance* d_{ij} between two vertices i and j in a graph G is the number of edges in a shortest path

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connecting i and j , while the *transmission* of a vertex i is the sum of the distances from i to all other vertices of G :

$$Tr(i) = \sum_{\substack{j \in V(G) \\ j \neq i}} d_{ij}.$$

The *distance matrix* $\mathcal{D}(G) = [\mathcal{D}_{ij}]_{n \times n}$ of G with n vertices is the square symmetric matrix of size n and the (i, j) -th entry of $\mathcal{D}(G)$ is denoted by $\mathcal{D}_{ij} = d_{ij}$ if $i \neq j$ and 0 otherwise; see [2]. The *distance Laplacian matrix* of a connected graph G is defined as $\mathcal{D}^L = \text{Diag}(Tr) - \mathcal{D}$, where $\text{Diag}(Tr)$ is the diagonal matrix whose diagonal entries are the transmissions of the vertices of G . This matrix was defined in [1].

The *reciprocal distance matrix* $\mathcal{R}_{\mathcal{D}} = [(\mathcal{R}_{\mathcal{D}})_{ij}]$ of a connected graph G with n vertices is a square, symmetric matrix of order n and the (i, j) -th entry of $\mathcal{R}_{\mathcal{D}}$ is defined by $(\mathcal{R}_{\mathcal{D}})_{ij} = \frac{1}{d_{ij}}$ if $i \neq j$ and 0 otherwise. The reciprocal distance matrix has been defined in [14]. The *reciprocal transmission* $RTr(i)$ of a vertex i is defined to be the sum of the reciprocal of the distances from i to all other vertices in G , that is,

$$RTr(i) = \sum_{\substack{j \in V(G) \\ j \neq i}} \frac{1}{d_{ij}}.$$

The *reciprocal distance Laplacian matrix* of a connected graph G is defined by $\mathcal{R}_{\mathcal{D}}^L = \text{Diag}(RTr) - \mathcal{R}_{\mathcal{D}}$, where $\text{Diag}(RTr)$ is the diagonal matrix whose diagonal entries are the reciprocal transmission of the vertices of G . This reciprocal distance Laplacian matrix was defined by Bapat and Panda in [4]. Let $\delta_1^L(G) \geq \delta_2^L(G) \geq \dots \geq \delta_{n-1}^L(G) \geq \delta_n^L(G)$ denote the spectrum of $\mathcal{R}_{\mathcal{D}}^L$. We call this sequence the reciprocal distance Laplacian spectrum of G . Sometimes we use $\mathcal{R}_{\mathcal{D}}^L$ -eigenvalues to denote the eigenvalues of the reciprocal distance Laplacian matrix. Bapat and Panda [4] proved that $\delta_1^L(G) \leq n$ with equality if and only if the complement of G is disconnected. Medina and Trigo [16] obtained some upper and lower bounds for $\delta_1^L(G)$ in terms of some graph invariants. Medina and Trigo [15] obtained some bounds for the reciprocal distance Laplacian energy in terms of some graph invariants.

Characterizing graphs with few distinct eigenvalues is a classical problem in spectral graph theory. This problem has been studied for several matrices associated with graphs, including the adjacency matrix [6, 10, 19, 20, 21], the Laplacian matrix [17, 18], the signless Laplacian matrix [3], the universal adjacency matrix [11], and the normalized Laplacian matrix [5, 7, 13, 23]. One of the main reasons for studying such graphs is that a small number of distinct eigenvalues often imposes strong structural restrictions on the graph [6]. Another motivation comes from the spectral determination problem: many graphs with few distinct eigenvalues are not determined by their spectrum, and this connects the topic with the broader question of which graphs are determined by their spectrum [22].

For distance-based matrices, the problem is different from the adjacency or Laplacian cases because the entries depend not only on adjacency, but also on the metric structure of the graph. The distance matrix and distance Laplacian matrix have been

studied in [1, 2]. The reciprocal distance Laplacian matrix was introduced in [4], and related bounds for its spectral radius and energy were obtained in [15, 16]. However, a characterization of connected graphs with few distinct reciprocal distance Laplacian eigenvalues has not been developed in the same way. Therefore, it is natural to search for such graph families.

Motivated by these works, in this article we consider the following problem: *Characterize connected graphs G such that G has $k \in \mathbb{N}$ distinct reciprocal distance Laplacian eigenvalues.* The case $k = 1$ is impossible for every connected graph with at least two vertices, since 0 is always an eigenvalue of $\mathcal{R}_D^L(G)$ and $\mathcal{R}_D^L(G) \neq 0$. For $k = 2$, we prove that the complete graph is the only connected graph with two distinct reciprocal distance Laplacian eigenvalues; see Lemma 2.1. For $k = 3$, we characterize the graphs with three distinct reciprocal distance Laplacian eigenvalues inside a natural class of connected graphs. For $k = 4$, we prove that $K_{n,n} - e$ has four distinct reciprocal distance Laplacian eigenvalues, where $K_{n,n} - e$ is the graph obtained from $K_{n,n}$ by deleting one edge.

We now explain the class used in the characterization for the case $k = 3$. In the reciprocal distance Laplacian setting, the largest eigenvalue $\delta_1^L(G)$ plays an important role. Its multiplicity affects the possible form of the spectrum when the graph has only a few distinct reciprocal distance Laplacian eigenvalues. If a connected graph has exactly three distinct reciprocal distance Laplacian eigenvalues, then its spectrum has the form

$$\delta_1^L(G), \quad \theta, \quad 0,$$

where one of the two nonzero eigenvalues must have large multiplicity. Thus, the case in which the largest reciprocal distance Laplacian eigenvalue is simple is a natural first case to study. This leads to the following class of graphs.

DEFINITION 1.1. Let Γ be the class of connected graphs G with $n \geq 3$ vertices such that the multiplicity of $\delta_1^L(G)$ is 1.

EXAMPLE 1.2. In this example, we show that the set Γ is non-empty. Consider the star graph G with four vertices. Then the $\mathcal{R}_D^L$ -eigenvalues of G are 4, 2.5, 2.5, 0. So the multiplicity of δ_1^L is 1. Hence $G \in \Gamma$. Consider the path P_4 of length 3. The $\mathcal{R}_D^L$ -eigenvalues of P_4 are $\frac{11}{3}, 3, 2, 0$. Hence $P_4 \in \Gamma$.

THEOREM 1.3. Let $G \in \Gamma$. Then G has exactly three distinct reciprocal distance Laplacian eigenvalues if and only if either $G = S_n$ or $G = K_{r,r}$, if $n = 2r$.

THEOREM 1.4. Let $K_{n,n}$ be the complete bipartite graph with $2n$ ($n \geq 2$) vertices and $e \in E(K_{n,n})$. Then the reciprocal distance Laplacian matrix of $K_{n,n} - e$ has four distinct eigenvalues.

In [4], the authors established a relation between the $\mathcal{R}_D^L$ -eigenvalues and the L -eigenvalues for graphs with diameter 2. In this article, we pose the following question: *Is there any relation between the $\mathcal{R}_D^L$ -eigenvalues and the L -eigenvalues for any connected graph?* In this article we establish such relation.

THEOREM 1.5. *Let G be a connected graph with n vertices. Then $\delta_i^L \geq \mu_i$, $i = 1, \dots, n$. Equality holds if and only if $G = K_n$.*

The article is organized as follows: In Section 2, we prove Theorem 1.3. In Section 3, we prove Theorem 1.4. In Section 4, we prove Theorem 1.5.

2. Proof of Theorem 1.3

Let us denote $e_i \in \mathbb{R}^m$ be the vector with the i -th position 1 and else 0.

LEMMA 2.1. *Let G be a connected graph with $n \geq 2$ vertices. Then $m(\delta_{n-1}^L) \leq n - 1$ with equality if and only if $G = K_n$.*

Proof. Since 0 is an eigenvalue of $\mathcal{R}_D^L(G)$, the multiplicity of δ_{n-1}^L is at most $n - 1$, that is, $m(\delta_{n-1}^L) \leq n - 1$. If $G = K_n$, then $m(\delta_{n-1}^L) = n - 1$. We now consider $m(\delta_{n-1}^L) = n - 1$. Let V_1 and V_2 be the eigenspaces corresponding to the eigenvalues 0 and δ_{n-1}^L , respectively. It is clear that $\mathbb{R}^n = V_1 \oplus V_2$. Any vector of the form $X = e_1 - e_l$, $1 \neq l$ is an eigenvector of $\mathcal{R}_D^L$ corresponding to the eigenvalue δ_{n-1}^L . Let $RTr_{\max} = RTr(1) \geq RTr(2) \geq \dots \geq RTr(n) = RTr_{\min}$. Since $\mathcal{R}_D^L X = \delta_{n-1}^L X$, we have $RTr_{\max} + \frac{1}{d_{1s}} = \delta_1^L$ for all $s = 2, \dots, n$. This implies $d_{1s} = 1$ for $s = 2, \dots, n$. Similarly, we can prove that $d_{ls} = 1$ for all $l, s = 1, \dots, n$ and $l \neq s$. Hence, G is complete. $\square$

LEMMA 2.2. [9] *Let G be a graph with $n \geq 2$ vertices. Let the Laplacian eigenvalues of G be $(n, \mu_2, \dots, \mu_2, \mu_2, 0)$, with $\mu_2 \neq n$. Then $G \cong S_n$ or $G \cong K_{r,r}$, where $n = 2r$.*

To proceed further, we need the following lemma and definition:

LEMMA 2.3. [12] *Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Let B be an $m \times m$ principal submatrix of A . Suppose B has eigenvalues $t_1 \geq t_2 \geq \dots \geq t_m$. Then $\lambda_i \geq t_i \geq \lambda_{n-m+i}$ for $i = 1, 2, \dots, m$.*

DEFINITION 2.4. [8] A graph G is said to be a *cograph* if it does not have an induced subgraph isomorphic to the path P_4 .

LEMMA 2.5. *Let G be a graph on $n (\geq 4)$ vertices that is not a cograph. Then $m(\delta_{n-1}^L) \leq n - 3$.*

Proof. Suppose that $m(\delta_{n-1}^L) > n - 3$. There are two possibilities.

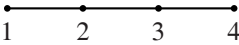
Case I. The multiplicity of δ_{n-1}^L is $n - 1$. Then, using Lemma 2.1, G is a complete graph. A contradiction to the fact that G is not a cograph.

Case II. The multiplicity of δ_{n-1}^L is $n - 2$. Since G is not a cograph, we have an induced subgraph of G which is P_4 . Let $P_4 = [1, 2, 3, 4]$. Let B be the principal submatrix of $\mathcal{R}_D^L$ generated by the columns and rows 1, 2, 3, 4. We use $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ to denote the eigenvalues of B . By Lemma 2.3, we have $\lambda_2 = \lambda_3 = \delta_{n-1}^L$.

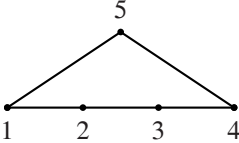
Let $x = [x_1, x_2, x_3, x_4]$ and $y = [y_1, y_2, y_3, y_4]$ be two eigenvectors corresponding to the eigenvalue $\lambda_2 = \lambda_3 = \delta_{n-1}^L$ in B . We now show that $x' = [x_1, x_2, x_3, x_4, 0, \dots, 0]$ and $y' = [y_1, y_2, y_3, y_4, 0, \dots, 0]$ are two eigenvectors corresponding to the eigenvalue δ_{n-1}^L in $\mathcal{R}_D^L$. We notice that $x'Bx = x'^t \mathcal{R}_D^L x'$ and $y'B y = y'^t \mathcal{R}_D^L y'$. Hence x' and y' are two eigenvector corresponding to the eigenvalue δ_{n-1}^L in $\mathcal{R}_D^L$ and both are perpendicular to the vector $\mathbb{1}$ (all ones vector).

CLAIM. There are no eigenvectors $z = [z_1, z_2, z_3, z_4]$ corresponding to the eigenvalue δ_{n-1}^L in B such that $[z_1, z_2, z_3, z_4, 0, \dots, 0] \perp \mathbb{1}$ and $z_3 = 0$.

Proof of the claim. Suppose that there is such type of eigenvector z . Then $z_1 + z_2 + z_4 = 0$. We notice that there are two options for the matrix B which are the following.



$$B_1 = \begin{bmatrix} RTr(1) & -1 & -\frac{1}{2} & -\frac{1}{3} \\ -1 & RTr(2) & -1 & -\frac{1}{2} \\ -\frac{1}{3} & -1 & RTr(3) & -1 \\ -\frac{1}{2} & -\frac{1}{2} & -1 & RTr(4) \end{bmatrix}$$



$$B_2 = \begin{bmatrix} RTr(1) & -1 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & RTr(2) & -1 & -\frac{1}{2} \\ -1 & -1 & RTr(3) & -1 \\ -\frac{1}{2} & -\frac{1}{2} & -1 & RTr(4) \end{bmatrix}$$

We have that $B_1 z = \delta_{n-1}^L z$. Then $-1/2z_1 - z_2 - z_4 = 0$. We observe that $z = [z_1, z_2, z_3, z_4] = [0, -1, 0, 1]$ is a solution of the equations $z_1 + z_2 + z_4 = 0$ and $-1/2z_1 - z_2 - z_4 = 0$. But $B_1 z \neq \delta_{n-1}^L z$ because the first component of the vector $B_1 z$ is $2/3$. A contradiction to the fact that z is an eigenvector of B_1 corresponding to the eigenvalue δ_{n-1}^L . Similarly, we get a contradiction for B_2 . Hence B has no eigenvector corresponding to the eigenvalue δ_{n-1}^L such that $[z_1, z_2, z_3, z_4, 0, \dots, 0] \perp \mathbb{1}$ and $z_3 = 0$. The third components x_3 and y_3 of x' and y' are nonzero. We now take the linear combination $z = cx' + dy'$, where $c = \frac{y_3}{x_3}$ and $d = -1$. Then z is an eigenvector of $\mathcal{R}_D^L$ corresponding to the eigenvalue δ_{n-1}^L and $z \perp \mathbb{1}$. This implies that $[z_1, z_2, 0, z_4]$ is an eigenvector of B corresponding to the eigenvalue δ_{n-1}^L which is a contradiction. Hence $m(\delta_{n-1}^L) \leq n - 3$. Hence the proof follows. $\square$

The *diameter* of a connected graph G is the maximum distance between the pair of vertices in G . We use $d(G)$ or simply d to denote the diameter of G .

LEMMA 2.6. [4] Let G be a graph of diameter 2 on n vertices. Let $\mu_1 \geq \dots \geq \mu_n$ be the Laplacian spectrum of G . Then $\delta_i^L = \frac{1}{2}(n + \mu_i)$ for $i = 1, \dots, n - 1$.

LEMMA 2.7. [4] Let G be a connected graph with n vertices. Then $\bar{G}$ is disconnected if and only if $\delta_1^L = n$.

LEMMA 2.8. [8] *Given a graph G , the following statements are equivalent:*

- (i) G is a cograph.
- (ii) The complement of any connected induced subgraph of G with at least two vertices is disconnected.
- (iii) Every connected induced subgraph of G has a diameter less than or equal to 2.

PROPOSITION 2.9. *Let S_n be the star with n vertices. Then the reciprocal distance Laplacian spectrum of S_n is $\left(0, \overbrace{\frac{n+1}{2}, \frac{n+1}{2}, \dots, \frac{n+1}{2}}^{(n-2)}, n\right)$.*

Proof. The reciprocal distance Laplacian matrix of S_n is

$$\mathcal{R}_{\mathcal{D}}^{\mathcal{L}}(S_n) = \begin{bmatrix} \frac{1}{2}((n+1)I_{n-1} - J_{n-1}) & x \\ x^t & n-1 \end{bmatrix},$$

where I_{n-1} is the identity matrix, J_{n-1} is the matrix with all entries equal to 1 and $x^t = (-1, -1, \dots, -1)_{1 \times (n-1)}$.

Let $\mathcal{P} = \{\xi_{1,i} = -e_1 + e_i : i = 2, 4, \dots, n-1\}$. Then we have $\mathcal{R}_{\mathcal{D}}^{\mathcal{L}}(S_n) \xi_{1,i} = \frac{n+1}{2} \xi_{1,i}$, $\forall \xi_{1,i} \in \mathcal{P}$. Therefore, $\frac{n+1}{2}$ is an eigenvalue of $\mathcal{R}_{\mathcal{D}}^{\mathcal{L}}(S_n)$ with multiplicity at least $n-2$.

Since the trace of $\mathcal{R}_{\mathcal{D}}^{\mathcal{L}}(S_n)$ is $\frac{1}{2}(n^2 - n - 2)$. The other two eigenvalues of $\mathcal{R}_{\mathcal{D}}^{\mathcal{L}}(S_n)$ are 0 and n . Hence, the reciprocal distance Laplacian spectrum of S_n is

$$\left(0, \overbrace{\frac{n+1}{2}, \frac{n+1}{2}, \dots, \frac{n+1}{2}}^{(n-2)}, n\right). \quad \square$$

PROPOSITION 2.10. *Let $K_{n,n}$ be the complete bipartite graph with $2n$ vertices.*

Then the reciprocal distance Laplacian spectrum of $K_{n,n}$ is $\left(0, \overbrace{\frac{3n}{2}, \dots, \frac{3n}{2}}^{2n-2}, 2n\right)$.

Proof. The reciprocal distance Laplacian matrix of $K_{n,n}$ is $\mathcal{R}_{\mathcal{D}}^{\mathcal{L}}(K_{n,n}) = \frac{3n}{2}I_{2n} - \begin{bmatrix} \frac{1}{2}J_n & J_n \\ J_n & \frac{1}{2}J_n \end{bmatrix}$, where I_{2n} is the identity matrix and J_n is the matrix with all entries equal to 1.

Let $\mathcal{S}_1 = \{\xi_{1,i} = -e_1 + e_i : i = 2, 3, 4, \dots, n\}$ and $\mathcal{S}_2 = \{\xi_{n+1,i} = -e_{n+1} + e_i : i = n+2, n+3, \dots, 2n\}$. Then we have the following relationships:

$$\mathcal{R}_{\mathcal{D}}^{\mathcal{L}}(K_{n,n}) \xi_{1,i} = \frac{3n}{2} \xi_{1,i}, \quad \forall i = 2, 3, 4, \dots, n$$

$$\mathcal{R}_{\mathcal{D}}^{\mathcal{L}}(K_{n,n}) \xi_{n+1,i} = \frac{3n}{2} \xi_{n+1,i}, \quad \forall i = n+2, n+3, \dots, 2n.$$

Therefore, $\frac{3n}{2}$ is an eigenvalues of $\mathcal{R}_D^{\mathcal{L}}(K_{n,n})$ with multiplicity at least $2n - 2$. Since the trace of $\mathcal{R}_D^{\mathcal{L}}(K_{n,n})$ is $2n\frac{3n-1}{2}$. The other reciprocal distance Laplacian eigenvalues of $K_{n,n}$ are 0 and $2n$. Hence, the reciprocal distance Laplacian spectrum of $K_{n,n}$ is

$$\left(0, \overbrace{\frac{3n}{2}, \dots, \frac{3n}{2}}^{2n-2}, 2n\right). \quad \square$$

Proof of Theorem 1.3. Suppose that G is either S_n or $K_{r,r}$. Then using Propositions 2.9 and 2.10, G has only three distinct reciprocal distance Laplacian eigenvalues.

Converse part. Suppose that $G \in \Gamma$ has three distinct $\mathcal{R}_D^{\mathcal{L}}$ -eigenvalues. Then using Lemma 2.1, the graph G cannot be a complete graph. Now there are two cases.

Case I. Suppose that $\delta_1^L(G) \neq n$. By Lemma 2.7, the complement $\bar{G}$ is connected. Since $G \in \Gamma$ and G has exactly three distinct reciprocal distance Laplacian eigenvalues, the spectrum has the form

$$\delta_1^L(G), \underbrace{\theta, \theta, \dots, \theta}_{n-2 \text{ times}}, 0,$$

where $\theta = \delta_{n-1}^L(G)$. Hence

$$m(\delta_{n-1}^L(G)) = n - 2.$$

If G is not a cograph, then Lemma 2.5 gives

$$m(\delta_{n-1}^L(G)) \leq n - 3,$$

which is a contradiction. Therefore, G must be a cograph. However, since $\bar{G}$ is connected, Lemma 2.8 implies that the complement of every connected induced subgraph of G with at least two vertices is disconnected. Consequently, this structural restriction is incompatible with the above multiplicity pattern unless $\delta_1^L(G) = n$. Therefore, Case I cannot occur.

Case II. Suppose that $\delta_1^L(G) = n$. The spectrum of $\mathcal{R}_D^{\mathcal{L}}$ is $(n, \delta_{n-1}^L, \delta_{n-1}^L, \dots, \delta_{n-1}^L, \delta_{n-1}^L, 0)$ and $\delta_{n-1}^L \neq n$ as G cannot be complete. Since G is a cograph, Lemma 2.8 implies that every connected induced subgraph of G has a diameter less than or equal to 2. As G is not complete, we conclude that $\text{diam}(G) = 2$. Using Lemma 2.6, the Laplacian spectrum of G is $(n, 2\delta_1^L - n, \dots, 2\delta_1^L - n, 2\delta_1^L - n, 0)$. Using Lemma 2.2, either $G \cong S_n$ or $G \cong K_{r,r}$ if $n = 2r$. $\square$

EXAMPLE 2.11. In this example, we provide a graph $G \notin \Gamma$ having exactly three distinct reciprocal distance Laplacian eigenvalues. Consider $G = K_n - e$, where $n \geq 4$ and $e \in E(K_n)$. We now calculate the spectrum of $G = K_n - e$. Notice that $\bar{G} \cong$

$K_2 \cup (n-2)K_1$. Hence the Laplacian spectrum of $\bar{G}$ is $\left(2, \overbrace{0, \dots, 0}^{n-1}\right)$. Therefore, the

Laplacian spectrum of G is $(\overbrace{n, \dots, n}^{n-2}, n-2, 0)$. Since G has diameter 2, Lemma 2.6 gives the reciprocal distance Laplacian spectrum of G is

$$(\overbrace{n, \dots, n}^{n-2}, n-1, 0).$$

Hence G has exactly three distinct reciprocal distance Laplacian eigenvalues. Moreover,

$$m(\delta_1^L(G)) = n-2 \geq 2.$$

Therefore, the largest reciprocal distance Laplacian eigenvalue is not simple, and so $G \notin \Gamma$.

REMARK 2.12. Theorem 1.3 characterizes graphs with three distinct reciprocal distance Laplacian eigenvalues under the assumption that $G \in \Gamma$. This assumption is used to control the multiplicity of the largest reciprocal distance Laplacian eigenvalue. If this assumption is removed, then the largest eigenvalue may have multiplicity greater than one, and the proof of Theorem 1.3 no longer applies directly. Therefore, obtaining a complete characterization graphs $G \notin \Gamma$ appears to be a more difficult problem and may involve additional graph families. We have seen that $K_n - e$ has exactly three distinct reciprocal distance Laplacian eigenvalues, although $K_n - e \notin \Gamma$. This naturally leads to the following question: Is $K_n - e$ the unique connected graph on n vertices having exactly three distinct reciprocal distance Laplacian eigenvalues such that the multiplicity of δ_1^L is $n-2$?

3. Proof of Theorem 1.4

The notion of equitable partitions for the Laplacian matrix was introduced in [18]. We extend this notion to the reciprocal distance Laplacian matrix.

DEFINITION 3.1. [6] Let H be a real symmetric matrix whose rows and columns are indexed by $X = 1, 2, \dots, n$. Let $\pi = \{X_1, X_2, \dots, X_t\}$ be a partition of X . The characteristic matrix C is the $n \times m$ matrix whose j -th column is the characteristic vector of $X_j (j = 1, 2, \dots, m)$. Let H be partitioned according to π as

$$H = \begin{bmatrix} H_{11} & H_{12} & \dots & H_{1m} \\ H_{21} & H_{22} & \dots & H_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ H_{m1} & H_{m2} & \dots & H_{mm} \end{bmatrix}.$$

where H_{ij} denotes the submatrix (block) of H formed by rows in X_i and the columns in X_j . If q_{ij} denotes the average row sum of H_{ij} , then the matrix $Q_\pi = [(q_{ij})]_{m \times m}$ is called the quotient matrix of H . If the row sum of each block H_{ij} is a constant, then the partition π is called equitable partition.

Equitable partition for reciprocal distance matrix. Let $G = (V, E)$ be a graph and $V_1, V_2, \dots, V_l$ be a partition of the vertex set V . Then the set $\pi = \{V_1, V_2, \dots, V_l\}$ is said to form an *equitable partition* for reciprocal distance matrix if for each $r, s \in \{1, 2, \dots, l\}$ there is a number q_{rs} such that for each $i \in V_r$, $q_{rs} = \sum_{j \in V_s} (\mathcal{R}_D)_{ij} = \sum_{j \in V_s} \frac{1}{d_{ij}}$, where q_{rs} are constant depending on r and s . For an equitable partition, the quotient matrix $Q_\pi(G) = [(q_{rs})]_{l \times l}$.

THEOREM 3.2. Let $V_1, V_2, \dots, V_l$ be an equitable partition for reciprocal distance matrix of the vertex set of a graph G with a constant d_{pq} ($p, q = 1, 2, \dots, l$) and let $B(G) = [(b_{pq})]$ be the $t \times t$ matrix defined by

$$b_{pq} = \begin{cases} -q_{pq}, & \text{if } p \neq q, \\ \left(\sum_{s=1}^l q_{ps} \right) - q_{pp}, & \text{if } p = q. \end{cases}$$

Then each eigenvalue of $B(G)$ is also an eigenvalue of $\mathcal{R}_D^L(G)$.

Proof. The proof is identical to that of [18, Theorem 2.3]. $\square$

Note. The matrix $B(G)$ in Theorem 3.2 is called the reciprocal distance Laplacian quotient matrix of the graph G .

Proof of Theorem 1.4. The reciprocal distance Laplacian matrix for $K_{n,n} - e$ is the following:

$$\mathcal{R}_D^L(K_{n,n} - e) = \begin{bmatrix} \frac{9n-7}{6} & -\frac{1}{2} & -\frac{1}{2} & \dots & -\frac{1}{2} & -\frac{1}{3} & -1 & -1 & \dots & -1 \\ -\frac{1}{2} & \frac{3n-1}{2} & -\frac{1}{2} & \dots & -\frac{1}{2} & -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \dots & \frac{3n-1}{2} & -1 & -1 & -1 & \dots & -1 \\ -\frac{1}{3} & -1 & -1 & \dots & -1 & \frac{9n-7}{6} & -\frac{1}{2} & -\frac{1}{2} & \dots & -\frac{1}{2} \\ -1 & -1 & -1 & \dots & -1 & -\frac{1}{2} & \frac{3n-1}{2} & -\frac{1}{2} & \dots & -\frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \dots & \frac{3n-1}{2} \end{bmatrix}$$

Let $\mathcal{S}_1 = \{\xi_{2,i} = -e_2 + e_i : i = 3, 4, \dots, n\}$ and $\mathcal{S}_2 = \{\xi_{n+2,i} = -e_{n+2} + e_i : i = n + 3, n + 4, \dots, 2n\}$.

Then we have the following relationships:

$$\mathcal{R}_D^L(K_{n,n} - e) \xi_{2,i} = \frac{3n}{2} \xi_{2,i}, \quad \forall i = 3, 4, \dots, n$$

$$\mathcal{R}_D^L(K_{n,n} - e) \xi_{n+2,i} = \frac{3n}{2} \xi_{n+2,i}, \quad \forall i = n + 3, n + 4, \dots, 2n.$$

Therefore, $\frac{3n}{2}$ is an eigenvalues of $\mathcal{R}_D^L(K_{n,n} - e)$ with multiplicity at least $2n - 4$.

Let $L_1 = \{1\}$, $L_2 = \{n+1\}$, $L_3 = \{2, 3, \dots, n\}$ and $L_4 = \{n+2, n+3, \dots, 2n\}$. Then the reciprocal distance Laplacian quotient matrix of G is

$$B(K_{n,n} - e) = \begin{bmatrix} \frac{9n-7}{6} & -\frac{1}{3} & -\frac{n-1}{2} & -(n-1) \\ -\frac{1}{3} & \frac{9n-7}{6} & -(n-1) & -\frac{n-1}{2} \\ -\frac{1}{2} & -1 & \frac{2n+1}{2} & -(n-1) \\ -1 & -\frac{1}{2} & -(n-1) & \frac{2n+1}{2} \end{bmatrix}$$

The characteristic polynomial of $B(K_{n,n} - e)$ is $P(t) = \frac{1}{12}t(2t - 3n)\left(6t^2 - (21n - 8)t + (18n^2 - 16n + 4)\right)$.

Then the roots of the polynomial $P(t) = 0$ are 0 , $\frac{3n}{2}$, $\frac{21n-8-\sqrt{9n^2+48n-32}}{12}$ and $\frac{21n-8+\sqrt{9n^2+48n-32}}{12}$ with multiplicities 1, 1, 1, and 1, respectively. Using Theorem 3.2, the roots of the polynomial $P(t) = 0$ are in the reciprocal distance Laplacian eigenvalues of $K_{n,n} - e$. Then, the reciprocal distance Laplacian eigenvalues of $K_{n,n} - e$ are 0 , $\frac{3n}{2}$, $\frac{21n-8-\sqrt{9n^2+48n-32}}{12}$, $\frac{21n-8+\sqrt{9n^2+48n-32}}{12}$, with multiplicities 1, at least $2n-4$, 1 and 1, respectively. The other eigenvalue can be calculated using the trace of $\mathcal{R}_D^{\mathcal{L}}(K_{n,n} - e)$. The trace of $\mathcal{R}_D^{\mathcal{L}}(K_{n,n} - e)$ is $\frac{9n^2-3n-4}{2}$. The other eigenvalue of $\mathcal{R}_D^{\mathcal{L}}(K_{n,n} - e)$ is $\frac{3n}{2}$. Hence, the proof follows. $\square$

REMARK 3.3. In the previous result, we have seen that there is a graph in Γ that has four distinct reciprocal distance Laplacian eigenvalues. This naturally leads to the following question: Is $K_{n,n} - e$ the unique connected graph on $2n$ vertices in Γ having exactly four distinct reciprocal distance Laplacian eigenvalues. Consider the graph $K_n - e - f$, where e and f are two distinct edges of K_n . Using an argument similar to that in Example 2.11, one can easily show that its reciprocal distance Laplacian eigenvalues are

$$\left(\overbrace{n, \dots, n}^{n-3}, \frac{2n-1}{2}, \frac{2n-3}{2}, 0\right).$$

Hence, $K_n - e - f$ has exactly four distinct reciprocal distance Laplacian eigenvalues, and the multiplicity of $\delta_1^{\mathcal{L}}$ is $n-3$. This raises another natural question: Is $K_n - e - f$ the unique connected graph on n vertices having exactly four distinct reciprocal distance Laplacian eigenvalues such that the multiplicity of $\delta_1^{\mathcal{L}}$ is $n-3$?

4. Proof of Theorem 1.5

LEMMA 4.1. [12] Let E and F be two Hermitian matrices. Let $\lambda_1(E) \geq \lambda_2(E) \geq \dots \geq \lambda_n(E)$, $\lambda_1(F) \geq \lambda_2(F) \geq \dots \geq \lambda_n(F)$ and $\lambda_1(E+F) \geq \lambda_2(E+F) \geq \dots \geq \lambda_n(E+F)$ be the ordering of the eigenvalues of E , F and $E+F$, respectively. Then

$$\lambda_j(E) + \lambda_n(F) \leq \lambda_j(E+F) \leq \lambda_j(E) + \lambda_1(F), \text{ for } j = 1, 2, \dots, n.$$

Proof of Theorem 1.5. We consider the matrix M which is defined below.

$$M_{ij} = \begin{cases} \frac{d_{ij}-1}{d_{ij}} & \text{for } d_{ij} > 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let $\pi(i) = \sum_{\substack{j=1 \\ j \neq i}}^n M_{ij}$, for $i = 1, \dots, n$. We have $\mathcal{R}_D(G) = A(G) + A(\bar{G}) - M$, where

$\text{Diag}(RTr) = D(G) + D(\bar{G}) - \text{Diag}(\pi)$, where $\text{Diag}(\pi)$ is the diagonal matrix whose diagonal entries are $\pi(i)$ for $i = 1, \dots, n$. Then the reciprocal distance Laplacian matrix can be written in the following way.

$$\begin{aligned} \mathcal{R}_D^L(G) &= \text{Diag}(RTr(G)) - \mathcal{R}_D(G) \\ &= D(G) + D(\bar{G}) - \text{Diag}(\pi) - A(G) - A(\bar{G}) + M \\ &= D(G) - A(G) + D(\bar{G}) - A(\bar{G}) - \text{Diag}(\pi) + M \\ &= L(G) + L(\bar{G}) - M', \end{aligned}$$

where $M' = \text{Diag}(\pi) - M$. Using Lemma 4.1, we have $\delta_i^L \geq \lambda_i(L - M')$. Again applying Lemma 4.1 on $L - M'$, we get $\lambda_i(L - M') \geq \mu_i$. Hence $\delta_i^L \geq \mu_i$ for $i = 1, \dots, n$.

We now prove the equality part. If $G = K_n$, then eigenvalues of $\mathcal{R}_D^L(K_n)$ are equal to $L(K_n)$. Hence $\delta_i^L = \mu_i$ for $i = 1, \dots, n$. We now assume that $\delta_i^L = \mu_i$ for $i = 1, \dots, n$. Then the $\text{trace}(\mathcal{R}_D^L(G)) = \text{trace}(L(G))$. We notice that $RTr(i) = \deg(i) + a_i$ for $i = 1, \dots, n$, where $a_i \geq 0$ for $i = 1, \dots, n$. Therefore, $\sum_{i=1}^n RTr(i) = \sum_{i=1}^n \deg(i) + \sum_{i=1}^n a_i$ which equals to $\sum_{i=1}^n \deg(i)$. Then $\sum_{i=1}^n a_i = 0$. Since each a_i is a nonnegative real number, we have $a_i = 0$ for $i = 1, \dots, n$. Then $RTr(i) = \deg(i)$ for $i = 1, \dots, n$ it is possible only when $\deg(i) = n - 1$ for $i = 1, \dots, n$. Hence $G = K_n$. $\square$

5. Conclusion

The results of this paper show that the number of distinct reciprocal distance Laplacian eigenvalues gives useful structural information about a graph. The case of two distinct eigenvalues forces the graph to be complete. Within the class Γ , the case of three distinct eigenvalues leads only to the star and the regular complete bipartite graphs. The graphs $K_{n,n} - e$ and $K_n - e - f$ show that graphs with four distinct reciprocal distance Laplacian eigenvalues can arise from simple modifications of complete or complete bipartite graphs. These examples indicate that the study of four distinct reciprocal distance Laplacian eigenvalues is a natural next step, especially for graphs obtained from complete multipartite graphs by deleting a small number of edges.

Statements and declarations. There is no conflict of interest. Our manuscript has no associated data.

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