

SOME GENERALIZED REFINEMENTS AND REVERSES OF CONVEX FUNCTIONS AND APPLICATIONS

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Abstract. In this paper, we establish several new inequalities associated with different classes of convexity, including convex, log-convex, and p -convex functions, together with their refinements and reverses. As applications, we establish several refinements and reverses of Hölder-type inequalities involving the quadratic weighted geometric mean.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$. Denote by $B(\mathcal{H})$ the $\mathbb{C}^*$ -algebra of all bounded linear operators acting on $\mathcal{H}$. The subset of all invertible elements in $B(\mathcal{H})$ is denoted by $B(\mathcal{H})^{-1}$. An operator $A \in B(\mathcal{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and denoted by $A \geq 0$. The collection of all positive operators on $\mathcal{H}$ is represented by $B(\mathcal{H})^+$. Furthermore, the set of all invertible positive operators is denoted by $B(\mathcal{H})^{++}$. In the finite-dimensional case where $\dim \mathcal{H} = n$, we naturally identify $B(\mathcal{H})$ with the algebra $\mathbb{M}_n$ of all $n \times n$ complex matrices. Given a matrix $A \in \mathbb{M}_n$, the singular values of A are defined as the eigenvalues of the positive semi-definite matrix $|A| := (A^*A)^{1/2}$, where A^* denotes the conjugate transpose (adjoint) of A . These singular values are denoted by $s_i(A)$ for $i = 1, 2, \dots, n$. A norm $||| \cdot |||$ on $\mathbb{M}_n$ is said to be unitarily invariant if it satisfies

$$|||UAV||| = |||A||| \quad \text{for all } A \in \mathbb{M}_n \text{ and all unitary matrices } U, V \in \mathbb{M}_n.$$

Two commonly used unitarily invariant norms are the trace norm and the Frobenius norm defined by

$$\|A\|_1 := \operatorname{tr}(|A|) = \sum_{i=1}^n s_i(A), \quad \text{where } \operatorname{tr} \text{ denotes the standard matrix trace}$$

and

$$\|A\|_F := (\operatorname{tr}(A^*A))^{1/2} = \left(\sum_{i,j} |a_{i,j}|^2 \right)^{1/2}, \quad \text{where } A = (a_{i,j})$$

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respectively.

Let $A, B \in B(\mathcal{H})^+$ and $v \in [0, 1]$, the weighted arithmetic mean of A and B is given by

$$A\nabla_v B := (1-v)A + vB.$$

In 1980, Kubo and Ando [10] introduced the weighted geometric mean for two operators $A \in B(\mathcal{H})^{++}$ and $B \in B(\mathcal{H})^+$, defined by

$$A\sharp_v B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^v A^{1/2}, \quad v \in [0, 1].$$

It is well known that this mean satisfies the operator inequality

$$A\sharp_v B \leq A\nabla_v B. \quad (1)$$

In the special case when $v = \frac{1}{2}$, we denote this two means by $A\nabla B$ and $A\sharp B$, respectively.

2016, Dragomir introduced in [4] a notion called the quadratic weighted geometric mean for $T \in B(\mathcal{H})^{-1}$ and $V \in B(\mathcal{H})$, given by

$$T\mathbb{S}_v V := \left| |VT^{-1}|^v T \right|^2 = |T|^2 \sharp_v |V|^2, \quad v \geq 0,$$

Consequently, the following inequality extends the relation (1):

$$T\mathbb{S}_v V \leq |T|^2 \nabla_v |V|^2.$$

Moreover, we denote

$$T\mathbb{S}V := T\mathbb{S}_{\frac{1}{2}}V = \left| |VT^{-1}|^{1/2} T \right|^2 = T^* |VT^{-1}| T = T^* \left((T^*)^{-1} V^* V T^{-1} \right)^{1/2} T.$$

McCarthy [11] established the following results

$$\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r, \quad r \in (0, 1) \quad (2)$$

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle, \quad r \in (1, \infty) \quad (3)$$

for $A \in \mathbb{M}_n^+$ all $x \in \mathcal{H}$ with $\|x\| = 1$. Employing McCarthy's inequality, Fujii et al. [6] proved a Hölder-type inequality for the weighted geometric mean:

$$\langle A\sharp_v Bx, x \rangle \leq \langle Ax, x \rangle^{1-v} \langle Bx, x \rangle^v, \quad A, B \in B(\mathcal{H})^{++}, \quad x \in \mathcal{H}. \quad (4)$$

2018, Dragomir [5] extended (4) to the quadratic weighted geometric mean:

$$\langle T\mathbb{S}_v Vx, x \rangle \leq \langle |T|^2 x, x \rangle^{1-v} \langle |V|^2 x, x \rangle^v, \quad T \in B(\mathcal{H})^{-1}, \quad V \in B(\mathcal{H}).$$

Convexity plays an essential role in deriving and refining matrix inequalities due to its strong connections with matrix analysis and operator theory. Let I be a real interval and $f : I \rightarrow \mathbb{R}$ be a given function. A function f is said to be convex on I if

$$f((1-v)x_1 + vx_2) \leq (1-v)f(x_1) + vf(x_2),$$

for all $x_1, x_2 \in I$ and $v \in [0, 1]$. A closely related class of functions is that of log-convex functions, defined by

$$f((1-v)x_1 + vx_2) \leq f(x_1)^{1-v} f(x_2)^v,$$

where $f : I \rightarrow (0, \infty)$, $x_1, x_2 \in I$, and $v \in [0, 1]$. This shows that log-convexity is a stronger property than ordinary convexity for positive-valued functions, and thus provides a useful framework for obtaining stronger inequalities and refined estimates in matrix analysis.

The theory of convexity has witnessed substantial development, leading to numerous extensions and generalized notions adapted to different mathematical settings. Among these generalizations, p -convexity has attracted considerable interest due to its ability to encompass several important convexity classes within a unified framework. Specifically, let $I \subseteq [0, \infty)$ be a p -set. A function $f : I \rightarrow \mathbb{R}$ is called p -convex [19] if, for every $x_1, x_2 \in I$ and $v \in [0, 1]$, one has

$$f\left(\left((1-v)x_1^p + vx_2^p\right)^{\frac{1}{p}}\right) \leq (1-v)f(x_1) + vf(x_2),$$

where $p \in \mathbb{R} \setminus \{0\}$. This concept provides a flexible framework that includes several well-known types of convexity as particular cases, such as classical convexity, harmonic convexity, and geometric convexity, obtained through suitable choices of the parameter p .

For some interesting papers on refinements involving certain classes of convexity, the reader is encouraged to consult the following works [7, 8, 9, 13, 14, 15, 16, 17, 18].

Very recently, Zuo *et al.* [20] established several important inequalities related to p -convex functions, providing new refinements and estimates that extend classical convexity results. For the reader's convenience, we recall below one of their main results.

THEOREM 1.1. ([20]) *Let $f : [x_1, x_2] \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a p -convex function, and let $m \in \mathbb{N}$ with $m \geq 1$. Suppose $0 < v \leq \eta < 1$ and $p \in \mathbb{R} \setminus \{0\}$.*

1. *If f is increasing, then*

$$\frac{\left((1-v)f(x_1) + vf(x_2)\right)^m - f^m\left(\left((1-v)x_1^p + vx_2^p\right)^{1/p}\right)}{\left((1-\eta)f(x_1) + \eta f(x_2)\right)^m - f^m\left(\left((1-\eta)x_1^p + \eta x_2^p\right)^{1/p}\right)} \leq \frac{1-v}{1-\eta}.$$

2. *If f is decreasing, then*

$$\frac{v}{\eta} \leq \frac{\left((1-v)f(x_1) + vf(x_2)\right)^m - f^m\left(\left((1-v)x_1^p + vx_2^p\right)^{1/p}\right)}{\left((1-\eta)f(x_1) + \eta f(x_2)\right)^m - f^m\left(\left((1-\eta)x_1^p + \eta x_2^p\right)^{1/p}\right)}.$$

A natural question arises: can the above theorem be extended by replacing the integer parameter m with an arbitrary real number? Motivated by this question, one of the main objectives of the present paper is to establish a more general result involving increasing convex functions, which enables a significant extension of the aforementioned theorem.

The remainder of this paper is organized as follows. In Section 2, we establish several new inequalities associated with different classes of convexity, including convex, log-convex, and p -convex functions, together with their refinements and reverses. Section 3 is devoted to deriving new refinements and reverses of Hölder-type inequalities involving the quadratic weighted geometric mean.

2. Further inequalities for various types of convexity

In this section, we establish several new inequalities associated with different classes of convexity, including convex, log-convex, and p -convex functions. Our approach is mainly based on the use of increasing convex functions, which enables us to derive new refinements and reversed inequalities extending several known results in the literature. These developments will serve as fundamental tools for obtaining applications to matrix inequalities in the subsequent sections.

2.1. New inequalities for convex and log-convex functions

In the sequel, we will frequently make use of some well-known refinements and reverses of the convexity inequality, which provide tighter bounds and reveal deeper structure in convex and log-convex functions.

LEMMA 2.1. ([12]) *Let $f : I \longrightarrow [0, \infty)$ be convex and $0 \leq v \leq 1$. Then*

$$\begin{aligned} & 2r_0 \left(\frac{f(x_1) + f(x_2)}{2} - f\left(\frac{x_1 + x_2}{2}\right) \right) \\ & \leq (1 - v)f(x_1) + vf(x_2) - f((1 - v)x_1 + vx_2) \\ & \leq 2R_0 \left(\frac{f(x_1) + f(x_2)}{2} - f\left(\frac{x_1 + x_2}{2}\right) \right), \end{aligned}$$

where $r_0 = \min\{v, 1 - v\}$ and $R_0 = \max\{v, 1 - v\}$.

LEMMA 2.2. ([1]) *Let ϕ be a strictly increasing convex function defined on an interval I . If x, y, z and w are points in I such that*

$$z - w \leq x - y,$$

where $w \leq z \leq x$ and $y \leq x$, then

$$(0 \leq) \quad \phi(z) - \phi(w) \leq \phi(x) - \phi(y).$$

THEOREM 2.3. *Let $f : I \longrightarrow [0, \infty)$ be convex, ϕ be a strictly increasing convex function defined on $\mathbb{R}$ and $0 \leq v \leq 1$. Then*

$$\begin{aligned} & \phi\left(2r_0 \frac{f(x_1) + f(x_2)}{2}\right) - \phi\left(2r_0 f\left(\frac{x_1 + x_2}{2}\right)\right) \\ & \leq \phi\left((1-v)f(x_1) + vf(x_2)\right) - \phi \circ f((1-v)x_1 + vx_2) \\ & \leq \phi\left(2R_0 \frac{f(x_1) + f(x_2)}{2}\right) - \phi\left(2R_0 f\left(\frac{x_1 + x_2}{2}\right)\right), \end{aligned}$$

where $r_0 = \min\{v, 1-v\}$ and $R_0 = \max\{v, 1-v\}$.

Proof. Let us define the following quantities:

$$\begin{aligned} x &= (1-v)f(x_1) + vf(x_2), & y &= f((1-v)x_1 + vx_2), \\ z &= 2r_0 \cdot \frac{f(x_1) + f(x_2)}{2}, & w &= 2r_0 \cdot f\left(\frac{x_1 + x_2}{2}\right), \\ z' &= 2R_0 \cdot \frac{f(x_1) + f(x_2)}{2}, & w' &= 2R_0 \cdot f\left(\frac{x_1 + x_2}{2}\right). \end{aligned}$$

Then, by invoking Lemma 2.1, we obtain the double inequality:

$$z - w \leq x - y \leq z' - w'.$$

On the other hand, $r_0 = \min\{v, 1-v\}$ and $R_0 = \max\{v, 1-v\}$ implies $w \leq z \leq x$ and $y \leq x$. This completes the proof by applying Lemma 2.2. $\square$

By substitute f by $\log f$, and setting $\phi(x) = \exp(x)$ in Theorem 2.3, we derive new refinements and corresponding reverses of classical inequalities for log-convex functions.

COROLLARY 2.4. *Let $f : [0, 1] \longrightarrow (0, \infty)$ be log-convex and $0 \leq v \leq 1$. Then*

$$\sqrt{f(1)f(0)}^{2r_0} - f^{2r_0}\left(\frac{1}{2}\right) \leq f^{1-v}(0)f^v(1) - f(v) \leq \sqrt{f(1)f(0)}^{2R_0} - f^{2R_0}\left(\frac{1}{2}\right),$$

where $r_0 = \min\{v, 1-v\}$ and $R_0 = \max\{v, 1-v\}$.

2.2. Further inequalities for p -convex functions

We begin by establishing a generalized inequality involving increasing convex functions, which will play a fundamental role in the subsequent developments. This result provides a useful tool for deriving several refinements and extensions of inequalities associated with different classes of convex functions.

THEOREM 2.5. *Let $x, y, z, w \geq 0$ with $x > y$, $z > w \geq y$, and $z \geq x$. Let ϕ be a strictly increasing and convex function defined on an interval J that includes the real numbers x, y, z , and w . Let $\alpha \geq 0$ be such that*

$$\frac{x-y}{z-w} \leq \alpha. \quad (5)$$

Then

$$\frac{\phi(x) - \phi(y)}{\phi(z) - \phi(w)} \leq \alpha. \quad (6)$$

Proof. As stated in [2, Theorem 1], any convex function can be approximated uniformly by smooth convex functions. Therefore, it is sufficient to establish (6) for a differentiable convex function ϕ . Furthermore, the subgradient inequality for a differentiable convex function ϕ defined on an interval J asserts that

$$\phi(u) - \phi(v) \geq \phi'(v)(u - v), \quad (7)$$

for all $u, v \in J$. For further details, see [3].

Assuming $w \geq x$, and utilizing the fact that ϕ is a convex and increasing function, we obtain

$$\begin{aligned} \alpha(\phi(z) - \phi(w)) + \phi(y) - \phi(x) &\geq \alpha(z-w)\phi'(w) + (y-x)\phi'(x) \\ &\geq (\alpha(z-w) + (y-x))\phi'(x) \\ &\geq 0, \end{aligned}$$

where we have used (7) to obtain the first inequality, while the second inequality arises from the fact that $\phi'(w) \geq \phi'(x)$ since ϕ is convex, and the third inequality follows from the assumption (5) and the fact that ϕ is increasing. This proves the desired inequality when $w \geq x$.

Now, suppose that $w \leq x$. Combining this with the inequality $x \leq z$, there exists a scalar $\beta \in [0, 1]$ such that $x = \beta w + (1 - \beta)z$. So, we have

$$\begin{aligned} x - y &= \beta w + (1 - \beta)z - y \\ &= (1 - \beta)(z - w) + w - y \\ &\geq (1 - \beta)(z - w). \end{aligned} \quad (8)$$

Combining this with (5), it follows that

$$\alpha \geq \frac{x-y}{z-w} \geq 1 - \beta,$$

which implies $\alpha + \beta - 1 \geq 0$. Furthermore, by the convexity of ϕ , we have

$$\begin{aligned} &\alpha[\phi(z) - \phi(w)] + \phi(y) - \phi(x) \\ &\geq \alpha[\phi(z) - \phi(w)] + \phi(y) - \beta\phi(w) - (1 - \beta)\phi(z) \\ &= (\alpha + \beta - 1)[\phi(z) - \phi(w)] + \phi(y) - \phi(w) \\ &\geq (\alpha + \beta - 1)(z - w)\phi'(w) + (y - w)\phi'(w) \\ &= [(\alpha + \beta - 1)(z - w) + y - w]\phi'(w) \\ &\geq 0, \end{aligned}$$

where

$$(\alpha + \beta - 1)(z - w) + y - w \geq 0$$

is derived from the inequality

$$\alpha(z - w) \geq (x - y) = (1 - \beta)(z - w) + w - y,$$

which can be justified using inequalities (5) and (8). $\square$

We now show a generalized extension of Theorem 1.1, where the integer m is replaced by an arbitrary strictly increasing and convex function.

THEOREM 2.6. *Let $f : [x_1, x_2] \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a p -convex function and ϕ be a strictly increasing convex function defined on an interval J . Suppose that $0 < v \leq \eta < 1$ and $p \in \mathbb{R} \setminus \{0\}$.*

1. *If f is increasing, then*

$$\frac{\phi((1-v)f(x_1) + vf(x_2)) - \phi\left(f\left(\left((1-v)x_1^p + vx_2^p\right)^{1/p}\right)\right)}{\phi((1-\eta)f(x_1) + \eta f(x_2)) - \phi\left(f\left(\left((1-\eta)x_1^p + \eta x_2^p\right)^{1/p}\right)\right)} \leq \frac{1-v}{1-\eta}.$$

2. *If f is decreasing, then*

$$\frac{v}{\eta} \leq \frac{\phi((1-v)f(x_1) + vf(x_2)) - \phi\left(f\left(\left((1-v)x_1^p + vx_2^p\right)^{1/p}\right)\right)}{\phi((1-\eta)f(x_1) + \eta f(x_2)) - \phi\left(f\left(\left((1-\eta)x_1^p + \eta x_2^p\right)^{1/p}\right)\right)}.$$

Proof. Assume that $p \in \mathbb{R} \setminus \{0\}$ and that f is increasing and p -convex. Define

$$x = (1-v)f(x_1) + vf(x_2),$$

$$y = f\left(\left((1-v)x_1^p + vx_2^p\right)^{1/p}\right),$$

$$z = (1-\eta)f(x_1) + \eta f(x_2),$$

and

$$w = f\left(\left((1-\eta)x_1^p + \eta x_2^p\right)^{1/p}\right).$$

Since f is p -convex, we have

$$y \leq x \quad \text{and} \quad w \leq z.$$

Moreover, because $0 < v \leq \eta < 1$, we obtain

$$1 - v \geq 1 - \eta.$$

Using the monotonicity of f , together with $x_1 \leq x_2$, yields

$$((1-\eta)x_1^p + \eta x_2^p)^{1/p} \geq ((1-\nu)x_1^p + \nu x_2^p)^{1/p},$$

and hence

$$w \geq y.$$

Similarly,

$$z - x = (\eta - \nu)(f(x_2) - f(x_1)) \geq 0,$$

which implies

$$z \geq x.$$

Next, observe that

$$\begin{aligned} x - y &= (1-\nu)f(x_1) + \nu f(x_2) - f\left(((1-\nu)x_1^p + \nu x_2^p)^{1/p}\right), \\ z - w &= (1-\eta)f(x_1) + \eta f(x_2) - f\left(((1-\eta)x_1^p + \eta x_2^p)^{1/p}\right). \end{aligned}$$

By Theorem 1.1, we know that

$$\frac{x-y}{z-w} \leq \frac{1-\nu}{1-\eta}.$$

Therefore, all assumptions of Theorem 2.5 are fulfilled with

$$\alpha = \frac{1-\nu}{1-\eta}.$$

Applying Theorem 2.5, we conclude that

$$\frac{\phi(x) - \phi(y)}{\phi(z) - \phi(w)} \leq \frac{1-\nu}{1-\eta},$$

that is,

$$\frac{\phi((1-\nu)f(x_1) + \nu f(x_2)) - \phi\left(f\left(((1-\nu)x_1^p + \nu x_2^p)^{1/p}\right)\right)}{\phi((1-\eta)f(x_1) + \eta f(x_2)) - \phi\left(f\left(((1-\eta)x_1^p + \eta x_2^p)^{1/p}\right)\right)} \leq \frac{1-\nu}{1-\eta}.$$

For part (b), using the decreasing property of f and applying Theorem 2.5 together with the corresponding inequality established in Theorem 1.1, we similarly obtain

$$\frac{\nu}{\eta} \leq \frac{\phi((1-\nu)f(x_1) + \nu f(x_2)) - \phi\left(f\left(((1-\nu)x_1^p + \nu x_2^p)^{1/p}\right)\right)}{\phi((1-\eta)f(x_1) + \eta f(x_2)) - \phi\left(f\left(((1-\eta)x_1^p + \eta x_2^p)^{1/p}\right)\right)}.$$

This completes the proof. $\square$

As a direct consequence of Theorem 2.6, by choosing $\phi(x) = x^\lambda$ for $\lambda \geq 1$, we obtain the following corollary.

COROLLARY 2.7. Let $f : [x_1, x_2] \subseteq [0, \infty) \rightarrow [0, \infty)$ be a p -convex function and $\lambda \geq 1$. Suppose that $0 < v \leq \eta < 1$ and $p \in \mathbb{R} \setminus \{0\}$.

1. If f is increasing, then

$$\frac{((1-v)f(x_1) + vf(x_2))^\lambda - \left(f\left(((1-v)x_1^p + vx_2^p)^{1/p}\right)\right)^\lambda}{((1-\eta)f(x_1) + \eta f(x_2))^\lambda - \left(f\left(((1-\eta)x_1^p + \eta x_2^p)^{1/p}\right)\right)^\lambda} \leq \frac{1-v}{1-\eta}.$$

2. If f is decreasing, then

$$\frac{v}{\eta} \leq \frac{((1-v)f(x_1) + vf(x_2))^\lambda - \left(f\left(((1-v)x_1^p + vx_2^p)^{1/p}\right)\right)^\lambda}{((1-\eta)f(x_1) + \eta f(x_2))^\lambda - \left(f\left(((1-\eta)x_1^p + \eta x_2^p)^{1/p}\right)\right)^\lambda}.$$

By specializing the parameter to $p = 1$ in Theorem 2.6, we immediately derive the following consequence for convex functions.

COROLLARY 2.8. Let $f : [x_1, x_2] \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a convex function, and let ϕ be a strictly increasing and convex function defined on an interval J . Suppose that $0 < v \leq \eta < 1$.

1. If f is increasing, then

$$\frac{\phi((1-v)f(x_1) + vf(x_2)) - \phi(f((1-v)x_1 + vx_2))}{\phi((1-\eta)f(x_1) + \eta f(x_2)) - \phi(f((1-\eta)x_1 + \eta x_2))} \leq \frac{1-v}{1-\eta}.$$

2. If f is decreasing, then

$$\frac{v}{\eta} \leq \frac{\phi((1-v)f(x_1) + vf(x_2)) - \phi(f((1-v)x_1 + vx_2))}{\phi((1-\eta)f(x_1) + \eta f(x_2)) - \phi(f((1-\eta)x_1 + \eta x_2))}.$$

The following proposition establishes sufficient conditions ensuring the monotonicity of the function

$$v \mapsto \langle A \sharp_v Bx, x \rangle$$

on the interval $[0, 1]$.

PROPOSITION 2.9. Let $A, B \in B(\mathcal{H})^{++}$ and let $x \in \mathcal{H}$. Then the function

$$f(v) = \langle A \sharp_v Bx, x \rangle, \quad v \in [0, 1],$$

is increasing whenever $B \geq A$, and decreasing whenever $B \leq A$.

Proof. Set

$$X = A^{-1/2}BA^{-1/2}.$$

Then

$$A\sharp_{\nu}B = A^{1/2}X^{\nu}A^{1/2},$$

and therefore

$$f(\nu) = \langle X^{\nu}A^{1/2}x, A^{1/2}x \rangle.$$

Assume that $B \geq A$. Then

$$X = A^{-1/2}BA^{-1/2} \geq I.$$

Since the scalar function $t \mapsto t^{\nu}$ is increasing with respect to ν for every $t \geq 1$, it follows from the functional calculus that

$$X^{\nu_1} \leq X^{\nu_2}, \quad 0 \leq \nu_1 \leq \nu_2 \leq 1.$$

Consequently,

$$A\sharp_{\nu_1}B = A^{1/2}X^{\nu_1}A^{1/2} \leq A^{1/2}X^{\nu_2}A^{1/2} = A\sharp_{\nu_2}B.$$

Hence,

$$\langle A\sharp_{\nu_1}Bx, x \rangle \leq \langle A\sharp_{\nu_2}Bx, x \rangle,$$

which shows that $f(\nu)$ is increasing.

Similarly, if $B \leq A$, then $X \leq I$. Since t^{ν} is decreasing in ν for every $0 < t \leq 1$, we obtain

$$X^{\nu_1} \geq X^{\nu_2}, \quad 0 \leq \nu_1 \leq \nu_2 \leq 1,$$

and thus

$$\langle A\sharp_{\nu_1}Bx, x \rangle \geq \langle A\sharp_{\nu_2}Bx, x \rangle.$$

Therefore, $f(\nu)$ is decreasing. $\square$

THEOREM 2.10. Let $A, B \in B(\mathcal{H})^{++}$, $x \in \mathcal{H}$, and define

$$f(\nu) = \langle A\sharp_{\nu}Bx, x \rangle, \quad \nu \in [0, 1].$$

Let ϕ be a strictly increasing and convex function defined on an interval J containing the range of f . Suppose that $0 < \nu \leq \eta < 1$.

1. If $B \geq A$, then

$$\frac{\phi((1-\nu)\langle Ax, x \rangle + \nu\langle Bx, x \rangle) - \phi(\langle A\sharp_{\nu}Bx, x \rangle)}{\phi((1-\eta)\langle Ax, x \rangle + \eta\langle Bx, x \rangle) - \phi(\langle A\sharp_{\eta}Bx, x \rangle)} \leq \frac{1-\nu}{1-\eta}.$$

2. If $B \leq A$, then

$$\frac{\nu}{\eta} \leq \frac{\phi((1-\nu)\langle Ax, x \rangle + \nu\langle Bx, x \rangle) - \phi(\langle A\sharp_{\nu}Bx, x \rangle)}{\phi((1-\eta)\langle Ax, x \rangle + \eta\langle Bx, x \rangle) - \phi(\langle A\sharp_{\eta}Bx, x \rangle)}.$$

Proof. Observe that

$$f(v) = \langle A \sharp_v Bx, x \rangle$$

is convex on $[0, 1]$. Moreover, by Proposition 2.9 and Corollary 2.8 to the function f on the interval $[0, 1]$, with $x_1 = 0$ and $x_2 = 1$, we obtain the desired inequalities. $\square$

COROLLARY 2.11. *Let $f : [x_1, x_2] \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a log-convex function. Suppose that $0 < v \leq \eta < 1$.*

1. *If f is increasing, then*

$$\frac{f(x_1)^{1-v} f(x_2)^v - f((1-v)x_1 + vx_2)}{f(x_1)^{1-\eta} f(x_2)^\eta - f((1-\eta)x_1 + \eta x_2)} \leq \frac{1-v}{1-\eta}.$$

2. *If f is decreasing, then*

$$\frac{v}{\eta} \leq \frac{f(x_1)^{1-v} f(x_2)^v - f((1-v)x_1 + vx_2)}{f(x_1)^{1-\eta} f(x_2)^\eta - f((1-\eta)x_1 + \eta x_2)}.$$

Proof. The result follows from the previous corollary by replacing f with $\log f$ and taking $\phi(x) = e^x$. $\square$

3. Refinements and reverses of Hölder type inequalities for quadratic weighted geometric mean

In this section, we derive several new refinements and reverses of operator inequalities related to the quadratic weighted geometric mean by choosing suitable log-convex functions. We begin this section with a lemma.

LEMMA 3.1. *Let $T \in B(\mathcal{H})^-$ and $V \in B(\mathcal{H})$. Then*

$$\phi(v) = \langle T \mathbb{S}_v Vx, x \rangle$$

is log-convex on $[0, 1]$ with $x \in \mathcal{H}$.

Proof. Notice that ϕ is continuous, so it is enough to prove the mid-point log-convexity, i.e.,

$$\phi\left(\frac{v_1 + v_2}{2}\right) \leq (\phi(v_1))^{1/2} (\phi(v_2))^{1/2}.$$

Put $A = |VT^{-1}|^2 \geq 0$ and $z = Tx$.

$$\begin{aligned} \phi\left(\frac{v_1 + v_2}{2}\right) &= \left\langle T \mathbb{S}_{\frac{v_1 + v_2}{2}} Vx, x \right\rangle = \left\langle A^{\frac{v_1 + v_2}{2}} z, z \right\rangle \\ &\leq \|A^{\frac{v_1}{2}} z\| \|A^{\frac{v_2}{2}} z\| = \langle A^{v_1} z, z \rangle^{\frac{1}{2}} \langle A^{v_2} z, z \rangle^{\frac{1}{2}} \\ &= \langle T \mathbb{S}_{v_1} Vx, x \rangle^{\frac{1}{2}} \langle T \mathbb{S}_{v_2} Vx, x \rangle^{\frac{1}{2}} \\ &= (\phi(v_1))^{1/2} (\phi(v_2))^{1/2}. \quad \square \end{aligned}$$

Combination Lemma 3.1 and Corollary 2.4, we have the following refinement and reverse of Hölder type inequality for quadratic weighted geometric mean.

THEOREM 3.2. *Let $T \in B(\mathcal{H})^{-1}$, $V \in B(\mathcal{H})$ and $v \in [0, 1]$. Then we have*

$$\begin{aligned} & \sqrt{\langle |V|^2 x, x \rangle \langle |T|^2 x, x \rangle}^{2r_0} - \langle T \otimes V x, x \rangle^{2r_0} \\ & \leq \langle |T|^2 x, x \rangle^{1-v} \langle |V|^2 x, x \rangle^v - \langle T \otimes_v V x, x \rangle \\ & \leq \sqrt{\langle |V|^2 x, x \rangle \langle |T|^2 x, x \rangle}^{2R_0} - \langle T \otimes V x, x \rangle^{2R_0}, \end{aligned}$$

where $r_0 = \min\{v, 1 - v\}$ and $R_0 = \max\{v, 1 - v\}$ and $x \in \mathcal{H}$.

Taking $T = A^{\frac{1}{2}}$ and $V = B^{\frac{1}{2}}$ in Theorem 3.2, then we get

COROLLARY 3.3. *Let $A, B \in B(\mathcal{H})^{++}$ and $v \in [0, 1]$. Then*

$$\begin{aligned} & \sqrt{\langle Ax, x \rangle \langle Bx, x \rangle}^{2r_0} - \langle A \sharp Bx, x \rangle^{2r_0} \\ & \leq \langle Ax, x \rangle^{1-v} \langle Bx, x \rangle^v - \langle A \sharp_v Bx, x \rangle \\ & \leq \sqrt{\langle Ax, x \rangle \langle Bx, x \rangle}^{2R_0} - \langle A \sharp Bx, x \rangle^{2R_0}, \end{aligned}$$

where $r_0 = \min\{v, 1 - v\}$, $R_0 = \max\{v, 1 - v\}$ and $x \in \mathcal{H}$.

Setting $A = I$ in Corollary 3.3, we get the following refinement and reverse of the McCarthy's inequality (2).

COROLLARY 3.4. *Let $B \in B(\mathcal{H})^{++}$ and $v \in [0, 1]$. Then*

$$\sqrt{\langle Bx, x \rangle}^{2r_0} - \langle B^{\frac{1}{2}} x, x \rangle^{2r_0} \leq \langle Bx, x \rangle^v - \langle B^v x, x \rangle \leq \sqrt{\langle Bx, x \rangle}^{2R_0} - \langle B^{\frac{1}{2}} x, x \rangle^{2R_0},$$

where $r_0 = \min\{v, 1 - v\}$, $R_0 = \max\{v, 1 - v\}$ and $x \in \mathcal{H}$.

Next, we further give n -tuple result about the quadratic weighted geometric mean, which generalized the conclusions mentioned above.

LEMMA 3.5. *Let $T_j \in B(\mathcal{H})^{-1}$, $V_j \in B(\mathcal{H})$ for $j = 1, \dots, n$ and $(p_1, \dots, p_n) \in (0, \infty)^n$ be an n -tuple of nonnegative weights not all of them equal to zero. Then*

$$\phi(v) = \left\langle \sum_{j=1}^n p_j T_j \otimes_v V_j x, x \right\rangle$$

is log-convex on $[0, 1]$ with $x \in \mathcal{H}$.

Proof. Notice that ϕ is continuous, so it is enough to prove the mid-point log-convexity. We have

$$\begin{aligned}
 \phi\left(\frac{v_1 + v_2}{2}\right) &= \left\langle \sum_{j=1}^n p_j T_j \mathbb{S}_{\frac{v_1+v_2}{2}} V_j x, x \right\rangle \\
 &= \sum_{j=1}^n p_j \left\langle T_j \mathbb{S}_{\frac{v_1+v_2}{2}} V_j x, x \right\rangle \\
 &\leq \sum_{j=1}^n p_j \left\langle T_j \mathbb{S}_{v_1} V_j x, x \right\rangle^{\frac{1}{2}} \left\langle T_j \mathbb{S}_{v_2} V_j x, x \right\rangle^{\frac{1}{2}} \quad (\text{by Lemma 3.1}) \\
 &= \sum_{j=1}^n \left\langle p_j T_j \mathbb{S}_{v_1} V_j x, x \right\rangle^{\frac{1}{2}} \left\langle p_j T_j \mathbb{S}_{v_2} V_j x, x \right\rangle^{\frac{1}{2}} \\
 &\leq \left\langle \sum_{j=1}^n p_j T_j \mathbb{S}_{v_1} V_j x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{j=1}^n p_j T_j \mathbb{S}_{v_2} V_j x, x \right\rangle^{\frac{1}{2}} \\
 &\quad (\text{by Cauchy-Schwarz inequality}) \\
 &= (\phi(v_1))^{1/2} (\phi(v_2))^{1/2}. \quad \square
 \end{aligned}$$

THEOREM 3.6. Let $T_j \in B(\mathcal{H})^{-1}$, $V_j \in B(\mathcal{H})$ for $j = 1, \dots, n$ and $(p_1, \dots, p_n) \in (0, \infty)^n$ be an n -tuple of nonnegative weights not all of them equal to zero and $v \in [0, 1]$. Then

$$\begin{aligned}
 &\sqrt{\left\langle \sum_{j=1}^n p_j |V_j|^2 x, x \right\rangle \left\langle \sum_{j=1}^n p_j |T_j|^2 x, x \right\rangle}^{2r_0} - \left\langle \sum_{j=1}^n p_j T_j \mathbb{S}_{v_1} V_j x, x \right\rangle^{2r_0} \\
 &\leq \left\langle \sum_{j=1}^n p_j |V_j|^2 x, x \right\rangle^v \left\langle \sum_{j=1}^n p_j |T_j|^2 x, x \right\rangle^{1-v} - \left\langle \sum_{j=1}^n p_j T_j \mathbb{S}_v V_j x, x \right\rangle \\
 &\leq \sqrt{\left\langle \sum_{j=1}^n p_j |V_j|^2 x, x \right\rangle \left\langle \sum_{j=1}^n p_j |T_j|^2 x, x \right\rangle}^{2R_0} - \left\langle \sum_{j=1}^n p_j T_j \mathbb{S}_{v_2} V_j x, x \right\rangle^{2R_0},
 \end{aligned}$$

where $r_0 = \min\{v, 1 - v\}$ and $R_0 = \max\{v, 1 - v\}$ and $x \in \mathcal{H}$.

Proof. Combination Lemma 3.5 and Corollary 2.4, we complete the proof. $\square$

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