

BOUNDEDNESS OF COMMUTATOR GENERATED BY FRACTIONAL INTEGRAL OPERATOR AND ORLICZ–BMO FUNCTION

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Abstract. For $\alpha \in (0, n)$ and a growth function $\varphi : [0, \infty) \rightarrow [0, \infty)$, it is proved that the commutator $[b, I_\alpha]$ generated by fractional integral operator I_α and Orlicz BMO function b is bounded from Orlicz-Hardy space $H_b^\varphi(\mathbb{R}^n)$ to Lebesgue space $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$, where $H_b^\varphi(\mathbb{R}^n)$ is a suitable Orlicz-Hardy space. Moreover, the authors also establish that the boundedness of commutator $[b, I_\alpha]$ from Orlicz-Hardy space $H^\varphi(\mathbb{R}^n)$ to weak Lebesgue space $L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)$.

1. Introduction

For $0 < \alpha < n$, the fractional integral operator I_α is defined as follows:

$$I_\alpha(g)(x) := \int_{\mathbb{R}^n} \frac{g(z)}{|x-z|^{n-\alpha}} dz \quad \text{for all } x \in \mathbb{R}^n.$$

Two classical results (see [11, Theorem 1.2.3] and [13, Theorem 4]) asserted that there exists a constant C such that

$$\|I_\alpha g\|_{L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)} \leq C \|g\|_{L^1(\mathbb{R}^n)} \quad (1.1)$$

and

$$\|I_\alpha g\|_{L^q(\mathbb{R}^n)} \leq C \|g\|_{L^p(\mathbb{R}^n)}$$

with $1 < p < q < \infty$ and $1/p - 1/q = \alpha/n$. In the framework of Hardy spaces, the boundedness of I_α is considered by Stein and Weiss [27, Theorem G] in their pioneering work on multi-variable harmonic functions, they established that

$$\|I_\alpha g\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \leq C \|g\|_{H^1(\mathbb{R}^n)}, \quad (1.2)$$

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which reveals the compatibility between the regularity of fractional integral operators and the atomic structure of Hardy spaces.

Given a locally integrable function b and a sublinear operator T , the commutator is denoted by $[b, T]$, it is given by

$$[b, T]g := bTg - T(bg),$$

where g is a continuous function whose support is compact. A famous theorem of Chanillo [4] characterized the $\text{BMO}(\mathbb{R}^n)$ space via the boundedness of $[b, I_\alpha]$, that is, $b \in \text{BMO}(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $0 < \alpha < n$ and $1 < p < \frac{n}{\alpha}$ with $1/q = 1/p - \alpha/n$. $1 < p < \frac{n}{\alpha}$ and $\frac{n}{n-\alpha} < q < \infty$ with Paluszyński [23, Theorem 1.1] proved that $[b, I_\alpha]$ does not map from $H^1(\mathbb{R}^n)$ into $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ continuously. Later, Pérez [24] introduced a subspace $\mathcal{H}_b^1(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ in the study of commutator of Calderón-Zygmund operator. Prompted by the bilinear splitting technique applied to the pair of Hardy space $H^1(\mathbb{R}^n)$ and BMO space, which is originally introduced by Bonami and Grellier [2], an important work is that Ky [15] found a suitable subspace $H_b^1(\mathbb{R}^n)$ of $H^1(\mathbb{R}^n)$ and established that $[b, I_\alpha]$ have boundedness from $H_b^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. Moreover, Ky [15] also proved that the $[b, I_\alpha]$ maps $H^1(\mathbb{R}^n)$ continuously into the weak Lebesgue space $L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)$. For more related results, we refer to the interested reader to [1, 5, 12, 14, 17, 22, 25, 26].

Motivated by the above works, the present paper studies the boundedness of commutator generated by Orlicz-BMO function and fractional integral operator within the framework of Orlicz-type spaces. To formulate our main results precisely, it is necessary to establish some definitions and notation.

A non-decreasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is termed Orlicz function if it satisfies: $\varphi(0) = 0$, $\varphi(s) > 0$ for all $s > 0$ and $\lim_{s \rightarrow \infty} \varphi(s) = \infty$. For Orlicz function φ , it is of lower type p_1 (with $p_1 \in [0, \infty)$) whenever one can find a constant $C_{p_1} > 1$ such that

$$\varphi(st) \leq C_{p_1} s^{p_1} \varphi(t) \quad \forall t \geq 0, s \in (0, 1].$$

The symbol

$$i(\varphi) := \sup\{p_1 \geq 0 : \varphi \text{ is of lower type } p_1\}$$

represents the critical lower type index of φ . For Orlicz function φ , it is of upper type p_2 (with $p_2 \in [0, \infty)$) whenever one can find a constant $C_{p_2} > 1$ such that

$$\varphi(st) \leq C_{p_2} s^{p_2} \varphi(t) \quad \forall t \geq 0, s \in [1, \infty).$$

The symbol

$$I(\varphi) := \inf\{p_2 \geq 0 : \varphi \text{ is of upper type } p_2\}$$

represents the critical upper type index of φ .

DEFINITION 1.1. We term $\varphi : [0, \infty) \rightarrow [0, \infty)$ a growth function if it is of upper type 1 and lower type p with $p \in (0, 1]$.

DEFINITION 1.2. ([16]) For a given Orlicz function φ , the space $\text{BMO}_\varphi(\mathbb{R}^n)$ stands for the set of all locally integrable functions g on $\mathbb{R}^n$ satisfying

$$\|g\|_{\text{BMO}_\varphi(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}} \int_B |g(x) - g_B| dx < \infty,$$

where the supremum ranges over all balls $B \subset \mathbb{R}^n$ and

$$g_B := \frac{1}{|B|} \int_B g(x) dx.$$

If $\varphi(t) = t$, $\text{BMO}_\varphi(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$. For more research on BMO type spaces, see [1, 6, 8, 9, 10, 28].

The Orlicz-Lebesgue space $L^\varphi(\mathbb{R}^n)$ contains all locally integrable functions g for which

$$\|g\|_{L^\varphi(\mathbb{R}^n)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \varphi(|g(x)|/\lambda) dx \leq 1 \right\} < \infty.$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual (space of tempered distributions). Define

$$\mathcal{S}_0(\mathbb{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) : \|\psi\| = \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq 1}} (1 + |x|)^{(n+1)} |\partial_x^\alpha \psi(x)| \leq 1 \right\}.$$

Let $g \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi_t(\cdot) = t^{-n} \psi(\cdot/t)$. The nontangential grand maximal function $\mathfrak{M}g$ is defined by

$$\mathfrak{M}(g)(x) := \sup_{\psi \in \mathcal{S}_0(\mathbb{R}^n)} \sup_{\substack{y \in \mathbb{R}^n \\ |y-x| < t}} |(g * \psi_t)(y)| \quad \text{for all } x \in \mathbb{R}^n.$$

DEFINITION 1.3. The Orlicz-Hardy space $H^\varphi(\mathbb{R}^n)$ is the collection of all tempered distributions g such that $\mathfrak{M}g$ belongs to $L^\varphi(\mathbb{R}^n)$. Its (quasi-)norm is

$$\|g\|_{H^\varphi(\mathbb{R}^n)} := \|\mathfrak{M}g\|_{L^\varphi(\mathbb{R}^n)}.$$

DEFINITION 1.4. Suppose that φ is a growth function satisfying

$$\frac{n}{n+1} < i(\varphi) \leq 1.$$

Given $b \in \text{BMO}_\varphi(\mathbb{R}^n)$, we denote by $H_b^\varphi(\mathbb{R}^n)$ the collection of all functions $g \in H^\varphi(\mathbb{R}^n)$ for which the function

$$[b, \mathfrak{M}]g := \mathfrak{M}((b - b(\cdot))g(\cdot))$$

is an element of $L^1(\mathbb{R}^n)$. For any $g \in H_b^\varphi(\mathbb{R}^n)$, define

$$\|g\|_{H_b^\varphi(\mathbb{R}^n)} := \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} + \|[b, \mathfrak{M}]\|_{L^1(\mathbb{R}^n)}.$$

The subsequent primary results offer a generalisation of the preceding results on the boundedness in Ky [15] to the Orlicz setting.

THEOREM 1.5. *Suppose that $0 < \alpha < n$ and φ is growth function with*

$$n/(n+1) < i(\varphi) \leq I(\varphi) < 1 \quad (1.3)$$

and

$$\begin{cases} 1/I(\varphi) > 1/i(\varphi) - 1 & \text{if } n = 1; \\ 1/I(\varphi) > (\lfloor n(1/i(\varphi) - 1) \rfloor + n - 1)/n & \text{if } n \geq 2. \end{cases} \quad (1.4)$$

If $b \in \text{BMO}_\varphi(\mathbb{R}^n)$ and $g \in H_b^\varphi(\mathbb{R}^n)$, then there exists a positive constant C such that

$$\| [b, I_\alpha] g \|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \leq C \| g \|_{H_b^\varphi(\mathbb{R}^n)}.$$

THEOREM 1.6. *Under the same assumptions on φ , α , and b as in Theorem 1.5, if $g \in H^\varphi(\mathbb{R}^n)$, then there exists constant $C > 0$ such that*

$$\| [b, I_\alpha] g \|_{L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)} \leq C \| g \|_{H^\varphi(\mathbb{R}^n)}.$$

REMARK 1.7. Theorems 1.5 and 1.6 are a further extension for the results of Ky [15, Corollary 8.1 and Theorem 8.2] in the Orlicz-type spaces framework. A study related to the Calderón-Zygmund operator, see [7].

In Section 2, we introduce the operator class $\mathcal{K}_{\varphi, \alpha}$ and prove $I_\alpha \in \mathcal{K}_{\varphi, \alpha}$. Then we establish the decomposition for commutator via wavelets in Section 3. Finally, Theorems 1.5 and 1.6 are verified in Section 4.

Finally, we give the notational rules adopted in this work. Let $\mathbb{N} := \{1, 2, \dots\}$. For every $t \in [0, \infty)$, $[t]$ denotes the supremum of all integers k with $k \leq t$. For $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$, define $B(x_0, r) := \{z \in \mathbb{R}^n : |z - x_0| < r\}$. The symbol $g_1 \lesssim g_2$ (or $g_2 \gtrsim g_1$) means $g_1 \leq Cg_2$ (or $g_2 \geq Cg_1$), and $g_1 \approx g_2$ means both $g_1 \lesssim g_2$ and $g_2 \lesssim g_1$ hold.

2. Class $\mathcal{K}_{\varphi, \alpha}$ and typical example

In order to give the definition of $\mathcal{K}_{\varphi, \alpha}$, we start by recalling from [16] the atomic description of Orlicz-Hardy spaces. Subsequently, a typical example is presented in $\mathcal{K}_{\varphi, \alpha}$.

For ball $B \subset \mathbb{R}^n$ and $q \in [1, \infty]$, the symbol L_B^q means the space of all measurable functions g on $\mathbb{R}^n$, which g satisfies $\text{supp } g \subset B$, the norm of L_B^q is defined by

$$\| g \|_{L_B^q} := \begin{cases} \left(\frac{1}{|B|} \int_B |g(x)|^q dx \right)^{1/q} < \infty, & 1 \leq q < \infty; \\ \text{ess sup}_{x \in B} |g(x)| < \infty, & q = \infty. \end{cases} \quad (2.5)$$

DEFINITION 2.1. Suppose that φ is growth function with the critical lower type exponent $i(\varphi)$. A triple (φ, q, s) is termed admissible if $q \in (1, \infty]$ and the positive integer s satisfies

$$s \geq m(\varphi) := \lfloor n(1/i(\varphi) - 1) \rfloor.$$

A locally integrable function a on $\mathbb{R}^n$ is termed (φ, q, s) -atom if the next conditions are valid:

- (i) $\text{supp } a \subset B$ for some ball $B \subset \mathbb{R}^n$;
- (ii) $\|a\|_{L_B^q} \leq \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}$;
- (iii) $\int_{\mathbb{R}^n} x^\gamma a(x) dx = 0$ for every multi-indices $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq s$.

DEFINITION 2.2. For $1 \leq q \leq \infty$ and $s \geq m(\varphi) := \lfloor n(1/i(\varphi) - 1) \rfloor$, suppose that $\{b_j\}_j$ is a sequence of multiples of (φ, q, s) -atom, each of which is supported in ball B_j . Define

$$\Lambda_q(\{b_j\}) := \inf \left\{ \lambda > 0 : \sum_j \varphi \left(\frac{\|b_j\|_{L_{B_j}^q}}{\lambda} \right) |B_j| \leq 1 \right\}.$$

DEFINITION 2.3. Let $0 < \alpha < n$ and φ be growth function such that $n/(n+1) < i(\varphi) \leq 1$. The $\mathcal{K}_{\varphi, \alpha}$ stands for the set consisting of all sublinear operator R that fulfills the following three conditions below:

- (i) R is bounded from $H^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$;
- (ii) R is bounded from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)$;
- (iii) For every $b \in \text{BMO}_\varphi(\mathbb{R}^n)$ and every $(\varphi, q, 0)$ -atom a whose support is contained in a ball $B \subset \mathbb{R}^n$ with $q \in (1, n/\alpha)$, there exists constant $C > 0$ such that

$$\|(b - b_B)Ra\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \leq C \|a\|_{L_B^q} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)}.$$

To facilitate subsequent proof, it is proposed that the result in [7, (2.3)] be employed as a lemma.

LEMMA 2.4. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ is of upper type $p_1 \in (0, 1]$ and lower type $p_2 \in (0, 1]$. Then, for ball $B(x_0, r)$ and $t > 0$,

$$\|\mathbf{1}_{B(x_0, tr)}\|_{L^\varphi(\mathbb{R}^n)} \leq \begin{cases} C^+ t^{n/p_1} \|\mathbf{1}_{B(x_0, r)}\|_{L^\varphi(\mathbb{R}^n)} & \text{as } t \in (0, 1]; \\ C^- t^{n/p_2} \|\mathbf{1}_{B(x_0, r)}\|_{L^\varphi(\mathbb{R}^n)} & \text{as } t \in [1, \infty), \end{cases}$$

where C^+ and C^- are positive constants.

LEMMA 2.5. Let $0 < \alpha < n$ and φ be growth function with $n/(n+1) < i(\varphi) \leq 1$. Then $I_\alpha \in \mathcal{K}_{\varphi, \alpha}$.

Proof. According to [20, Theorem 3.1] and [11, Theorem 1.2.3], I_α has boundedness from $H^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$, and I_α also bounded from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)$. In order to prove this lemma, it suffices to show that I_α satisfies Definition 2.3(iii).

Now, we begin verify that I_α satisfies Definition 2.3(iii). From the condition $n/(n+1) < i(\varphi)$, we should choose σ such that φ is of lower type σ with $n/(n+1) < \sigma < i(\varphi)$. So,

$$n \left(\frac{1}{\sigma} - 1 \right) < 1. \quad (2.6)$$

Suppose $b \in \text{BMO}_\varphi(\mathbb{R}^n)$ and a is $(\varphi, q, 0)$ atom with $\text{supp } a \subset B$. Denote

$$p_\alpha = \frac{n}{n-\alpha}.$$

Write

$$\begin{aligned} \|(b - b_B)I_\alpha a\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} &\leq \|(b - b_B)\mathbf{1}_{2B}I_\alpha a\|_{L^{p_\alpha}(\mathbb{R}^n)} + \|(b - b_B)\mathbf{1}_{(2B)^c}I_\alpha a\|_{L^{p_\alpha}(\mathbb{R}^n)} \\ &=: \text{I} + \text{II}. \end{aligned} \quad (2.7)$$

To estimate I, if $q \in (1, n/\alpha)$ and $1/q_0 = 1/q - \alpha/n$, then we see that I_α is bounded from $L^q(\mathbb{R}^n)$ to $L^{q_0}(\mathbb{R}^n)$. Taking $r > 1$ with $1/r + 1/q_0 = 1/p_\alpha$, by Hölder inequality, it follows that

$$\begin{aligned} \text{I} &\leq \left(\left(\int_{2B} |b(x) - b_B|^{p_\alpha \frac{r}{p_\alpha}} dx \right)^{\frac{p_\alpha}{r}} \cdot \left(\int_{2B} |I_\alpha a(x)|^{p_\alpha \frac{q_0}{p_\alpha}} dx \right)^{\frac{p_\alpha}{q_0}} \right)^{\frac{1}{p_\alpha}} \\ &= \left(\int_{2B} |b(x) - b_B|^r dx \right)^{\frac{1}{r}} \left(\int_{2B} |I_\alpha a(x)|^{q_0} dx \right)^{\frac{1}{q_0}} \\ &\lesssim \|a\|_{L^q(\mathbb{R}^n)} \left(\left(\int_{2B} |b(x) - b_{2B}|^r dx \right)^{\frac{1}{r}} + \left(\int_{2B} |b_{2B} - b_B|^r dx \right)^{\frac{1}{r}} \right). \end{aligned} \quad (2.8)$$

Further, based on the fact that φ is of lower type σ , we recall [7, Lemma 3.4 and Lemma 5.5] and obtain

$$\|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \approx \sup_{2B \subset \mathbb{R}^n} \frac{|2B|}{\|\mathbf{1}_{2B}\|_{L^\varphi(\mathbb{R}^n)}} \left(\frac{1}{|2B|} \int_{2B} |b(x) - b_{2B}|^r dx \right)^{1/r}$$

and

$$|b_{2B} - b_B| \lesssim 2^{n/\sigma} |2B|^{-1} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}.$$

From (2.8) and the two previously established estimates, it follows that

$$\begin{aligned} \text{I} &\lesssim \|a\|_{L^q(\mathbb{R}^n)} \left(|2B|^{\frac{1}{r}} \left(\frac{1}{|2B|} \int_{2B} |b(x) - b_{2B}|^r dx \right)^{1/r} + |2B|^{\frac{1}{r}} |b_{2B} - b_B| \right) \\ &\lesssim \|a\|_{L^q(\mathbb{R}^n)} \left(|2B|^{\frac{1}{r}-1} \|\mathbf{1}_{2B}\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} + |2B|^{\frac{1}{r}} 2^{n/\sigma} |2B|^{-1} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \right) \end{aligned} \quad (2.9)$$

$$\begin{aligned}
&\lesssim \|a\|_{L^q(\mathbb{R}^n)} \|B|^{\frac{1}{r}-1} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \\
&= \|a\|_{L_B^q} \|B|^{\frac{1}{q}+\frac{1}{r}-1} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \\
&= \|a\|_{L_B^q} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)},
\end{aligned}$$

where the penultimate step used $\|a\|_{L^q(\mathbb{R}^n)} = \|a\|_{L_B^q} |B|^{1/q}$ (see (2.5)) and the last step used

$$\frac{1}{q} + \frac{1}{r} - 1 = \frac{1}{q_0} + \frac{\alpha}{n} + \frac{1}{p_\alpha} - \frac{1}{q_0} - 1 = \frac{\alpha}{n} + \frac{1}{p_\alpha} - 1 = 0.$$

Below, we consider II. Denote $B = B(x_0, r)$. Due to the fact that a is a $(\varphi, q, 0)$ -atom supported on a ball $B \subset \mathbb{R}^n$, we have $\text{supp } a \subset B$ and $\int_{\mathbb{R}^n} a(x) dx = 0$. For any $z \in B$ and $x \in (2^{j+1}B) \setminus (2^jB)$ with $j \in \mathbb{N}$, we find a point, which is labeled by ξ_z , lying on the straight segment joining z to x_0 , satisfying that

$$\begin{aligned}
|I_\alpha a(x)| &= \left| \int_B a(z) \left(\frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|x-x_0|^{n-\alpha}} \right) dz \right| \\
&\leq \int_B \left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|x-x_0|^{n-\alpha}} \right| |a(z)| dz \\
&\leq \int_B |z-x_0| \left| (n-\alpha) |x-\xi_z|^{-(n-\alpha+1)} \right| |a(z)| dz \\
&\leq (n-\alpha)r \cdot \left(\frac{2^j r}{2} \right)^{-(n-\alpha+1)} \int_B |a(z)| dz,
\end{aligned} \tag{2.10}$$

where the last step utilized

$$|x - \xi_z| \geq |x - x_0| - |x_0 - \xi_z| \geq |x - x_0| - \frac{|x - x_0|}{2} = \frac{|x - x_0|}{2} \geq \frac{2^j r}{2}.$$

In addition, by $\|a\|_{L^q(\mathbb{R}^n)} = \|a\|_{L_B^q} |B|^{1-1/q'}$, we get

$$\int_B |a(z)| dz \leq |B|^{1/q'} \|a\|_{L^q} = |B| \|a\|_{L_B^q},$$

which and (2.10) imply

$$|I_\alpha a(x)| \lesssim r \cdot r^n \cdot r^{-(n-\alpha+1)} \cdot 2^{-j(n-\alpha+1)} \|a\|_{L_B^q} = r^\alpha \cdot 2^{-j(n-\alpha+1)} \|a\|_{L_B^q}.$$

This, along with Lemma 2.4 and [7, Lemma 3.4 and Lemma 5.5], yields

$$\begin{aligned}
\Pi &= \left(\int_{(2B)^c} |I_\alpha a(x) (b(x) - b_B)|^{p_\alpha} dx \right)^{\frac{1}{p_\alpha}} \\
&\leq \sum_{j=1}^{\infty} \left(\int_{(2^{j+1}B) \setminus (2^jB)} |I_\alpha a(x) (b(x) - b_B)|^{p_\alpha} dx \right)^{\frac{1}{p_\alpha}}
\end{aligned}$$

$$\begin{aligned}
&\lesssim r^\alpha \|a\|_{L_B^q} \sum_{j=1}^{\infty} 2^{-j(n-\alpha+1)} \left[\left(\int_{2^{j+1}B} |b(x) - b_{2^{j+1}B}|^{p_\alpha} dx \right)^{\frac{1}{p_\alpha}} \right. \\
&\quad \left. + \left(\int_{2^{j+1}B} |b_B - b_{2^{j+1}B}|^{p_\alpha} dx \right)^{\frac{1}{p_\alpha}} \right] \\
&\lesssim r^\alpha \|a\|_{L_B^q} \sum_{j=1}^{\infty} 2^{-j(n-\alpha+1)} \left(|2^{j+1}B|^{\frac{1}{p_\alpha}-1} \|\mathbf{1}_{2^{j+1}B}\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \right. \\
&\quad \left. + |2^{j+1}B|^{\frac{1}{p_\alpha}} 2^{\frac{(j+1)n}{\sigma}} |2^{j+1}B|^{-1} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \right) \\
&\lesssim r^\alpha \|a\|_{L_B^q} \sum_{j=1}^{\infty} 2^{-j(n-\alpha+1)} |2^{j+1}B|^{\frac{1}{p_\alpha}-1} 2^{\frac{(j+1)n}{\sigma}} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \\
&\approx r^\alpha r^{n(\frac{1}{p_\alpha}-1)} \|a\|_{L_B^q} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \left(\sum_{j=1}^{\infty} 2^{-j(1+n-\alpha)+n(j+1)(\frac{1}{p_\alpha}-1)+\frac{(j+1)n}{\sigma}} \right).
\end{aligned}$$

Further, by $\alpha = -n(1/p_\alpha - 1)$ and

$$-j(1+n-\alpha) + n(j+1) \left(\frac{1}{p_\alpha} - 1 \right) + \frac{(j+1)n}{\sigma} = -j \left(1 - n \left(\frac{1}{\sigma} - 1 \right) \right) + \frac{n}{\sigma} - \alpha,$$

we use (2.6) to obtain that

$$\Pi \lesssim \|a\|_{L_B^q} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \sum_{j=1}^{\infty} 2^{-j(1-n(\frac{1}{\sigma}-1))} \lesssim \|a\|_{L_B^q} \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)}. \quad (2.11)$$

Substituting (2.9) and (2.11) into (2.7) yields that I_α satisfies Definition 2.3(iii). $\square$

3. Decomposition of commutator

This section is devoted to find a wavelet decomposition of commutator, where the wavelet basis is introduced in [21].

Denote $F = \{0, 1\}^n \setminus \{(0, \dots, 0)\}$, the function of all dyadic cubes in $\mathbb{R}^n$ is written as $\mathcal{D}$. Let $\{\psi_I^\lambda\}_{I \in \mathcal{D}, \lambda \in F}$ be a normalized and orthogonal wavelet basis of $L^2(\mathbb{R}^n)$ with compact support and sufficient smoothness (see [21]).

Any $g \in L^2(\mathbb{R}^n)$ can be expanded as

$$g = \sum_{I \in \mathcal{D}} \sum_{\lambda \in F} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda \quad \text{for all } g \in L^2(\mathbb{R}^n).$$

According to [2], we define the bilinear operators Π_i ($i = 1, 2, 3, 4$) by

$$\Pi_1(g_1, g_2) = \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in F} \langle g_1, \varphi_I \rangle \langle g_2, \psi_{I'}^\lambda \rangle \varphi_I \psi_{I'}^\lambda, \quad (3.12)$$

$$\Pi_2(g_1, g_2) = \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in F} \langle g_1, \psi_I^\lambda \rangle \langle g_2, \varphi_{I'} \rangle \psi_I^\lambda \varphi_{I'}, \quad (3.13)$$

$$\Pi_3(g_1, g_2) = \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\substack{\lambda, \lambda' \in F \\ (I, \lambda) \neq (I', \lambda')}} \langle g_1, \psi_I^\lambda \rangle \langle g_2, \psi_{I'}^{\lambda'} \rangle \psi_I^\lambda \psi_{I'}^{\lambda'} \quad (3.14)$$

and

$$\Pi_4(g_1, g_2) = \sum_{I \in \mathcal{D}} \sum_{\lambda \in F} \langle g_1, \psi_I^\lambda \rangle \langle g_2, \psi_I^\lambda \rangle (\psi_I^\lambda)^2. \quad (3.15)$$

PROPOSITION 3.1. *Suppose that φ is a growth function, its lower and upper critical indices satisfy (1.3). Respectively, each Π_i ($i = 1, 2, 3, 4$) corresponds to the expression in (3.12), (3.13), (3.14) and (3.15). Consequently, the statements below are true.*

- (i) Both Π_1 and Π_3 are bounded from $H^\varphi(\mathbb{R}^n) \times \text{BMO}_\varphi(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$;
- (ii) Π_4 is bounded from $H^\varphi(\mathbb{R}^n) \times \text{BMO}_\varphi(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$;
- (iii) Suppose that $I(\varphi) \in (n/(n+1), 1)$ and (1.4) holds for φ , a sequence of dyadic cubes are denoted by R_j . Given $b \in \text{BMO}_\varphi$ and a finite atomic decomposition $g = \sum_{j=1}^M a_j$ with $g \in H^\varphi(\mathbb{R}^n)$, where each a_j can be written as a constant times a (φ, ∞, s) atom, and $\text{supp } a_j \subset R_j$. Then

$$\sum_{j=1}^M \|\Pi_2(a_j, b - b_{B_j})\|_{H^1(\mathbb{R}^n)} \lesssim \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)},$$

where b_{B_j} is the average of b over ball B_j and $B_j \supseteq R_j$ is the smallest ball.

Proof. Note that (i) and (ii) follow from [7, Propositions 3.14, 3.16 and 3.17]. By [7, (5.17)], we obtain that (iii) holds. $\square$

Next we will establish a decomposition for the commutator $[b, T]$. The proof relies on a wavelet orthonormal basis $\{\psi_I^\lambda\}$ (see [21]) and the associated bilinear operators Π_i introduced in [3].

LEMMA 3.2. *Given a growth function φ and its lower and upper critical indices satisfy (1.3) and (1.4). Let Π_4 be defined as (3.15) and $T \in \mathcal{K}_{\varphi, \alpha}$ with $0 < \alpha < n$. For any $b \in \text{BMO}_\varphi(\mathbb{R}^n)$ and any $g \in H^\varphi(\mathbb{R}^n)$, there exists an operator*

$$\mathfrak{T} : H^\varphi(\mathbb{R}^n) \times \text{BMO}_\varphi(\mathbb{R}^n) \rightarrow L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$$

such that

$$[b, T]g = \mathfrak{T}(g, b) - T(\Pi_4(g, b)). \quad (3.16)$$

Proof. Based on the argument of [19, Proposition 6.7], the following three steps will be used to prove the desired result.

Step 1: Verifying (3.16) under the case that g has a finite linear combination of wavelets. We apply [7, Lemma 3.3, (c) and (a)] to obtain that g has finite atomic decomposition

$$g = \sum_{k=1}^M a_k$$

with

$$a_k = \sum_{\substack{I \in \mathcal{D} \\ I \subset R_k}} \sum_{\lambda \in F} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda \quad (k = 1, \dots, M)$$

and

$$\Lambda_\infty(\{a_k\}_{k=1}^M) \lesssim \|g\|_{H^\varphi(\mathbb{R}^n)},$$

where $\text{supp } a_k \subseteq R_k$ and $\langle g, \psi_I^\lambda \rangle$ is the coefficients of a_k . Further, let B_k be the smallest ball containing dyadic cube R_k . According to [7, Proposition 2.5], we have

$$\sum_{k=1}^M \|a_k\|_{L_{B_k}^\infty} \|\mathbf{1}_{B_k}\|_{L^\varphi(\mathbb{R}^n)} \lesssim \Lambda_\infty(\{a_k\}_{k=1}^M) \lesssim \|g\|_{H^\varphi(\mathbb{R}^n)}. \quad (3.17)$$

Denote

$$\mathcal{W}(g, b) := T \left(\sum_{k=1}^M [(b - b_{B_k})a_k - \Pi_2(a_k, b - b_{B_k})] \right).$$

For almost every $x \in \mathbb{R}^n$, using a standard algebraic identity (see [19, (6.9)]), it holds

$$\begin{aligned} b(x)g(\cdot) - b(\cdot)g(\cdot) &= \sum_{k=1}^M \left[(b(x) - b_{B_k})a_k(\cdot) - \Pi_2(a_k, b - b_{B_k})(\cdot) \right] \\ &\quad - \Pi_1(g, b)(\cdot) - \Pi_3(g, b)(\cdot) - \Pi_4(g, b)(\cdot). \end{aligned}$$

This implies

$$[b, T]g = \mathcal{W}(g, b) - T(\Pi_1(g, b)) - T(\Pi_3(g, b)) - T(\Pi_4(g, b)). \quad (3.18)$$

Now, if we define

$$\mathfrak{T}(g, b) = \mathcal{W}(g, b) - T(\Pi_1(g, b)) - T(\Pi_3(g, b)),$$

then immediately obtain

$$[b, T]g = \mathfrak{T}(g, b) - T(\Pi_4(g, b)),$$

which is the desired equality.

Step 2: For $b \in \text{BMO}_\varphi(\mathbb{R}^n)$ and $g \in H^\varphi(\mathbb{R}^n)$ with finite linear combination of wavelets, it follows that

$$\|\mathfrak{T}(g, b)\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \lesssim \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)}.$$

By the definitions of $\mathfrak{T}$ and $\mathcal{W}$, it follows that

$$\begin{aligned} & \|\mathfrak{T}(g, b)\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \\ & \leq \|\mathcal{W}(g, b)\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \|T(\Pi_1(g, b))\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \|T(\Pi_3(g, b))\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \quad (3.19) \\ & \leq \sum_{k=1}^M \|(b(\cdot) - b_{B_k})Ta_k\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \left\| T\left(\sum_{k=1}^M \Pi_2(a_k, b - b_{B_k})\right) \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \\ & \quad + \|T(\Pi_1(g, b))\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \|T(\Pi_3(g, b))\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \\ & \lesssim \sum_{k=1}^M \|a_k\|_{L_{B_k}^q} \|\mathbf{1}_{B_k}\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} + \sum_{k=1}^M \|\Pi_2(a_k, b - b_{B_k})\|_{H^1(\mathbb{R}^n)} \\ & \quad + \|\Pi_1(g, b)\|_{H^1(\mathbb{R}^n)} + \|\Pi_3(g, b)\|_{H^1(\mathbb{R}^n)}, \end{aligned}$$

where the last step used $T \in \mathcal{K}_{\varphi, \alpha}$ and the definition of $\mathcal{K}_{\varphi, \alpha}$. In addition, applying Proposition 3.1, we have

$$\|\Pi_i(g, b)\|_{H^1(\mathbb{R}^n)} \lesssim \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)}, \quad i = 1, 3.$$

By this, (3.19), Proposition 3.1(iii), (3.17) and $\|a_k\|_{L_{B_k}^q} \leq \|a_k\|_{L_{B_k}^\infty}$ with $q \in (1, n/\alpha)$, we see that

$$\begin{aligned} & \|\mathfrak{T}(g, b)\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \\ & \lesssim \sum_{k=1}^M \|a_k\|_{L_{B_k}^q} \|\mathbf{1}_{B_k}\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} + \sum_{k=1}^M \|\Pi_2(a_k, b - b_{B_k})\|_{H^1(\mathbb{R}^n)} \\ & \quad + \|\Pi_1(g, b)\|_{H^1(\mathbb{R}^n)} + \|\Pi_3(g, b)\|_{H^1(\mathbb{R}^n)} \\ & \lesssim \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} + \sum_{k=1}^M \|a_k\|_{L_{B_k}^q} \|\mathbf{1}_{B_k}\|_{L^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} \\ & \quad + \sum_{k=1}^M \|\Pi_2(a_k, b - b_{B_k})\|_{H^1(\mathbb{R}^n)} \\ & \lesssim \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)}, \end{aligned}$$

as desired.

Step 3: The proof of (3.16). Based on the facts obtained in Steps 1 and 2, we repeat the argument of [7, (5.18)] to obtain (3.16). This implies Lemma 3.2. $\square$

4. Proofs of Theorems 1.5 and 1.6

The primary objective of this section is to complete the proofs of Theorems 1.5 and 1.6 by using Lemma 4.1. We first recall the following lemma from [7, Theorem 5.10].

LEMMA 4.1. *Suppose that φ is a growth function and it satisfies (1.3) and (1.4). If $b \in \text{BMO}_\varphi(\mathbb{R}^n)$, then $g \in H_b^\varphi(\mathbb{R}^n)$ is equivalent to $\Pi_4(g, b) \in H^1(\mathbb{R}^n)$ with*

$$\|g\|_{H_b^\varphi(\mathbb{R}^n)} \approx \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} + \|\Pi_4(g, b)\|_{H^1(\mathbb{R}^n)}.$$

Proof of Theorem 1.5. Under the assumptions that $b \in \text{BMO}_\varphi(\mathbb{R}^n)$ and $g \in H_b^\varphi(\mathbb{R}^n)$, by Lemma 4.1, we have $\Pi_4(g, b) \in H^1(\mathbb{R}^n)$ and

$$\|g\|_{H_b^\varphi(\mathbb{R}^n)} \approx \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} + \|\Pi_4(g, b)\|_{H^1(\mathbb{R}^n)}. \quad (4.20)$$

Based on Lemma 2.5, we use Lemma 3.2 to obtain that there exists operator

$$\mathfrak{T} : H^\varphi(\mathbb{R}^n) \times \text{BMO}_\varphi(\mathbb{R}^n) \rightarrow L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$$

such that

$$[b, I_\alpha]g = \mathfrak{T}(g, b) - I_\alpha(\Pi_4(g, b)).$$

From this and (1.2), it follows that

$$\begin{aligned} \|[b, I_\alpha]g\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} &= \|\mathfrak{T}(g, b) - I_\alpha(\Pi_4(g, b))\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \\ &\leq \|\mathfrak{T}(g, b)\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \|I_\alpha(\Pi_4(g, b))\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \\ &\lesssim \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)} + \|\Pi_4(g, b)\|_{H^1(\mathbb{R}^n)} \\ &\approx \|g\|_{H_b^\varphi(\mathbb{R}^n)}, \end{aligned}$$

which proves Theorem 1.5. $\square$

Proof of Theorem 1.6. By Lemma 3.2, we find that there exists an operator

$$\mathfrak{T} : H^\varphi(\mathbb{R}^n) \times \text{BMO}_\varphi(\mathbb{R}^n) \rightarrow L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$$

such that for any $b \in \text{BMO}_\varphi(\mathbb{R}^n)$ and $g \in H^\varphi(\mathbb{R}^n)$,

$$[b, I_\alpha]g(y) = \mathfrak{T}(g, b)(y) - I_\alpha(\Pi_4(g, b))(y) \quad \text{for a. e. } y \in \mathbb{R}^n.$$

So, using (1.1) and Proposition 3.1, we have

$$\begin{aligned} \|[b, I_\alpha]g\|_{L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)} &\lesssim \|\mathfrak{T}(g, b)\|_{L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)} + \|I_\alpha(\Pi_4(g, b))\|_{L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)} \\ &\lesssim \|\mathfrak{T}(g, b)\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \|\Pi_4(g, b)\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \|g\|_{H^\varphi(\mathbb{R}^n)} \|b\|_{\text{BMO}_\varphi(\mathbb{R}^n)}. \end{aligned}$$

This implies Theorem 1.6. $\square$

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