

SHARP CONSTANTS FOR A WEIGHTED GEOMETRIC–MEAN FAMILY ON BOXES

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Abstract. We determine the optimal constant in a family of weighted geometric-mean inequalities on a compact box when the right-hand side is any premean that dominates the weighted geometric mean and agrees with it on constant vectors. The proof reduces the n -variable problem to a scalar extremal problem on the box. The resulting sharp constant depends only on the box endpoints and the exponent, and not on the dimension, the weights, or the particular dominating premean. As applications, we obtain a complete phase diagram for the two-parameter weighted AM-GM family, recover the original one-parameter case as a diagonal slice, characterize equality, and derive Lambert W representations and asymptotic expansions for the boundary curves.

1. Introduction

Let $n \in \mathbb{N}$, let $\alpha_i > 0$ with $\sum_{i=1}^n \alpha_i = 1$, and define the weighted geometric and arithmetic means by

$$G_\alpha(a) := \prod_{i=1}^n a_i^{\alpha_i}, \quad A_\alpha(a) := \sum_{i=1}^n \alpha_i a_i, \quad a = (a_1, \dots, a_n) \in (0, \infty)^n.$$

The weighted arithmetic-geometric mean inequality

$$G_\alpha(a) \leq A_\alpha(a) \tag{1.1}$$

is one of the most classical inequalities in analysis; see Hardy, Littlewood and Pólya [5]. Its relations with Hölder's inequality and weighted power means, as well as many refinements and reverse estimates, have been studied extensively; see, for instance, [1, 2, 4, 6, 7, 8, 9].

In this paper we consider the parameter family

$$\left(\prod_{i=1}^n a_i^{k_i} \right)^t \leq t \sum_{i=1}^n k_i a_i, \quad a_i > 0, \tag{1.2}$$

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where $k_i > 0$ are not assumed to be normalized. Writing

$$K := \sum_{i=1}^n k_i, \quad \alpha_i := \frac{k_i}{K}, \quad s := Kt,$$

we may rewrite (1.2) as

$$G_\alpha(a)^s \leq sA_\alpha(a). \quad (1.3)$$

When the variables range over all of $(0, \infty)^n$, the only admissible exponent is $s = 1$, so (1.3) reduces exactly to (1.1). The situation changes under box constraints. If $a_i \in [m, M]$, then the entire problem is controlled by the scalar variable $G_\alpha(a) \in [m, M]$.

The main purpose of the paper is to isolate this phenomenon in a sharper and more flexible form. We study

$$G_\alpha(a)^s \leq c\mathcal{M}(a), \quad a \in [m, M]^n, \quad (1.4)$$

where $\mathcal{M}$ is any G_α -dominating premean, namely, a reflexive map that dominates the weighted geometric mean. Our main theorem determines the best constant in (1.4) and shows that it depends only on the interval $[m, M]$ and the exponent s . In particular, it is independent of the dimension, the weight vector and the particular choice of the dominating premean.

Specializing to $\mathcal{M} = A_\alpha$ yields two consequences. First, we obtain the optimal coefficient in the two-parameter family

$$\left(\prod_{i=1}^n a_i^{k_i} \right)^t \leq \lambda \sum_{i=1}^n k_i a_i, \quad a \in [m, M]^n.$$

Second, the original one-parameter inequality (1.2) appears as the diagonal slice $\lambda = t$ of this sharp phase boundary. We also derive Lambert W representations and asymptotic expansions for the boundary curves.

The novelty therefore lies not in a new refinement term for (1.1), but in a universal sharp-constant principle for a whole weighted geometric-mean family under box constraints.

2. A universal sharp-constant theorem

DEFINITION 2.1. Fix $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. A map $\mathcal{M} : (0, \infty)^n \rightarrow (0, \infty)$ is called a G_α -dominating premean if

$$G_\alpha(a) \leq \mathcal{M}(a) \quad \text{for all } a \in (0, \infty)^n, \quad \mathcal{M}(x\mathbf{1}) = x \quad (x > 0). \quad (2.1)$$

It is called *strict* if equality in $G_\alpha(a) \leq \mathcal{M}(a)$ holds only for constant vectors.

The term *premean* is used here because no internality condition

$$\min_{1 \leq i \leq n} a_i \leq \mathcal{M}(a) \leq \max_{1 \leq i \leq n} a_i$$

is imposed. Thus Definition 2.1 requires reflexivity and domination of G_α , but not the full mean-value property.

The weighted arithmetic mean A_α is a strict G_α -dominating premean by (1.1). More generally, every weighted power mean

$$P_{r,\alpha}(a) := \left(\sum_{i=1}^n \alpha_i a_i^r \right)^{1/r} \quad (r > 0)$$

is a strict G_α -dominating premean by the monotonicity of power means; see [6, 8].

THEOREM 2.2. (Universal sharp constant) *Let $0 < m \leq M < \infty$, let $s > 0$, and let $\mathcal{M}$ be a G_α -dominating premean. Then the following are equivalent:*

(i) *The inequality*

$$G_\alpha(a)^s \leq c \mathcal{M}(a) \quad (2.2)$$

holds for every $a \in [m, M]^n$.

(ii) *One has*

$$c \geq C_{m,M}(s) := \max_{x \in [m,M]} x^{s-1} = \begin{cases} m^{s-1}, & 0 < s < 1, \\ 1, & s = 1, \\ M^{s-1}, & s > 1. \end{cases} \quad (2.3)$$

In particular, $C_{m,M}(s)$ is the sharp constant in (2.2).

Proof. Assume that $c \geq C_{m,M}(s)$ and let $a \in [m, M]^n$. Since every coordinate of a lies in $[m, M]$, the weighted geometric mean also belongs to $[m, M]$. Hence

$$G_\alpha(a)^{s-1} \leq C_{m,M}(s) \leq c.$$

Multiplying by $G_\alpha(a) > 0$ and using (2.1), we obtain

$$G_\alpha(a)^s \leq c G_\alpha(a) \leq c \mathcal{M}(a).$$

Thus (2.2) holds for every $a \in [m, M]^n$.

Conversely, suppose that (2.2) holds for all $a \in [m, M]^n$. Taking $a = x\mathbf{1}$ with $x \in [m, M]$ and using $G_\alpha(x\mathbf{1}) = \mathcal{M}(x\mathbf{1}) = x$, we get

$$x^s \leq cx, \quad x \in [m, M],$$

that is,

$$x^{s-1} \leq c, \quad x \in [m, M].$$

Taking the maximum over $[m, M]$ gives $C_{m,M}(s) \leq c$. $\square$

COROLLARY 2.3. (Equality at the sharp constant) *Let $\mathcal{M}$ be a strict G_α -dominating premean and let $c = C_{m,M}(s)$ in (2.2). Then equality holds in (2.2) if and only if one of the following mutually exclusive conditions is satisfied:*

$$\begin{aligned} a &= m\mathbf{1} & (0 < s < 1), \\ \text{or } a &= x\mathbf{1} \text{ for some } x \in [m, M] & (s = 1), \\ \text{or } a &= M\mathbf{1} & (s > 1). \end{aligned}$$

Proof. The proof is obtained by tracking the two inequalities

$$G_\alpha(a)^s \leq cG_\alpha(a) \leq c\mathcal{M}(a).$$

If $0 < s < 1$, then $c = m^{s-1}$ and equality in the first step forces $G_\alpha(a) = m$. Since every coordinate is at least m , we must have $a = m\mathbf{1}$. The case $s > 1$ is analogous and yields $a = M\mathbf{1}$. If $s = 1$, then $c = 1$ and equality is equivalent to $G_\alpha(a) = \mathcal{M}(a)$, hence to $a_1 = \cdots = a_n$ by strictness. $\square$

COROLLARY 2.4. (Global unbounded case) *Let $\mathcal{M}$ be a G_α -dominating premean. Then*

$$G_\alpha(a)^s \leq c\mathcal{M}(a) \quad (a \in (0, \infty)^n)$$

holds for all positive vectors if and only if $s = 1$ and $c \geq 1$.

Proof. Testing the inequality on constant vectors gives $x^{s-1} \leq c$ for all $x > 0$, which is possible only when $s = 1$. If $s = 1$, the claim follows from (2.1). $\square$

3. Arithmetic consequences and the phase diagram

Let $k_i > 0$ and let $K = \sum_{i=1}^n k_i$, $\alpha_i = k_i/K$. Then

$$A_\alpha(a) = \frac{1}{K} \sum_{i=1}^n k_i a_i, \quad G_\alpha(a)^K = \prod_{i=1}^n a_i^{k_i}.$$

THEOREM 3.1. (Sharp coefficient for the two-parameter family) *Let $0 < m \leq M < \infty$ and $t > 0$. Then*

$$\left(\prod_{i=1}^n a_i^{k_i} \right)^t \leq \lambda \sum_{i=1}^n k_i a_i, \quad a \in [m, M]^n, \quad (3.1)$$

holds for every $a \in [m, M]^n$ if and only if

$$\lambda \geq \Lambda_{m,M}(t) := \frac{1}{K} C_{m,M}(Kt) = \frac{1}{K} \max\{m^{Kt-1}, M^{Kt-1}\}. \quad (3.2)$$

Equivalently,

$$\Lambda_{m,M}(t) = \begin{cases} \frac{m^{Kt-1}}{K}, & 0 < t < 1/K, \\ \frac{1}{K}, & t = 1/K, \\ \frac{M^{Kt-1}}{K}, & t > 1/K. \end{cases}$$

For the sharp constant $\lambda = \Lambda_{m,M}(t)$, equality holds precisely at $m\mathbf{1}$, any constant vector $x\mathbf{1}$ with $x \in [m, M]$, and $M\mathbf{1}$ according as $Kt < 1$, $Kt = 1$, and $Kt > 1$.

Proof. Rewrite (3.1) as

$$G_\alpha(a)^{Kt} \leq (\lambda K) A_\alpha(a).$$

Now apply Theorem 2.2 and Corollary 2.3 with $s = Kt$ and $c = \lambda K$. $\square$

COROLLARY 3.2. (Global phase diagram) *The inequality*

$$\left(\prod_{i=1}^n a_i^{k_i} \right)^t \leq \lambda \sum_{i=1}^n k_i a_i, \quad a_i > 0,$$

holds for all positive variables if and only if

$$t = \frac{1}{K}, \quad \lambda \geq \frac{1}{K}.$$

In particular, for the diagonal choice $\lambda = t$, the inequality holds for every $a \in (0, \infty)^n$ if and only if $t = 1/K$.

Proof. This is Corollary 2.4 with $\mathcal{M} = A_\alpha$, $s = Kt$ and $c = \lambda K$. $\square$

To describe the diagonal slice $\lambda = t$, define

$$\phi(s) := \begin{cases} s^{1/(s-1)}, & s > 0, s \neq 1, \\ e, & s = 1. \end{cases} \quad (3.3)$$

The function $\phi : (0, \infty) \rightarrow (1, \infty)$ is continuous and strictly decreasing, with

$$\lim_{s \downarrow 0} \phi(s) = \infty, \quad \phi(1) = e, \quad \lim_{s \rightarrow \infty} \phi(s) = 1.$$

COROLLARY 3.3. (Exact admissible interval for the diagonal family) *Let $0 < m \leq M < \infty$. Define*

$$s_-(m) := \begin{cases} 1, & 0 < m \leq e, \\ \text{the unique solution of } m^{s-1} = s \text{ in } (0, 1), & m > e, \end{cases}$$

and

$$s_+(M) := \begin{cases} \infty, & 0 < M \leq 1, \\ \text{the unique solution of } M^{s-1} = s \text{ in } (1, \infty), & 1 < M < e, \\ 1, & M \geq e. \end{cases}$$

Then (1.2) holds for every $a \in [m, M]^n$ if and only if

$$\frac{s_-(m)}{K} \leq t \leq \frac{s_+(M)}{K}. \quad (3.4)$$

Equivalently,

$$\begin{cases} m^{Kt-1} \leq Kt, & 0 < Kt < 1, \\ C_{m,M}(1) = 1 = Kt, & Kt = 1, \\ M^{Kt-1} \leq Kt, & Kt > 1. \end{cases}$$

If $m \leq e \leq M$, then the admissible interval collapses to the single point $t = 1/K$.

Proof. By Theorem 3.1, the diagonal family is admissible exactly when $t \geq \Lambda_{m,M}(t)$, or equivalently,

$$Kt \geq C_{m,M}(Kt).$$

The piecewise criterion follows from (2.3), and the description (3.4) follows from the monotonicity of ϕ . $\square$

PROPOSITION 3.4. (Log-convex phase boundary) For every fixed box $[m, M]$, the function $\Lambda_{m,M} : (0, \infty) \rightarrow (0, \infty)$ is continuous and log-convex. In fact,

$$\log(K\Lambda_{m,M}(t)) = \max\{(Kt-1)\log m, (Kt-1)\log M\},$$

so $\log(K\Lambda_{m,M}(t))$ is the maximum of two affine functions of t .

Proof. The formula is an immediate consequence of (3.2). The maximum of affine functions is convex. $\square$

4. Boundary curves and Lambert W formulas

PROPOSITION 4.1. (Lambert W representations) For $1 < x < e$ and $x > e$, respectively, let $s_+(x) > 1$ and $s_-(x) \in (0, 1)$ be the unique nontrivial solutions of $x^{s-1} = s$. Then

$$s_+(x) = -\frac{1}{\log x} W_{-1}\left(-\frac{\log x}{x}\right), \quad 1 < x < e, \quad (4.1)$$

and

$$s_-(x) = -\frac{1}{\log x} W_0\left(-\frac{\log x}{x}\right), \quad x > e. \quad (4.2)$$

Proof. Starting from $x^{s-1} = s$, we write

$$e^{(s-1)\log x} = s, \quad se^{-s\log x} = x^{-1}.$$

Multiplying by $-\log x$ gives

$$(-s\log x)e^{-s\log x} = -\frac{\log x}{x}.$$

Hence

$$-s\log x = W\left(-\frac{\log x}{x}\right) \implies s = -\frac{1}{\log x}W\left(-\frac{\log x}{x}\right).$$

For $1 < x < e$, the relevant root satisfies $s > 1$ and therefore corresponds to the branch W_{-1} . For $x > e$, the relevant root lies in $(0, 1)$ and corresponds to the principal branch W_0 ; see [3]. $\square$

PROPOSITION 4.2. (Monotonicity and asymptotics) *The functions $s_+(x)$ on $(1, e)$ and $s_-(x)$ on (e, ∞) are strictly decreasing. Moreover,*

$$\begin{aligned} s_+(x) &= 1 + \frac{2}{e}(e-x) + O((e-x)^2), & x \uparrow e, \\ s_-(x) &= 1 - \frac{2}{e}(x-e) + O((x-e)^2), & x \downarrow e, \\ s_+(x) &= \frac{\log \frac{1}{x-1} + \log \log \frac{1}{x-1} + O(1)}{x-1}, & x \downarrow 1, \\ s_-(x) &= \frac{1}{x} + \frac{\log x}{x^2} + O\left(\frac{(\log x)^2}{x^3}\right), & x \rightarrow \infty. \end{aligned}$$

Consequently,

$$t_{\min}(m) := \frac{s_-(m)}{K}, \quad t_{\max}(M) := \frac{s_+(M)}{K},$$

inherit the same monotonicity and asymptotic behavior up to the factor $1/K$.

Proof. Since ϕ in (3.3) is strictly decreasing and $s_{\pm}$ are the corresponding inverse branches, both functions are strictly decreasing.

For the expansions near e , write $s = 1 + u$. Then

$$\phi(1+u) = (1+u)^{1/u} = e\left(1 - \frac{u}{2} + O(u^2)\right), \quad u \rightarrow 0,$$

which yields the first two formulas after inversion. The asymptotic behavior as $x \downarrow 1$ and $x \rightarrow \infty$ follows from (4.1), (4.2) and the standard branch expansions of the Lambert W function in [3]. $\square$

5. Examples and final remarks

EXAMPLE 5.1. Let $k_1 = k_2 = k_3 = 1$, so $K = 3$. On the box $1 \leq a, b, c \leq 2$, Corollary 3.3 gives

$$(abc)^t \leq t(a+b+c) \quad (1 \leq a, b, c \leq 2)$$

if and only if $1/3 \leq t \leq 2/3$, because $2^{s-1} = s$ has the nontrivial root $s = 2$.

EXAMPLE 5.2. With the same weights, replace the arithmetic mean by the quadratic mean. Theorem 2.2, applied to the quadratic mean, yields

$$(abc)^t \leq 3t \left(\frac{a^2 + b^2 + c^2}{3} \right)^{1/2} \quad (1 \leq a, b, c \leq 2)$$

for all $a, b, c \in [1, 2]$ if and only if $1/3 \leq t \leq 2/3$. Thus the admissible t -range is unchanged when the arithmetic mean is replaced by any dominating power mean.

REMARK 5.3. Theorem 2.2 shows that the box-constrained phase diagram is determined solely by the interval $[m, M]$ and the exponent s . The dimension, the weights and the particular dominating premean affect only the normalization $s = Kt$ and the equality structure. This suggests analogous sharp-constant problems for other families built from lower and upper means.

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