

REFINEMENTS OF THE HERMITE–HADAMARD INEQUALITY BY INCREASING FUNCTIONS AND THEIR PSEUDO-INVERSES

LÁSZLÓ HORVÁTH

(Communicated by S. Ivelić Bradanović)

Abstract. In this paper, we establish some generalizations and extensions of a recent refinement of the Hermite-Hadamard inequality by using the notion of pseudo-inverse. The relationship between the obtained refinements are studied. We generate sequences of refinements and analyze their convergence. Finally, some applications are given.

1. Introduction

The classical Hermite-Hadamard inequality provides a lower and an upper estimations for the integral average of a convex function $f : [a, b] \rightarrow \mathbb{R}$, involving the midpoint and the endpoints of the domain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}.$$

There are many papers dealing with generalizations, extensions and refinements of these inequalities (see without claiming to be exhaustive [1], [2], [5], [7], [8], [10], [11], [12], [14], [17], [19], [21], [22], [23]).

Recently, Miao and Dragomir [17] established the following new refinement of the Hermite–Hadamard inequality:

THEOREM 1. *Let f be a real-valued convex function on $[a, b]$. If ψ is a real-valued, continuous, and strictly increasing function on $[a, b]$ such that $g(a) = a$, $g(b) = b$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f\left(\frac{\psi(t)+\psi^{-1}(t)}{2}\right) dt \leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{b-a} \int_a^b \frac{f(\psi(t))+f(\psi^{-1}(t))}{2} dt \leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

Mathematics subject classification (2020): 26D15, 26A51, 26A42.

Keywords and phrases: Convex function, Hermite-Hadamard inequality, refinement, majorization type inequality, Young's inequality, pseudo-inverse.

In this paper, we establish some generalizations of the above result by using the notion of pseudo-inverse (see [15]). The main tools used in the proofs are majorization type theorems (see [11], [12], [13]) and a general form of the remarkable Young's inequality. We only assume that the function ψ is increasing, and provide a unified proof for the second and third inequalities. Due to the weak condition on ψ , a range of either integral or discrete specific refinements can be obtained from the new parametric refinement. Assume that the function ψ itself generates a refinement of one side of the Hermite-Hadamard inequality. Then we study the relationship between the refinement obtained in this way and the new refinements derived from it. From the refinements obtained, we can arrive at a chain of refinements that converge to the integral mean. As applications, we give discrete refinements, and refine the Hermite-Hadamard inequality for strongly convex functions.

2. Preliminary results

Let $a < b$ and $\psi : [a, b] \rightarrow [a, b]$ be an increasing function. We define the functions $\psi_l^{-1} : [a, b] \rightarrow [a, b]$ and $\psi_r^{-1} : [a, b] \rightarrow [a, b]$ by, respectively,

$$\psi_l^{-1}(t) := \sup \{s \in [a, b] \mid \psi(s) < t\}$$

and

$$\psi_r^{-1}(t) := \inf \{s \in [a, b] \mid \psi(s) > t\},$$

where, as usual, $\sup \emptyset = a$ and $\inf \emptyset = b$.

Then $\psi_l^{-1}(t) \leq \psi_r^{-1}(t)$ for each $t \in [a, b]$. Any function $\chi : [a, b] \rightarrow [a, b]$ satisfying $\psi_l^{-1}(t) \leq \chi(t) \leq \psi_r^{-1}(t)$ for all $t \in [a, b]$ is called a pseudo-inverse of ψ (for a more general concept of the pseudo-inverse and quasi-inverse, see [6] and [15]). Since

$$\chi(t_1) \leq \psi_r^{-1}(t_1) \leq \psi_l^{-1}(t_2) \leq \chi(t_2), \quad a \leq t_1 < t_2 \leq b,$$

χ is increasing.

When ψ is a strictly increasing bijection and it is continuous, $\psi_l^{-1} = \psi_r^{-1} = \psi^{-1}$, where ψ^{-1} is the inverse function of ψ .

$\psi_l^{-1} = \psi_r^{-1}$ if and only if ψ is strictly increasing, so in this case the pseudo-inverse of ψ is unique and it will be denoted by ψ^* . It is a continuous function.

We need the following version of Young's inequality for increasing functions (weighted form can be found in [18]).

THEOREM 2. (see [6]) *Let $\eta : [0, \infty[\rightarrow [0, \infty[$ be an increasing function such that $\eta(0) = 0$ and $\lim_{\infty} \eta = \infty$, and let ζ be a pseudo-inverse of η . Then for every pair of nonnegative numbers a and b we have*

$$ab \leq \int_0^a \eta(t) dt + \int_0^b \zeta(t) dt,$$

and equality holds if and only if

$$\eta(a-) \leq b \leq \eta(a+).$$

The following consequences of the previous inequality will be important in our proofs.

PROPOSITION 1. Let $a < b$, $\psi : [a, b] \rightarrow [a, b]$ be an increasing function, and let χ be a pseudo-inverse of ψ . Then

(a)

$$uv - a^2 \leq \int_a^u \psi(t) dt + \int_a^v \chi(t) dt \quad (1)$$

for all $u, v \in [a, b]$, and equality holds if and only if

$$\psi(u-) \leq v \leq \psi(u+),$$

where $\psi(a-) := a$ and $\psi(b+) := b$.

(b)

$$b^2 - a^2 = \int_a^b \psi(t) dt + \int_a^b \chi(t) dt \quad (2)$$

Proof. (a) Let η and ζ be defined on $[0, \infty[$ by

$$\eta(t) := \begin{cases} 0, & t = 0 \\ \psi(t+a) - a, & t \in]0, b-a] \\ t, & t > b-a \end{cases}$$

and

$$\zeta(t) := \begin{cases} \chi(t+a) - a, & t \in [0, b-a] \\ t, & t > b-a \end{cases}.$$

Clearly, η is increasing, $\eta(0) = 0$, $\lim_{\infty} \eta = \infty$ and ζ is a pseudo-inverse of η .

Letting $s = t - a$, we obtain by Proposition 1,

$$\begin{aligned} \int_a^u \psi(t) dt + \int_a^v \chi(t) dt &= \int_0^{u-a} \psi(s+a) ds + \int_0^{v-a} \chi(s+a) ds \\ &= \int_0^{u-a} (\psi(s+a) - a) ds + a(u-a) + \int_0^{v-a} (\chi(s+a) - a) ds \\ &\quad + a(v-a) \geq (u-a)(v-a) + a(u-a) + a(v-a) = uv - a^2. \end{aligned}$$

The equality condition follows from the appropriate part of Proposition 1.

(b) Since $\eta((b-a)-) \leq \eta((b-a)) = b-a = \eta((b-a)+)$, the equality part of Proposition 1 implies that (2) holds.

The proof is complete. $\square$

Let $(X, \mathcal{A})$ be a measurable space ($\mathcal{A}$ always means a σ -algebra of subsets of X). If μ is a measure on $\mathcal{A}$, then the real vector space of μ -integrable real functions on X is denoted by $L(\mu)$.

The following majorization type inequality is a special case of Theorem 3 of [13]. The essence of such inequalities can be found in Theorem 6 of [12]. It plays a crucial role in proving the main result of this paper.

The σ -algebra of Borel sets on an interval $C \subset \mathbb{R}$ is denoted by $\mathcal{B}_C$. The Lebesgue measure on $\mathcal{B}_C$ is denoted by λ_C .

THEOREM 3. (see Theorem 3 of [13]) *Let $a < b$, and let μ_i ($i = 1, \dots, m$) and ν_j ($j = 1, \dots, n$) be finite measures on $\mathcal{B}_{[a,b]}$. Let $\varphi_i : [a, b] \rightarrow [a, b]$ and $\psi_j : [a, b] \rightarrow [a, b]$ be Borel measurable functions ($i = 1, \dots, m$, $j = 1, \dots, n$). Then for every convex function $f : [a, b] \rightarrow \mathbb{R}$ the inequality*

$$\sum_{i=1}^m \int_{[a,b]} f \circ \varphi_i d\mu_i \leq \sum_{j=1}^n \int_{[a,b]} f \circ \psi_j d\nu_j$$

holds if and only if

$$\sum_{i=1}^m \mu_i([a, b]) = \sum_{j=1}^n \nu_j([a, b]), \quad \sum_{i=1}^m \int_{[a,b]} \varphi_i d\mu_i = \sum_{j=1}^n \int_{[a,b]} \psi_j d\nu_j \quad (3)$$

and

$$\sum_{i=1}^m \int_{\{\varphi_i < w\}} (w - \varphi_i) d\mu_i \leq \sum_{j=1}^n \int_{\{\psi_j < w\}} (w - \psi_j) d\nu_j, \quad w \in]a, b[. \quad (4)$$

We note explicitly two special cases of the previous result. Interesting results concerning the refinement of the Hermite-Hadamard inequality follow immediately from them.

LEMMA 1. *Let $a < b$, let ν_j ($j = 1, \dots, n$) be finite measures on $\mathcal{B}_{[a,b]}$, and let $\psi_j : [a, b] \rightarrow [a, b]$ ($j = 1, \dots, n$) be Borel measurable functions. Then for every convex function $f : [a, b] \rightarrow \mathbb{R}$ the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \sum_{j=1}^n \int_{[a,b]} f \circ \psi_j d\nu_j \quad (5)$$

holds if and only if

$$\sum_{j=1}^n \nu_j([a, b]) = 1 \quad \text{and} \quad \sum_{j=1}^n \int_{[a,b]} \psi_j d\nu_j = \frac{a+b}{2}. \quad (6)$$

Proof. Since the inequality (5) can be rewritten in the form

$$\frac{1}{b-a} \int_{[a,b]} f\left(\frac{a+b}{2}\right) d\lambda_{[a,b]} \leq \sum_{j=1}^n \int_{[a,b]} f \circ \psi_j d\nu_j,$$

we can apply Theorem 3.

It follows from (3) that the conditions in (6) are necessary.

To prove that (6) is sufficient, it must be shown that the condition corresponding to (4) is also satisfied, i.e.

$$\begin{aligned} & \begin{cases} 0, & a < w \leq \frac{a+b}{2} \\ w - \frac{a+b}{2}, & \frac{a+b}{2} < w < b \end{cases} \\ & = \int_{\{\frac{a+b}{2} < w\}} \left(w - \frac{a+b}{2}\right) d\lambda_{[a,b]} \leq \sum_{j=1}^n \int_{\{\psi_j < w\}} (w - \psi_j) d\nu_j, \quad w \in]a, b[. \end{aligned}$$

Suppose that the inequality

$$w - \frac{a+b}{2} \leq \sum_{j=1}^n \int_{\{\psi_j < w\}} (w - \psi_j) d\nu_j, \quad w \in \left] \frac{a+b}{2}, b \right[$$

is false for some $w_c \in \left] \frac{a+b}{2}, b \right[$. Then by $\sum_{j=1}^n \nu_j([a, b]) = 1$,

$$\begin{aligned} \sum_{j=1}^n \int_{[a,b]} \psi_j d\nu_j &= \sum_{j=1}^n \int_{\{\psi_j < w_c\}} \psi_j d\nu_j + \sum_{j=1}^n \int_{\{\psi_j \geq w_c\}} \psi_j d\nu_j \\ &> -w_c + \frac{a+b}{2} + w_c \sum_{j=1}^n \nu_j(\{\psi_j < w_c\}) + w_c \sum_{j=1}^n \nu_j(\{\psi_j \geq w_c\}) = \frac{a+b}{2} \end{aligned}$$

and this contradicts the second condition in (6).

The proof is complete. $\square$

Let $(X, \mathcal{A})$ be a measurable space. The unit mass at $x \in X$ (the Dirac measure at x) is denoted by ε_x .

LEMMA 2. Let $a < b$, let μ_i ($i = 1, \dots, m$) be finite measures on $\mathcal{B}_{[a,b]}$, and let $\varphi_i : [a, b] \rightarrow [a, b]$ ($i = 1, \dots, m$) be Borel measurable functions. Then for every convex function $f : [a, b] \rightarrow \mathbb{R}$ the inequality

$$\sum_{i=1}^m \int_{[a,b]} f \circ \varphi_i d\mu_i \leq \frac{f(a) + f(b)}{2} \quad (7)$$

holds if and only if

$$\sum_{i=1}^m \mu_i([a, b]) = 1 \quad \text{and} \quad \sum_{i=1}^m \int_{[a, b]} \varphi_i d\mu_i = \frac{a+b}{2}. \quad (8)$$

Proof. The inequality (7) is equivalent to

$$\sum_{i=1}^m \int_{[a, b]} f \circ \varphi_i d\mu_i \leq \int_{[a, b]} f d_{\frac{1}{2}\varepsilon_a + \frac{1}{2}\varepsilon_b},$$

and therefore Theorem 3 shows that the conditions in (8) are necessary.

Assume (8) is satisfied. By the convexity of f ,

$$f(\varphi_i(t)) \leq \frac{b - \varphi_i(t)}{b - a} f(a) + \frac{\varphi_i(t) - a}{b - a} f(b), \quad t \in [a, b], \quad i = 1, \dots, m,$$

and hence, using (8), we obtain (7).

The proof is complete. $\square$

REMARK 1. It follows from the previous lemmas that if inequalities

$$\sum_{j=1}^n \int_{[a, b]} f \circ \psi_j d\nu_j \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \sum_{i=1}^m \int_{[a, b]} f \circ \varphi_i d\mu_i$$

hold for every convex function $f : [a, b] \rightarrow \mathbb{R}$ (the parameters are the same as in the lemmas), then we automatically obtain a refinement of the Hermite-Hadamard inequality. Indeed, Theorem 3 implies that if either of the two inequalities considered is fulfilled, then (6) and (8) must be satisfied.

In the following statement, we specify sufficient but not necessary conditions for refining both sides of the Hermite-Hadamard inequality. The result is contained in a more general form in Theorem 8 of [11].

THEOREM 4. (see Theorem 8 of [11]) *Let $[a, b] \subset \mathbb{R}$ with $a < b$. Assume $\varphi_1, \varphi_0, \psi_1, \psi_0 : [a, b] \rightarrow [a, b]$ are increasing functions such that*

$$\int_a^x \varphi_0(t) dt \leq \int_a^x \varphi_1(t) dt \leq \int_a^x t dt \leq \int_a^x \psi_1(t) dt \leq \int_a^x \psi_0(t) dt, \quad x \in]a, b[$$

and

$$\int_a^b \varphi_0(t) dt = \int_a^b \varphi_1(t) dt = \int_a^b \psi_1(t) dt = \int_a^b \psi_0(t) dt = \int_a^b t dt = \frac{b^2 - a^2}{2}$$

are satisfied. Then for all convex function $f : [a, b] \rightarrow \mathbb{R}$ we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)(b-a) &\leq \int_a^b f(\psi_0(t)) dt \leq \int_a^b f(\psi_1(t)) dt \leq \int_a^b f(t) dt \\ &\leq \int_a^b f(\varphi_1(t)) dt \leq \int_a^b f(\varphi_2(t)) dt \leq \frac{f(a)+f(b)}{2}(b-a). \end{aligned}$$

We conclude this section with results related to the notion of pseudo-inverse.

LEMMA 3. Let $a < b$, and let $\psi, \chi : [a, b] \rightarrow [a, b]$ be increasing functions.

(a) If $\psi(t) < t$ for some $t \in [a, b]$, then $t \leq \psi_l^{-1}(t)$. If $\psi(t) > t$ for some $t \in [a, b]$, then $t \geq \psi_r^{-1}(t)$. If $\psi(t) = t$ for some $t \in [a, b]$, then $\psi_l^{-1}(t) \leq t \leq \psi_r^{-1}(t)$.

(b) If ψ and χ are strictly increasing, and χ is between the functions ψ and ψ^* , then χ^* is also between ψ and ψ^* .

Proof. (a) They come from the definitions of ψ_l^{-1} and $\psi_r^{-1}(t)$.

(b) It follows from the construction of the graph of the pseudo-inverse ψ^* : draw vertical line segments at the discontinuities of ψ (including at a and at b , if either $\psi(a) > a$ or $\psi(b) < b$), and then reflect this extended graph of ψ through the line $y = x$.

The proof is complete. $\square$

LEMMA 4. Let $a < b$, and let $\psi : [a, b] \rightarrow [a, b]$ be a strictly increasing function. Let $\psi_0 := \psi$, and define the functions $\psi_n : [a, b] \rightarrow [a, b]$ by

$$\psi_n := \frac{\psi_{n-1} + \psi_{n-1}^*}{2}, \quad n = 1, 2, \dots$$

Then $\psi_n(t) \rightarrow t$ and $\psi_n^*(t) \rightarrow t$ uniformly on $[a, b]$.

Proof. Since ψ is strictly increasing, its pseudo-inverse ψ^* is unique. The function ψ_1 is the arithmetic mean of ψ and ψ^* , and ψ^* is increasing, and hence $\psi_1 : [a, b] \rightarrow [a, b]$ is a strictly increasing function between ψ and ψ^* . It follows that the pseudo-inverse ψ_1^* is unique, and by Lemma 3 (b), it is also between ψ and ψ^* . We can prove by induction on n that $\psi_n : [a, b] \rightarrow [a, b]$ is strictly increasing, and ψ_n and ψ_n^* are both between ψ_{n-1} and ψ_{n-1}^* .

Since ψ_n and ψ_n^* are both between ψ_{n-1} and ψ_{n-1}^* , and ψ_n is the arithmetic mean of ψ_{n-1} and ψ_{n-1}^* , we have

$$|\psi_n(t) - \psi_n^*(t)| \leq \frac{1}{2} |\psi_{n-1}(t) - \psi_{n-1}^*(t)|, \quad t \in [a, b], \quad n = 1, 2, \dots,$$

and this implies that

$$|\psi_n(t) - \psi_n^*(t)| \leq \frac{1}{2^n} |\psi(t) - \psi^*(t)| \leq \frac{b-a}{2^n}, \quad t \in [a, b], \quad n = 1, 2, \dots$$

From this inequality and from Lemma 3 (a) we deduce that

$$\left\{ \begin{array}{l} |\psi_n(t) - t| \\ |\psi_n^*(t) - t| \end{array} \right\} \leq |\psi_n(t) - \psi_n^*(t)| \leq \frac{b-a}{2^n}, \quad t \in [a, b], \quad n = 1, 2, \dots,$$

and this implies that the sequences (ψ_n) and (ψ_n^*) are uniformly convergent on $[a, b]$ and $\psi_n(t) \rightarrow t$, $\psi_n^*(t) \rightarrow t$ for all $t \in [a, b]$.

The proof is complete. $\square$

3. Main results

We are now in a position to state the following result, in which both sides of the Hermite-Hadamard inequality are refined.

THEOREM 5. *Let $a < b$, $\psi : [a, b] \rightarrow [a, b]$ be an increasing function, and let χ be a pseudo-inverse of ψ . Then, for all convex function $f : [a, b] \rightarrow \mathbb{R}$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f\left(\frac{\psi(t) + \chi(t)}{2}\right) dt \leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{b-a} \int_a^b \frac{f(\psi(t)) + f(\chi(t))}{2} dt \leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

Proof. The first and the fourth inequalities come from Lemma 1 and Lemma 2, respectively.

We prove the middle two inequalities using Theorem 3.

(i) We first consider the second inequality.

Theorem 3 can be applied with $\mu_1 = \nu_1 := \lambda_{[a,b]}$, $\varphi_1(t) := \frac{\psi(t) + \chi(t)}{2}$ and $\psi_1(t) := t$ ($t \in [a, b]$).

Since the function φ_1 is increasing, for each $w \in]a, b[$ there exists a number $s_w \in [a, b]$ such that either $\{\varphi_1 < w\} = [a, s_w[$ or $\{\varphi_1 < w\} = [a, s_w]$.

By Proposition 1 (b),

$$\int_a^b \frac{\psi(t) + \chi(t)}{2} dt = \frac{b^2 - a^2}{2} = \int_a^b t dt, \quad (9)$$

so it remained to be proven that

$$\int_a^{s_w} (w - \varphi_1(t)) dt \leq \int_a^w (w - t) dt = \frac{(w-a)^2}{2}, \quad w \in]a, b[.$$

By Proposition 1 (a),

$$\int_a^{s_w} (w - \varphi_1(t)) dt \leq w(s_w - a) - \frac{1}{2}(s_w^2 - a^2) = (s_w - a) \left(w - \frac{s_w + a}{2} \right)$$

It is easy to verify that the inequality

$$(s_w - a) \left(w - \frac{s_w + a}{2} \right) \leq \frac{(w - a)^2}{2}$$

is equivalent to the valid inequality $0 \leq (w - s_w)^2$.

(ii) Now we consider the third inequality.

In this case Theorem 3 can be applied with $\mu_1 := \lambda_{[a,b]}$, $\nu_1 = \nu_2 := \frac{1}{2}\lambda_{[a,b]}$, $\varphi_1(t) := t$ ($t \in [a, b]$), $\psi_1 := \psi$ and $\psi_2 := \chi$.

Similar to part (a), for each $w \in]a, b[$ there exist numbers $s_{i,w} \in [a, b]$ ($i = 1, 2$) such that either $\{\psi_i < w\} = [a, s_{i,w}[$ or $\{\psi_i < w\} = [a, s_{i,w}]$ ($i = 1, 2$).

Based on the above, only the inequality

$$\frac{(w - a)^2}{2} = \int_a^w (w - t) dt \leq \frac{1}{2} \int_a^{s_{1,w}} (w - \psi(t)) dt + \frac{1}{2} \int_a^{s_{2,w}} (w - \chi(t)) dt \quad (10)$$

is not obvious among the conditions.

Since $\psi(s_{1,w}-) \leq w \leq \psi(s_{1,w}+)$, we obtain from the equality condition in Proposition 1 (a) that

$$\begin{aligned} & \frac{1}{2} \int_a^{s_{1,w}} (w - \psi(t)) dt + \frac{1}{2} \int_a^{s_{2,w}} (w - \chi(t)) dt \\ &= \frac{1}{2} \int_a^{s_{1,w}} (w - \psi(t)) dt + \frac{1}{2} \int_a^w (w - \chi(t)) dt + \frac{1}{2} \int_w^{s_{2,w}} (w - \chi(t)) dt \\ &= \frac{1}{2} w(s_{1,w} - a) + \frac{1}{2} w(w - a) - \frac{1}{2} (ws_{1,w} - a^2) + \frac{1}{2} \int_w^{s_{2,w}} (w - \chi(t)) dt, \end{aligned}$$

therefore, after some calculations, we get that (10) is equivalent to

$$0 \leq \frac{1}{2} \int_w^{s_{2,w}} (w - \chi(t)) dt$$

which is obviously true.

The proof is complete. $\square$

REMARK 2. (a) When ψ is a continuous, and strictly increasing function such that $\psi(a) = a$ and $\psi(b) = b$, we obtain Theorem 1. All inequalities in Theorem 5 are proven using the same method, while in Theorem 5 of [17], different treatments are used to prove the second and third inequalities. The technic used to prove the third inequality cannot be applied in our case.

(b) Our only condition is that ψ is increasing, so a lot of special refinements (whether integral or discrete) can be obtained from this parametric refinement.

In the following statements, we assume that the function ψ itself generates a refinement of one side of the Hermite-Hadamard inequality, and then we examine the relationship between the refinement obtained in this way and the refinements derived from it in Theorem 5.

First, we consider the left-hand side of the inequality.

THEOREM 6. *Let $a < b$, and let $\psi : [a, b] \rightarrow [a, b]$ be a strictly increasing function. If*

$$\int_a^s \psi(t) dt \geq \frac{s^2 - a^2}{2}, \quad s \in]a, b[\quad (11)$$

and

$$\int_a^b \psi(t) dt = \frac{b^2 - a^2}{2}, \quad (12)$$

then, for all convex function $f : [a, b] \rightarrow \mathbb{R}$, we have

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(\psi(t)) dt \leq \int_a^b f\left(\frac{\psi(t) + \psi^*(t)}{2}\right) dt \leq \int_a^b f(t) dt.$$

Proof. It follows from Proposition 1 (b) that

$$\int_a^b \psi^*(t) dt = \frac{b^2 - a^2}{2},$$

and therefore by Theorem 4, it is enough to show that

$$\int_a^s \frac{\psi(t) + \psi^*(t)}{2} dt \leq \int_a^s \psi(t) dt, \quad s \in]a, b[,$$

or equivalently that the function

$$h : [a, b] \rightarrow \mathbb{R}, \quad h(s) := \int_a^s (\psi(t) - \psi^*(t)) dt$$

is nonnegative.

The function h is continuous, its right-hand derivative $h'_+(s)$ and left-hand derivative $h'_-(s)$ exist for all $s \in]a, b[$ and $h'_+(s) = \psi(s+) - \psi^*(s)$ and $h'_-(s) = \psi(s-) - \psi^*(s)$ (ψ^* is continuous).

Let

$$H := \{t \in [a, b] \mid \psi^*(t) = t\} \cup \{a, b\}.$$

If $s \in H \cap]a, b[$, then $\psi(s-) \leq s \leq \psi(s+)$. It follows from Proposition 1 (a) that for such points s ,

$$\int_a^s (\psi(t) + \psi^*(t)) dt = s^2 - a^2,$$

and hence (11) implies that

$$\int_a^s \psi^*(t) dt \leq \frac{s^2 - a^2}{2},$$

which shows that $h(s) \geq 0$. Since $h(a) = h(b) = 0$, $h(s) \geq 0$ for all $s \in H$.

Since ψ^* is continuous, H is a closed set, and therefore its complement H^c can be expressed as a countable union of pairwise disjoint open intervals whose endpoints belong to H .

If H^c is the empty set, then $\psi(s) = \psi^*(s) = s$ for all $s \in [a, b]$ which is a trivial case.

If H^c is not the empty set, then let $]s_1, s_2[\subset H^c$ be an open interval with endpoints $s_1 < s_2$ ($s_1, s_2 \in H$). Since ψ^* is continuous, either $\psi^*(s) < s$ or $\psi^*(s) > s$ for all $s \in]s_1, s_2[$, and therefore, in the first case, $\psi(s) \geq s$, while in the second case, $\psi(s) \leq s$ for all $s \in]s_1, s_2[$ (see Lemma 3 (a)). It follows that either $h'_+(s) > 0$ or $h'_-(s) < 0$ for all $s \in]s_1, s_2[$. This implies (see [9]) that h is either increasing or decreasing on $[s_1, s_2]$. Since h is continuous and $h(s_1) \geq 0$ and $h(s_2) \geq 0$, we obtain that $h(s) \geq 0$ for all $s \in [s_1, s_2]$.

The proof is complete. $\square$

COROLLARY 1. Assume the conditions of Theorem 6 are satisfied. Let $\psi_0 := \psi$, and define the function $\psi_n : [a, b] \rightarrow [a, b]$ by

$$\psi_n := \frac{\psi_{n-1} + \psi_{n-1}^*}{2}, \quad n = 1, 2, \dots$$

Then for all convex function $f : [a, b] \rightarrow \mathbb{R}$, we have

$$\begin{aligned} (b-a)f\left(\frac{a+b}{2}\right) &\leq \int_a^b f(\psi(t)) dt \leq \int_a^b f(\psi_1(t)) dt \\ &\leq \dots \int_a^b f(\psi_n(t)) dt \leq \dots \leq \int_a^b f(t) dt. \end{aligned}$$

Proof. Using Proposition 1 (b), it is easy to prove by induction on n that

$$\int_a^b \psi_n(t) dt = \frac{b^2 - a^2}{2}.$$

Another induction argument shows that ψ_n ($n = 0, 1, \dots$) is strictly increasing, and hence by Proposition 1 (a),

$$\int_a^s \psi_n(t) dt \geq \frac{s^2 - a^2}{2}, \quad s \in]a, b[.$$

Applying Theorem 6, we obtain the result.

The proof is complete. $\square$

Now we turn to studying the right-hand side of the inequality.

THEOREM 7. Let $a < b$.

(a) Let $\psi: [a, b] \rightarrow [a, b]$ be an increasing function, and let χ be a pseudo-inverse of ψ . If

$$\int_a^s \psi(t) dt \leq \frac{s^2 - a^2}{2}, \quad s \in]a, b[\quad (13)$$

and (12) is satisfied, then, for all convex function $f: [a, b] \rightarrow \mathbb{R}$, we have

$$\int_a^b f(t) dt \leq \int_a^b \frac{f(\psi(t)) + f(\chi(t))}{2} dt \leq \int_a^b f(\psi(t)) dt \leq (b-a) \frac{f(a) + f(b)}{2}. \quad (14)$$

(b) Let $\psi: [a, b] \rightarrow [a, b]$ be an absolutely continuous, and strictly increasing function such that $\psi(a) = a$ and $\psi(b) = b$. If

$$\int_a^b f(t) dt \leq \int_a^b f(\psi(t)) dt \quad (15)$$

for all convex function $f: [a, b] \rightarrow \mathbb{R}$, then (14) holds with $\chi = \psi^{-1}$.

Proof. (a) By Proposition 1 (a) and (13),

$$s^2 - a^2 \leq \int_a^s \psi(t) dt + \int_a^s \chi(t) dt \leq \frac{s^2 - a^2}{2} + \int_a^s \chi(t) dt, \quad s \in]a, b[,$$

which yields that

$$\int_a^s \chi(t) dt \geq \frac{s^2 - a^2}{2}, \quad s \in]a, b[.$$

By Proposition 1 (b) and (12),

$$\int_a^b \chi(t) dt = \frac{b^2 - a^2}{2}.$$

It now follows from Theorem 4 that

$$\int_a^b f(\chi(t)) dt \leq \int_a^b f(t) dt \leq \int_a^b f(\psi(t)) dt$$

for all convex function $f : [a, b] \rightarrow \mathbb{R}$, and therefore Theorem 5 implies the result.

(b) By Theorem 5 and Lemma 2, it is sufficient to prove the second inequality in (14).

Using (15) and the conditions relating to ψ , it follows from Theorem 3 that

$$\int_a^b \psi(t) dt = \frac{b^2 - a^2}{2} \quad (16)$$

and

$$\int_a^{\psi^{-1}(w)} (w - \psi(t)) dt \geq \frac{(w - a)^2}{2}, \quad w \in]a, b[. \quad (17)$$

In view of (16) and Proposition 1 (b), we have

$$\int_a^b \psi^{-1}(t) dt = \frac{b^2 - a^2}{2},$$

and so by Theorem 3 the proof will be complete upon showing that

$$\begin{aligned} & \frac{1}{2} \int_a^{\psi^{-1}(w)} (w - \psi(t)) dt + \frac{1}{2} \int_a^{\psi(w)} (w - \psi^{-1}(t)) dt \\ & \leq \frac{1}{2} \int_a^{\psi^{-1}(w)} (w - \psi(t)) dt, \quad w \in]a, b[, \end{aligned}$$

which is equivalent to

$$\int_a^{\psi(w)} (w - \psi^{-1}(t)) dt \leq \int_a^{\psi^{-1}(w)} (w - \psi(t)) dt, \quad w \in]a, b[. \quad (18)$$

Define the set $H := \{t \in [a, b] \mid \psi(t) = t\}$.

Assume $w \in H$ that is $\psi(w) = \psi^{-1}(w) = w$. Then by Proposition 1 (a), we have

$$\int_a^w (w - \psi^{-1}(t)) dt + \int_a^w (w - \psi(t)) dt = 2w(w - a) - (w^2 - a^2),$$

and therefore (17) implies that

$$\int_a^w (w - \psi^{-1}(t)) dt \leq 2w(w - a) - (w^2 - a^2) - \frac{(w - a)^2}{2} = \frac{(w - a)^2}{2}.$$

This establishes (18) in this case.

If $H = [a, b]$, then (18) is trivial. Thus suppose that $H \neq [a, b]$.

Since H is closed and $a, b \in H$, the complement of H is the disjoint union of a countable collection of open intervals.

Let $w_1, w_2 \in H$ such that $w_1 < w_2$ and $]w_1, w_2[\subset [a, b] \setminus H$. Define the function h on $[w_1, w_2]$ by

$$h(w) := \frac{(w - a)^2}{2} - w(\psi(w) - a) + \int_a^{\psi(w)} \psi^{-1}(t) dt.$$

Since the functions

$$s \rightarrow \int_a^s \psi^{-1}(t) dt, \quad s \in [w_1, w_2]$$

and ψ are absolutely continuous and their composition is of finite variation (it is easy to check), h is also absolutely continuous. It follows that h is differentiable almost everywhere and

$$h'(w) = w - \psi(w) \quad \text{a.e. on }]w_1, w_2[.$$

It is clear that either $h'(w) > 0$ a.e. on $]w_1, w_2[$ or $h'(w) < 0$ a.e. on $]w_1, w_2[$, and therefore (see [9]) h is either increasing or decreasing on $[w_1, w_2]$. Thus it follows from $h(w_i) \geq 0$ ($i = 1, 2$) that h is nonnegative, that is,

$$\int_a^{\psi(w)} (w - \psi^{-1}(t)) dt \leq \frac{(w - a)^2}{2}.$$

Thus we obtain (18) at once from (17).

The proof is complete. $\square$

COROLLARY 2. Assume the conditions of Theorem 7 (b) are satisfied, and assume that $\psi'(t) \neq 0$ a.e. on $[a, b]$. Let $\psi_0 := \psi$, and define the function $\psi_n : [a, b] \rightarrow [a, b]$ by

$$\psi_n := \frac{\psi_{n-1} + \psi_{n-1}^{-1}}{2}, \quad n = 1, 2, \dots$$

Then for all convex function $f : [a, b] \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \int_a^b f(\psi(t)) dt &\leq \dots \leq \int_a^b \frac{f(\psi_n(t)) + f(\psi_n^{-1}(t))}{2} dt \leq \dots \\ &\leq \int_a^b \frac{f(\psi_0(t)) + f(\psi_0^{-1}(t))}{2} dt \leq \int_a^b f(\psi(t)) dt \leq (b-a) \frac{f(a) + f(b)}{2}. \end{aligned}$$

Proof. Since $\psi'(t) > 0$ a.e. on $[a, b]$, ψ^{-1} is also absolutely continuous, and $\psi'_1(t) > 0$ a.e. on $[a, b]$. Proceeding in this manner, we can prove by induction on n that ψ_n is an absolutely continuous, and strictly increasing function such that $\psi(a) = a$ and $\psi(b) = b$, and $\psi'_n(t) > 0$ a.e. on $[a, b]$. Applying Theorem 7 (b), we deduce the result.

The proof is complete. $\square$

REMARK 3. We can see (not surprisingly) that the results obtained by “symmetrization” are sharper. Perhaps the above results are also true under the conditions of Theorem 5, but this could not be verified.

Finally, we study the convergence of the sequences in Corollary 1 and Corollary 2.

THEOREM 8. Let $a < b$, and let $\psi : [a, b] \rightarrow [a, b]$ be a strictly increasing function. Let $\psi_0 := \psi$, and define the functions $\psi_n : [a, b] \rightarrow [a, b]$ by

$$\psi_n := \frac{\psi_{n-1} + \psi_{n-1}^*}{2}, \quad n = 1, 2, \dots$$

Then, for all convex function $f : [a, b] \rightarrow \mathbb{R}$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f\left(\frac{\psi_{n-1}(t) + \psi_{n-1}^*(t)}{2}\right) dt \leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{b-a} \int_a^b \frac{f(\psi_{n-1}(t)) + f(\psi_{n-1}^*(t))}{2} dt \leq \frac{f(a) + f(b)}{2}, \quad n = 2, 3, \dots \end{aligned}$$

and

$$\int_a^b f\left(\frac{\psi_{n-1}(t) + \psi_{n-1}^*(t)}{2}\right) dt \xrightarrow{n \rightarrow \infty} \int_a^b f(t) dt$$

and

$$\int_a^b \frac{f(\psi_{n-1}(t)) + f(\psi_{n-1}^*(t))}{2} dt \xrightarrow{n \rightarrow \infty} \int_a^b f(t) dt.$$

Proof. It follows from Theorem 5 and Lemma 4. $\square$

REMARK 4. Convergence is confirmed under conditions that are much more general than those specified in the previous two corollaries. This also reinforces our conjecture that Corollary 1 and Corollary 2 are fulfilled even under more general conditions.

4. Applications

The following result deals with discrete refinements of the Hermite-Hadamard inequalities that can be obtained from Theorem 5.

PROPOSITION 2. Let $a < b$, let $a = \tau_0 < \tau_1 < \dots < \tau_n = b$, and let $a \leq c_1 < \dots < c_n \leq b$ ($n = 1, 2, \dots$). Set $c_0 := a$ and $c_{n+1} := b$. Let $a = \rho_0 < \rho_1 < \dots < \rho_k = b$, where $\{\rho_0, \dots, \rho_k\} = \{\tau_0, \dots, \tau_n\} \cup \{c_1, \dots, c_n\}$. Then, for all convex function $f : [a, b] \rightarrow \mathbb{R}$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \sum_{i=1}^k f\left(\frac{\rho_{i-1} + \rho_i}{2}\right) (\rho_i - \rho_{i-1}) \leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{2(b-a)} \left(\sum_{i=1}^n f(c_i) (\tau_i - \tau_{i-1}) + \sum_{j=1}^{n+1} f(\tau_{j-1}) (c_j - c_{j-1}) \right) \leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

Proof. Let $\psi : [a, b] \rightarrow [a, b]$ be an increasing step-function such that

$$\psi(t) = c_i, \quad \tau_{i-1} < t < \tau_i, \quad i = 1, \dots, n$$

and let $\chi : [a, b] \rightarrow [a, b]$ be a pseudo-inverse of ψ . Then χ is also an increasing step-function such that

$$\chi(t) = \tau_{j-1}, \quad c_{j-1} < t < c_j, \quad j = 1, \dots, n+1.$$

The result follows from Theorem 5.

The proof is complete. $\square$

REMARK 5. The refinement of the left-hand side can also be obtained by applying the Hermite-Hadamard inequality to every element of the partition of $[a, b]$. This is a usual method, but our approach is novel. However, the refinement of the right-hand side is new.

Let $C \subset \mathbb{R}$ be an interval with nonempty interior. A function $f : C \rightarrow \mathbb{R}$ is called strongly convex with modulus $d > 0$ if

$$f(\lambda s + (1-\lambda)t) \leq \lambda f(s) + (1-\lambda)f(t) - d\lambda(1-\lambda)(s-t)^2$$

for all $s, t \in C$ and $\lambda \in [0, 1]$.

The following statement is a special case of Lemma 2.1 in paper [20], which describes the close relationship between convex and strongly convex functions.

LEMMA 5. Let $C \subset \mathbb{R}$ be an interval with nonempty interior and $d > 0$. A function $f : C \rightarrow \mathbb{R}$ is strongly convex with modulus d if and only if the function $t \rightarrow f(t) - dt^2$ ($t \in C$) is convex.

The following result contains the counterpart of the Hermite–Hadamard inequality for strongly convex functions.

THEOREM 9. (see [16]) Let $a < b$. Then for all strongly convex function $f : [a, b] \rightarrow \mathbb{R}$ with modulus $d > 0$, we have

$$f\left(\frac{a+b}{2}\right) + \frac{d}{12}(b-a)^2 \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2} - \frac{d}{6}(b-a)^2.$$

By applying Theorem 5, we give a refinement of Hermite–Hadamard inequality for strongly convex functions.

THEOREM 10. Let $a < b$, $\psi : [a, b] \rightarrow [a, b]$ be an increasing function, and let χ be a pseudo-inverse of ψ . Then, for all strongly convex function $f : [a, b] \rightarrow \mathbb{R}$ with modulus $d > 0$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) + \frac{d}{12}(b-a)^2 &\leq \frac{1}{b-a} \int_a^b f\left(\frac{\psi(t)+\chi(t)}{2}\right) dt \\ &+ d \left(\frac{a^2+ab+b^2}{3} - \frac{1}{b-a} \int_a^b \left(\frac{\psi(t)+\chi(t)}{2}\right)^2 dt \right) \\ &\leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{b-a} \int_a^b \frac{f(\psi(t))+f(\chi(t))}{2} dt \\ &+ d \left(\frac{a^2+ab+b^2}{3} - \frac{1}{b-a} \int_a^b \frac{\psi^2(t)+\chi^2(t)}{2} dt \right) \leq \frac{f(a)+f(b)}{2} - \frac{d}{6}(b-a)^2. \end{aligned}$$

Proof. These inequalities follow from Theorem 5, applying it to the convex function $t \rightarrow f(t) - dt^2$ ($t \in [a, b]$).

The proof is complete. $\square$

REMARK 6. (a) With the special choice of ψ , we can obtain a lot of different refinements (whether integral or discrete), as opposed to a single specific refinement (see [3]).

REFERENCES

- [1] T. ALI, M. A. KHAN, A. KILICMAN AND Q. DIN, *On the refined Hermite-Hadamard inequalities*, Math. Sci. Appl. E-Notes **6** (1) (2018), 85–92.
- [2] M. U. AWAN, M. A. NOOR, T.-S. DU, K. I. NOOR, *New refinements of fractional Hermite-Hadamard inequality*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **113** (1) (2019), 21–29.
- [3] A. AZÓCAR, K. NIKODEM AND G. ROA, *Fejér-type inequalities for strongly convex functions*, Annal. Math. Silesianae **26** (2012), 43–54.
- [4] M. K. BAKULA AND K. NIKODEM, *On the converse Jensen inequality for strongly convex functions*, J. Math. Anal. Appl. **434** (1) (2016), 516–522.
- [5] H. BARSAM, S. M. RAMEZANI, Y. SAYYARI, *On the new Hermite-Hadamard type inequalities for s -convex functions*, Afr. Mat. **32** (2021), 1355–1367.
- [6] F. CUNNINGHAM, JR. AND N. GROSSMAN, *On Young's inequality*, The American Mathematical Monthly **78** (7) (1971), 781–783.
- [7] S. S. DRAGOMIR AND C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities*, RGMIA Monographs, Victoria University, (2000)
- [8] A. E. FARISSI, *Simple proof and refinement of Hermite-Hadamard inequality*, J. Math. Inequal. **4** (3) (2010), 365–369.
- [9] T. M. FLETT, *Differential Analysis*, Cambridge University Press, Cambridge, England (1980)
- [10] X. GAO, *A note on the Hermite-Hadamard inequality*, J. Math. Inequal. **4** (2010), 587–591.
- [11] L. HORVÁTH, *Uniform treatment of integral majorization inequalities with applications to Hermite-Hadamard-Fejér type inequalities and f -divergences*, Entropy, 2023, **25**, 954.
- [12] L. HORVÁTH, *Integral inequalities using signed measures corresponding to majorization*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117** (2) (2023), 80.
- [13] L. HORVÁTH, *Majorization type integral inequalities related to a result of Bennett with applications*, Mathematics **13** (10) (2025), 1563.
- [14] M. A. KHAN, Y.-M. CHU, T. U. KHAN AND J. KHAN, *Some new inequalities of Hermite-Hadamard type for s -convex functions with applications*, Open Math. **15** (1) (2017), 1414–1430.
- [15] E. P. KLEMENT, R. MESIAR AND E. PAP, *Quasi- and pseudo-inverses of monotone functions, and the construction of t -norms*, Fuzzy Sets and Systems **104** (1999), 3–13.
- [16] N. MERENTES AND K. NIKODEM, *Remarks on strongly convex functions*, Aequationes Mathematicae **80** (1–2) (2010), 193–199.
- [17] J. MIAO AND S. S. DRAGOMIR, *Refinement of Hermite-Hadamard inequality*, Rocky Mountain J. Math., to appear.
- [18] F.-C. MITROI AND C. P. NICULESCU, *An Extension of Young's Inequality*, Abstract and Applied Analysis **2011**, 162049, 18 pages, 2011.
- [19] C. P. NICULESCU AND L.-E. PERSSON, *Old and new on the Hermite-Hadamard inequality*, Real Anal. Exch. **29** (2) (2003/04), 663–685.
- [20] K. NIKODEM AND ZS. PÁLES, *Characterizations of inner product spaces by strongly convex functions*, Banach J. Math. Anal. **5** (1) (2011), 83–87.
- [21] Z. PAVIC, *Improvements of the Hermite-Hadamard inequality*, J. Inequal. Appl. **2015**, 1, 222.
- [22] K. YANAGI, *Refined Hermite-Hadamard inequalities and some norm inequalities*, Symmetry **2022**, 14, 2522.
- [23] G. ZABANDAN, *A new refinement of the Hermite-Hadamard inequality for convex functions*, J. Inequal. Pure Appl. Math. **10** (2) (2009), Article ID 45.

(Received March 12, 2026)

László Horváth
 Department of Mathematics
 University of Pannonia
 Egyetem u. 10, 8200 Veszprém, Hungary
 e-mail: horvath.laszlo@mik.uni-pannon.hu