

ESTIMATES FOR THE NORM OF THE VOLTERRA OPERATOR IN WEIGHTED LEBESGUE SPACES

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Abstract. In this paper, we study the Volterra integral operator with so-called Oinarov's kernel acting between weighted Lebesgue spaces $L_{p,v}$ and $L_{q,u}$, where $1 < p, q < \infty$. We derive new two-sided estimates for the operator norm and provide refined upper and lower estimates explicitly expressed in terms of the kernel and the associated weights.

1. Introduction

Let $1 \leq q, p < \infty$, and let u and v be *weight functions*, that is, nonnegative measurable functions defined almost everywhere on an interval $(a, b) \subset \mathbb{R}$. Denote by $L_{q,u}$ the weighted Lebesgue space consisting of all measurable functions f such that

$$\|f\|_{q,u} = \left(\int_a^b |f(x)|^q u(x) dx \right)^{1/q} < \infty.$$

The space $L_{p,v}$ is defined analogously by replacing q with p and u with v .

Consider the *Volterra operator* (or *Hardy–Volterra operator*)

$$K : L_{p,v} \rightarrow L_{q,u}, \quad (Kf)(x) = \int_a^x k(x,t) f(t) dt, \quad (1)$$

where $k(x,t)$ is a measurable kernel defined on the domain $D = \{(x,t) : a < t \leq x < b\}$.

A fundamental problem in the theory of integral operators is to determine conditions on the parameters p, q , the weight functions u, v , and the kernel k that guarantee the boundedness of the operator K . More precisely, one seeks necessary and sufficient conditions for the existence of a finite constant $C > 0$ such that

$$\|Kf\|_{q,u} \leq C \|f\|_{p,v} \quad \text{for all } f \in L_{p,v}. \quad (2)$$

This problem has been extensively studied by many authors, including K. Andersen, S. Bloom, H. Heinig, R. Kerman, A. Kufner, M. Martin-Reyes, R. Oinarov, L.-E. Persson, E. Sawyer, and V. D. Stepanov; see, for example, [1]–[16] and the references therein.

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In the majority of these works, the primary focus was on deriving necessary and sufficient conditions for the boundedness of the operator, whereas comparatively little attention was devoted to determining the operator norm or obtaining explicit estimates for it. However, in recent years, especially in the study of qualitative properties of differential equations, the role of the operator norm has become increasingly significant. In particular, sharp or explicit estimates of operator norms are often crucial in applications. Beginning around 2020, a growing body of literature has emerged that is specifically devoted to deriving various estimates for operator norm. Below, we briefly summarize some results in this direction.

In these studies, the kernel of the operator is typically assumed to satisfy the so-called *Oinarov conditions*. Namely, the continuous function $k(x, t)$ is assumed to be nondecreasing with respect to the first argument and nonincreasing with respect to the second argument, and there exists a constant $h \geq 1$ such that

$$k(x, t) \leq h(k(x, s) + k(s, t)), \quad a < t < s < x < b. \quad (3)$$

A kernel satisfying these properties is referred to as an *Oinarov kernel*.

In 2021, A. Kalybay and A. Baiarystanov [5] established the following two-sided estimates for the operator norm in the case $1 < p \leq q < \infty$:

$$A \leq \|K\| \leq (h+1)^3 p^{1/q} (p')^{1/p'} A, \quad (4)$$

where $\|K\|$ denotes the operator norm, $p' = \frac{p}{p-1}$ and $A = \max\{A_1, A_2\}$. Here,

$$A_1 = \sup_{x \in (a, b)} \left(\int_x^b k^q(t, x) u(t) dt \right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'}(t) dt \right)^{\frac{1}{p'}},$$

and

$$A_2 = \sup_{x \in (a, b)} \left(\int_x^b u(t) dt \right)^{\frac{1}{q}} \left(\int_a^x k^{p'}(x, t) v^{1-p'}(t) dt \right)^{\frac{1}{p'}}.$$

Later, in 2023, K. Kuliev, G. Kulieva, and M. Eshimova [9, 8] obtained the estimates

$$A \leq \|K\| \leq X,$$

where X denotes the positive solution of the nonlinear equation

$$X^{q'} - h q^{\frac{1}{q-1}} (q')^{\frac{1}{q-1}} A^{\frac{1}{q-1}} X = h q^{\frac{1}{q-1}} (p')^{\frac{q'}{p'}} A^{q'}, \quad (5)$$

with $q' = \frac{q}{q-1}$. More recently, in 2024, A. Kufner, K. Kuliev, G. Kulieva, and M. Eshimova [7] derived new equivalent conditions that led to further improvements in the estimates of the operator norm.

Additionally, in 2022, K. Kuliev [10] obtained estimates for the operator norm in the case $1 < q \leq p < \infty$. In these results, the upper estimates is again expressed in terms of the positive solution of a nonlinear equation of the form (5). Upper and lower

estimates for the norm of the operator (1) were also established in [3] for a special choice of the kernel.

In this paper, we obtain explicit and readily applicable estimates for the operator norm. The principal advantage of our results lies in the fact that the upper estimates are given in explicit form and improve upon the estimate in (4). The paper is organized as follows. In Section 2, we present the main results, where two-sided estimates for the operator norm are established (see Theorems 1 and 2). Section 3 is devoted to the proofs of these results.

2. Main part

This section presents the main results of the paper, together with the auxiliary results.

Let denote

$$B_1 := \left(\int_a^b \left(\int_x^b k^q(t, x) u(t) dt \right)^{\frac{r}{q}} \left(\int_a^x v^{1-p'}(t) dt \right)^{\frac{r}{q'}} v^{1-p'}(x) dx \right)^{\frac{1}{r}}$$

and

$$B_2 := \left(\int_a^b \left(\int_x^b u(t) dt \right)^{\frac{r}{p}} \left(\int_a^x k^{p'}(x, t) v^{1-p'}(t) dt \right)^{\frac{r}{p'}} u(x) dx \right)^{\frac{1}{r}}.$$

Then our first result reads:

THEOREM 1. *Let $1 < q < p < \infty$, and let $k(x, t)$ be an Oinarov kernel. Assume that the operator K defined by (1) is bounded; that is,*

$$B_1 < \infty \quad \text{and} \quad B_2 < \infty.$$

Then the norm of K satisfies the following two-sided estimate:

$$\max \left\{ q \left(\frac{p'}{r} \right)^{\frac{1}{q'}} B_1, p' \left(\frac{q}{r} \right)^{\frac{1}{p}} B_2 \right\} \leq \|K\| \leq 4h \left[\left(\frac{r}{p'} \right)^{\frac{1}{r}} B_1 + hq^{\frac{1}{pq'}} (p')^{\frac{1}{p'}} B_2 \right].$$

Our second result reads as follows.

THEOREM 2. *Let $1 < p \leq q < \infty$, and let $k(x, t)$ be an Oinarov kernel. Assume that the operator (1) is bounded; that is,*

$$A_1 < \infty \quad \text{and} \quad A_2 < \infty.$$

Then the operator norm of K satisfies the two-sided estimate

$$\max \{A_1, A_2\} \leq \|K\| \leq 4h \left[A_1 + h(q')^{\frac{1}{p'}} q^{\frac{1}{q}} A_2 \right].$$

REMARK 1. We note the following results.

1. If $1 < q < p < \infty$, then the conditions $B_1 < \infty$ and $B_2 < \infty$ are equivalent to the boundedness of the operator (1).
2. If $1 < p \leq q < \infty$, then the conditions $A_1 < \infty$ and $A_2 < \infty$ are equivalent to the boundedness of the operator (1).

For further details and proofs, we refer the reader to [11].

Duality principle. The concept of duality plays a crucial role in the proofs of the main results when working with conjugate inequalities.

LEMMA 1. Let $1 < p, q < \infty$, $0 < C < \infty$, and let $k(x, t)$ be a nonnegative measurable function on $(a, b) \times (a, b)$. Then the inequality

$$\left(\int_a^b \left| \int_a^x k(x, t) f(t) dt \right|^q u(x) dx \right)^{1/q} \leq C \left(\int_a^b |f(x)|^p v(x) dx \right)^{1/p} \quad (6)$$

holds for all $f \in L_{p,v}$ if and only if the dual inequality

$$\left(\int_a^b \left| \int_x^b k(t, x) g(t) dt \right|^{p'} v^{1-p'}(x) dx \right)^{1/p'} \leq C \left(\int_a^b |g(x)|^{q'} u^{1-q'}(x) dx \right)^{1/q'} \quad (7)$$

holds for all $g \in L_{q',u^{1-q'}}$, with the same constant C .

The proof of the lemma can be found in [11].

REMARK 2. Using the duality principle together with the main results, we also derive estimates for the norm of the operator

$$\mathcal{K} : L_{p,v} \rightarrow L_{q,u}, \quad (\mathcal{K}g)(t) = \int_t^b k(x, t) g(x) dx. \quad (8)$$

- Let $1 < q < p < \infty$ and set

$$\mathcal{B}_1 := \left(\int_a^b \left(\int_a^x k^q(x, t) u(t) dt \right)^{\frac{r}{q}} \left(\int_x^b v^{1-p'}(t) dt \right)^{\frac{r}{q'}} v^{1-p'}(x) dx \right)^{\frac{1}{r}}$$

and

$$\mathcal{B}_2 := \left(\int_a^b \left(\int_a^x u(t) dt \right)^{\frac{r}{p}} \left(\int_x^b k^{p'}(t, x) v^{1-p'}(t) dt \right)^{\frac{r}{p'}} u(x) dx \right)^{\frac{1}{r}}.$$

If the operator $\mathcal{K}$ defined by (8) is bounded, that is,

$$\mathcal{B}_1 < \infty \quad \text{and} \quad \mathcal{B}_2 < \infty,$$

then its norm satisfies the following two-sided estimate:

$$\max \left\{ q \left(\frac{p'}{r} \right)^{\frac{1}{q'}} \mathcal{B}_1, p' \left(\frac{q}{r} \right)^{\frac{1}{p}} \mathcal{B}_2 \right\} \leq \| \mathcal{K} \| \leq 4h \left[\left(\frac{r}{p'} \right)^{\frac{1}{r}} \mathcal{B}_1 + h q^{\frac{1}{pq'}} (p')^{\frac{1}{p'}} \mathcal{B}_2 \right].$$

- Let $1 < p \leq q < \infty$ and define

$$\mathcal{A}_1 := \sup_{x \in (a,b)} \left(\int_a^x k^q(x,t) u(t) dt \right)^{\frac{1}{q}} \left(\int_x^b v^{1-p'}(t) dt \right)^{\frac{1}{p'}}$$

and

$$\mathcal{A}_2 := \sup_{x \in (a,b)} \left(\int_a^x u(t) dt \right)^{\frac{1}{q}} \left(\int_x^b k^{p'}(t,x) v^{1-p'}(t) dt \right)^{\frac{1}{p'}}.$$

If the operator $\mathcal{K}$ defined by (8) is bounded, that is,

$$\mathcal{A}_1 < \infty \quad \text{and} \quad \mathcal{A}_2 < \infty,$$

then its norm satisfies the following estimates:

$$\max \{ \mathcal{A}_1, \mathcal{A}_2 \} \leq \| \mathcal{K} \| \leq 4h \left[\mathcal{A}_1 + h (q')^{\frac{1}{p'}} q^{\frac{1}{q}} \mathcal{A}_2 \right].$$

3. Proofs

In this section, we present proofs of the main theorems.

3.1. Proof of Theorem 1

Proof. (Upper estimate). Using the principle of duality, in order to obtain the upper estimates for the operator norm, we work with the adjoint operator

$$K^* : L_{q', u^{1-q'}} \rightarrow L_{p', v^{1-p'}}, \quad (K^* g)(t) = \int_t^b k(x,t) g(x) dx. \quad (9)$$

Further, we divide the proof into two parts.

I. Assume that $g \in C_c(a,b)$, that is, g is a continuous function with compact support contained in (a,b) , and that g is nonnegative. Then the function $(K^* g)(t)$ is bounded and nonincreasing on (a,b) . Let $d \geq h$ be arbitrary but fixed. Then there exists an integer $m \in \mathbb{Z}$ such that

$$(K^* g)(t) \leq (d+1)^{-m}, \quad \text{for all } t \in (a,b).$$

Define

$$m_0 = \max \{ m \in \mathbb{Z} : (K^* g)(t) \leq (d+1)^{-m} \text{ for all } t \in (a,b) \},$$

and set

$$t_{m_0} = a, \quad t_k = \sup \left\{ t \in (a, b) : (K^*g)(t) = (d+1)^{-k} \right\}, \quad k > m_0.$$

Since $(K^*g)(t)$ is continuous, it follows that

$$(K^*g)(t_k) = (d+1)^{-k}.$$

Furthermore, for every integer $k \geq m_0$ we have $t_k < t_{k+1}$. Indeed,

$$(K^*g)(t_k) = (d+1)^{-k} > (d+1)^{-(k+1)} = (K^*g)(t_{k+1}),$$

and the monotonicity of (K^*g) implies $t_k < t_{k+1}$.

Thus, the sequence $\{t_k\}_{k=m_0}^\infty \subset (a, b)$ satisfies

$$(a, b) = \bigcup_{k=m_0+1}^\infty (t_{k-1}, t_k],$$

where the intervals $(t_{k-1}, t_k]$ are pairwise disjoint.

Next, observe that

$$\begin{aligned} (d+1)^{-(k+1)}(d+1-h) &= (d+1)^{-k} - h(d+1)^{-(k+1)} \\ &= (K^*g)(t_k) - h(K^*g)(t_{k+1}) \\ &= \int_{t_k}^b k(x, t_k)g(x)dx - h \int_{t_{k+1}}^b k(x, t_{k+1})g(x)dx \\ &= \int_{t_k}^{t_{k+1}} k(x, t_k)g(x)dx + \int_{t_{k+1}}^b k(x, t_k)g(x)dx - h \int_{t_{k+1}}^b k(x, t_{k+1})g(x)dx \\ &\leq \int_{t_k}^{t_{k+1}} k(x, t_k)g(x)dx + hk(t_{k+1}, t_k) \int_{t_{k+1}}^b g(x)dx. \end{aligned}$$

Using the monotonicity of $K^*g(t)$ together with the above estimate, we obtain

$$\begin{aligned} \|K^*g\|_{p', v^{1-p'}}^{p'} &= \int_a^b (K^*g(s))^{p'} v^{1-p'}(s) ds \\ &= \sum_{k=m_0+1}^\infty \int_{t_{k-1}}^{t_k} (K^*g(s))^{p'} v^{1-p'}(s) ds \\ &\leq \sum_{k=m_0+1}^\infty (K^*g(t_{k-1}))^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \\ &= \sum_{k=m_0+1}^\infty (d+1)^{-p'(k-1)} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \\ &= \frac{(d+1)^{2p'}}{(d+1-h)^{p'}} \sum_{k=m_0+1}^\infty (d+1)^{-p'(k+1)} (d+1-h)^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \\ &\leq \frac{(d+1)^{2p'}}{(d+1-h)^{p'}} \sum_{k=m_0+1}^\infty \left[\int_{t_k}^{t_{k+1}} k(x, t_k)g(x)dx + hk(t_{k+1}, t_k) \int_{t_{k+1}}^b g(x)dx \right]^{p'} \\ &\quad \times \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds, \end{aligned}$$

i.e.,

$$\|K^*g\|_{p', v^{1-p'}} \leq \frac{(d+1)^2}{d+1-h} \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k(x, t_k) g(x) dx + \right. \right. \\ \left. \left. + h k(t_{k+1}, t_k) \int_{t_{k+1}}^b g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{\frac{1}{p'}}.$$

Applying Minkowski's inequality we get

$$\|K^*g\|_{p', v^{1-p'}} \leq \frac{(d+1)^2}{d+1-h} (I_1 + hI_2), \quad (10)$$

where

$$I_1 = \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k(x, t_k) g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{1/p'}$$

and

$$I_2 = \left(\sum_{k=m_0+1}^{\infty} \left[k(t_{k+1}, t_k) \int_{t_{k+1}}^b g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{1/p'}.$$

II. We further estimate the quantities I_1 and I_2 separately.

Applying Hölder's inequality twice, we obtain

$$\begin{aligned} I_1 &= \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k(x, t_k) g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \\ &\leq \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k^q(x, t_k) u(x) dx \right]^{\frac{p'}{q}} \left[\int_{t_k}^{t_{k+1}} g^{q'}(x) u^{1-q'}(x) dx \right]^{\frac{p'}{q'}} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \\ &= \left(\sum_{k=m_0+1}^{\infty} \left\{ \left[\int_{t_k}^{t_{k+1}} k^q(x, t_k) u(x) dx \right]^{\frac{p'}{q}} \left[\int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right] \right\} \right. \\ &\quad \left. \times \left\{ \left[\int_{t_k}^{t_{k+1}} g^{q'}(x) u^{1-q'}(x) dx \right]^{\frac{p'}{q'}} \right\} \right)^{\frac{1}{p'}} \\ &\leq \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k^q(x, t_k) u(x) dx \right]^{\frac{r}{q}} \left[\int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right]^{\frac{r}{p'}} \right)^{\frac{1}{r}} \\ &\quad \times \left(\sum_{k=m_0+1}^{\infty} \int_{t_k}^{t_{k+1}} g^{q'}(x) u^{1-q'}(x) dx \right)^{\frac{1}{q'}} \\ &= \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k^q(x, t_k) u(x) dx \right]^{\frac{r}{q}} \left[\int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right]^{\frac{r}{p'}} \right)^{\frac{1}{r}} \cdot \|g\|_{q', u^{1-q'}}. \end{aligned}$$

Using the equality

$$\left(\int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{\frac{r}{p'}} = \frac{r}{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(t) \left(\int_{t_{k-1}}^t v^{1-p'}(s) ds \right)^{\frac{r}{q'}} dt$$

and monotonicity of $k(x, t)$ we have:

$$\begin{aligned} & \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k^q(x, t_k) u(x) dx \right]^{\frac{r}{q}} \left[\int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right]^{\frac{r}{p'}} \right)^{\frac{1}{r}} = \\ &= \left(\frac{r}{p'} \right)^{\frac{1}{r}} \left(\sum_{k=m_0+1}^{\infty} \int_{t_{k-1}}^{t_k} \left[\int_{t_k}^{t_{k+1}} k^q(x, t_k) u(x) dx \right]^{\frac{r}{q}} \left(\int_{t_{k-1}}^t v^{1-p'}(s) ds \right)^{\frac{r}{q'}} v^{1-p'}(t) dt \right)^{\frac{1}{r}} \\ &\leq \left(\frac{r}{p'} \right)^{\frac{1}{r}} \left(\sum_{k=m_0+1}^{\infty} \int_{t_{k-1}}^{t_k} \left[\int_t^b k^q(x, t) u(x) dx \right]^{\frac{r}{q}} \left(\int_a^t v^{1-p'}(s) ds \right)^{\frac{r}{q'}} v^{1-p'}(t) dt \right)^{\frac{1}{r}} \\ &= \left(\frac{r}{p'} \right)^{\frac{1}{r}} \left(\int_a^b \left[\int_t^b k^q(x, t) u(x) dx \right]^{\frac{r}{q}} \left(\int_a^t v^{1-p'}(s) ds \right)^{\frac{r}{q'}} v^{1-p'}(t) dt \right)^{\frac{1}{r}} \\ &= \left(\frac{r}{p'} \right)^{\frac{1}{r}} B_1. \end{aligned}$$

Thus, we get

$$I_1 \leq \left(\frac{r}{p'} \right)^{\frac{1}{r}} B_1 \|g\|_{q', u^{1-q'}}. \quad (11)$$

Further, we estimate the term I_2 . To this end, we rewrite I_2 as

$$\begin{aligned} I_2 &= \left(\sum_{k=m_0+1}^{\infty} \left[k(t_{k+1}, t_k) \int_{t_{k+1}}^b g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{1/p'} \\ &= \left(\int_a^b \left[\int_t^b g(x) dx \right]^{p'} d\mu(t) \right)^{1/p'}, \end{aligned}$$

where

$$\mu([t, \infty)) = \sum_{l=m_0+1}^k \int_{t_{l-1}}^{t_l} k^{p'}(t_{l+1}, t_l) v^{1-p'}(s) ds.$$

and then applying the generalized Hardy inequality (see page 48 in [13]), we obtain

$$I_2 \leq q^{\frac{1}{p'q}} (p')^{\frac{1}{p'}} B \|g\|_{q', u^{1-q'}}.$$

The validity of this estimate follows from the monotonicity of the kernel,

$$k(t_{k+1}, t_k) \leq k(t, s) \quad \text{for} \quad t_{k-1} \leq s \leq t_k \leq t_{k+1} \leq t,$$

and from the estimate

$$\begin{aligned} B &= \left(\int_a^b (\mu([t, \infty)))^{\frac{r}{p'}} \left(\int_t^b u(x) dx \right)^{\frac{r}{p}} u(t) dt \right)^{\frac{1}{r}} \\ &\leq \left(\int_a^b \left(\int_a^t k^{p'}(t, s) v^{1-p'}(s) ds \right)^{\frac{r}{p'}} \left(\int_t^b u(x) dx \right)^{\frac{r}{p}} u(t) dt \right)^{\frac{1}{r}} = B_2 < \infty. \end{aligned}$$

Consequently, we conclude that

$$I_2 \leq q^{\frac{1}{pq'}} (p')^{\frac{1}{p'}} B \leq q^{\frac{1}{pq'}} (p')^{\frac{1}{p'}} B_2.$$

Using this together with (11) in (10), we obtain

$$\|K^* g\|_{p', v^{1-p'}} \leq \frac{(d+1)^2}{d+1-h} \left[\left(\frac{r}{p'} \right)^{\frac{1}{r}} B_1 + h q^{\frac{1}{pq'}} (p')^{\frac{1}{p'}} B_2 \right] \|g\|_{q', u^{1-q'}}. \quad (12)$$

Choosing $d = 2h - 1$, which minimizes the expression $\frac{(d+1)^2}{d+1-h}$, we arrive at

$$\|K^* g\|_{p', v^{1-p'}} \leq 4h \left[\left(\frac{r}{p'} \right)^{\frac{1}{r}} B_1 + h q^{\frac{1}{pq'}} (p')^{\frac{1}{p'}} B_2 \right] \|g\|_{q', u^{1-q'}}.$$

From this and by the definition of the norm we have

$$\begin{aligned} \|K^*\| &= \sup_{g \neq 0, g \in L_{q', u^{1-q'}}} \frac{\|K^* g\|_{p', v^{1-p'}}}{\|g\|_{q', u^{1-q'}}} \\ &= \sup_{g \geq 0, g \in C_c(a, b)} \frac{\|K^* g\|_{p', v^{1-p'}}}{\|g\|_{q', u^{1-q'}}} \\ &\leq 4h \left[\left(\frac{r}{p'} \right)^{\frac{1}{r}} B_1 + h q^{\frac{1}{pq'}} (p')^{\frac{1}{p'}} B_2 \right]. \end{aligned}$$

The upper estimate is proved.

(Lower estimate). Assume that the operator (1) is bounded, that is, inequality (2) holds. Then the following Hardy-type inequality

$$\left(\int_a^b \left(\int_a^x k(x, t) f(t) dt \right)^q u(x) dx \right)^{1/q} \leq C \left(\int_a^b f^p(x) v(x) dx \right)^{1/p} \quad (13)$$

holds for all nonnegative functions $f \in L_{p, v}$.

Using the monotonicity of the kernel $k(x, t)$ and integration by parts, we obtain

$$\begin{aligned}
 C^q \left(\int_a^b f^p(x) v(x) dx \right)^{q/p} &\geq \int_a^b \left(\int_a^x k(x, t) f(t) dt \right)^q u(x) dx \\
 &= q \int_a^b \left[\int_a^x \left(\int_a^t k(x, s) f(s) ds \right)^{q-1} k(x, t) f(t) dt \right] u(x) dx \\
 &\geq q \int_a^b \left[\int_a^x \left(\int_a^t f(s) ds \right)^{q-1} k^q(x, t) f(t) dt \right] u(x) dx \\
 &= q \int_a^b \left(\int_a^t f(s) ds \right)^{q-1} \left[\int_t^b k^q(x, t) u(x) dx \right] f(t) dt \\
 &= \int_a^b \left[\int_t^b k^q(x, t) u(x) dx \right] d \left(\int_a^t f(s) ds \right)^q \\
 &= \int_a^b \left(\int_a^t f(s) ds \right)^q \frac{d}{dt} \left[- \int_t^b k^q(x, t) u(x) dx \right] dt.
 \end{aligned}$$

Hence, we arrive at the generalized Hardy inequality

$$\left(\int_a^b \left(\int_a^t f(s) ds \right)^q \frac{d}{dt} \left[- \int_t^b k^q(x, t) u(x) dx \right] dt \right)^{1/q} \leq C \left(\int_a^b f^p(x) v(x) dx \right)^{1/p}. \quad (14)$$

The necessary and sufficient conditions for the validity of inequality (14) are given in [15], see Theorem 1.15 on page 13. In particular, the following lower estimate for the constant C holds:

$$q^{\frac{1}{q}} \left(\frac{p'q}{r} \right)^{1/q'} B_1 \leq C.$$

Next, we prove the second part of the lower estimate, namely,

$$(p')^{1/p'} \left(\frac{p'q}{r} \right)^{1/p} B_2 \leq C.$$

To this end, we apply Lemma 1. From the boundedness of the operator (1), that is, from inequality (2), we deduce the validity of the dual Hardy-type inequality (7). Repeating the same arguments as in the proof of the first part of the lower estimate, we obtain the following generalized Hardy inequality:

$$\left(\int_a^b \left(\int_x^b g(t) dt \right)^{p'} \frac{d}{dx} \left[\int_a^x k^{p'}(x, t) v^{1-p'}(t) dt \right] dx \right)^{1/p'} \leq C \left(\int_a^b g(x)^{q'} u^{1-q'}(x) dx \right)^{1/q'} \quad (15)$$

which holds for all $g \in L_{q', u^{1-q'}}$ with the same constant C .

The necessary and sufficient conditions for the validity of inequality (14), as well as lower and upper estimates for the best constant C , follow from Theorem 1.15 on page 13 in [15] and Lemma 1.

The proof is complete. $\square$

3.2. Proof of Theorem 2

Proof. (Upper estimate). Repeating same procedure as the part I of the proof of Theorem 1 we get

$$\|K^*g\|_{p', v^{1-p'}} \leq \frac{(d+1)^2}{d+1-h} (I_1 + hI_2), \quad (16)$$

where

$$I_1 = \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k(x, t_k) g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{1/p'}$$

and

$$I_2 = \left(\sum_{k=m_0+1}^{\infty} \left[k(t_{k+1}, t_k) \int_{t_{k+1}}^b g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{1/p'}.$$

Then we estimate the integrals I_1 and I_2 separately.

In I_1 by applying Hölder's inequality, we obtain that

$$\begin{aligned} I_1 &= \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k(x, t_k) g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \\ &\leq \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} k^q(x, t_k) u(x) dx \right]^{\frac{p'}{q}} \left[\int_{t_k}^{t_{k+1}} g^{q'}(x) u^{1-q'}(x) dx \right]^{\frac{p'}{q'}} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \\ &= \left(\sum_{k=m_0+1}^{\infty} \left\{ \left[\int_{t_k}^{t_{k+1}} k^q(x, t_k) u(x) dx \right]^{\frac{p'}{q}} \left[\int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right] \right\} \right. \\ &\quad \left. \times \left\{ \left[\int_{t_k}^{t_{k+1}} g^{q'}(x) u^{1-q'}(x) dx \right]^{\frac{p'}{q'}} \right\} \right)^{\frac{1}{p'}} \\ &\leq \left(\sum_{k=m_0+1}^{\infty} \left\{ \left[\int_{t_k}^b k^q(x, t_k) u(x) dx \right]^{\frac{p'}{q}} \left[\int_a^{t_k} v^{1-p'}(s) ds \right] \right\} \right. \\ &\quad \left. \times \left\{ \left[\int_{t_k}^{t_{k+1}} g^{q'}(x) u^{1-q'}(x) dx \right]^{\frac{p'}{q'}} \right\} \right)^{\frac{1}{p'}} \\ &\leq A_1 \left(\sum_{k=m_0+1}^{\infty} \left[\int_{t_k}^{t_{k+1}} g^{q'}(x) u^{1-q'}(x) dx \right]^{\frac{p'}{q'}} \right)^{\frac{1}{p'}} \\ &\leq A_1 \left(\sum_{k=m_0+1}^{\infty} \int_{t_k}^{t_{k+1}} g^{q'}(x) u^{1-q'}(x) dx \right)^{\frac{1}{q'}} \\ &= A_1 \left(\int_a^b g^{q'}(x) u^{1-q'}(x) dx \right)^{\frac{1}{q'}}. \end{aligned}$$

Thus, we get

$$I_1 \leq A_1 \cdot \|g\|_{q', u^{1-q'}}. \quad (17)$$

Next, we estimate the quantity I_2 . To this end, we apply Hardy's inequality (5.7) from [11], which yields

$$\begin{aligned} I_2 &= \left(\sum_{k=m_0+1}^{\infty} \left[k(t_{k+1}, t_k) \int_{t_{k+1}}^b g(x) dx \right]^{p'} \int_{t_{k-1}}^{t_k} v^{1-p'}(s) ds \right)^{1/p'} \\ &= \left(\int_a^b \left[\int_t^b g(x) dx \right]^{p'} d\mu(t) \right)^{1/p'} \leq C \left(\int_a^b g^{q'}(x) u^{1-q'}(x) dx \right)^{1/q'}. \end{aligned}$$

In deriving this estimate, we used the monotonicity property of the kernel, namely,

$$k(t_{k+1}, t_k) \leq k(t, s) \quad \text{for} \quad t_{k-1} \leq s \leq t_k \leq t_{k+1} \leq t.$$

Consequently, we have

$$\mu([t, \infty)) = \sum_{l=m_0+1}^k \int_{t_{l-1}}^{t_l} k^{p'}(t_{l+1}, t_l) v^{1-p'}(s) ds \leq \int_a^t k^{p'}(t, s) v^{1-p'}(s) ds,$$

and

$$A = \sup_{t \in (a, b)} (\mu([t, \infty)))^{1/q} \left(\int_t^b u(x) dx \right)^{1/p'} \leq A_2 < \infty.$$

Therefore, we obtain

$$I_2 \leq (q')^{1/p'} q^{1/q} A_2 \|g\|_{q', u^{1-q'}}.$$

Using this together with (17) in (10), we get

$$\|K^* g\|_{p', v^{1-p'}} \leq \frac{(d+1)^2}{d+1-h} \left[A_1 + h(q')^{\frac{1}{p'}} q^{\frac{1}{q}} A_2 \right] \cdot \|g\|_{q', u^{1-q'}}. \quad (18)$$

Choosing $d = 2h - 1$, which minimizes the expression $\frac{(d+1)^2}{d+1-h}$, we arrive at

$$\|K^* g\|_{p', v^{1-p'}} \leq 4h \left[A_1 + h(q')^{\frac{1}{p'}} q^{\frac{1}{q}} A_2 \right] \cdot \|g\|_{q', u^{1-q'}}.$$

From this and by the definition of the norm we have

$$\begin{aligned} \|K\| &= \|K^*\| = \sup_{g \neq 0, g \in L_{q', u^{1-q'}}} \frac{\|K^* g\|_{p', v^{1-p'}}}{\|g\|_{q', u^{1-q'}}} \\ &= \sup_{g \geq 0, g \in C_c(a, b)} \frac{\|K^* g\|_{p', v^{1-p'}}}{\|g\|_{q', u^{1-q'}}} \leq 4h \left[A_1 + h(q')^{\frac{1}{p'}} q^{\frac{1}{q}} A_2 \right]. \end{aligned}$$

The upper estimate is proved.

(*Lower estimate*). The lower estimate is proved analogously to the proof of the lower estimate in the previous theorem. In particular, it follows from the generalized Hardy inequalities (14) and (15), Theorem 1.14 on page 13 of [15], and Lemma 1.

The proof is complete. $\square$

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