

REFINEMENTS OF SOME CLASSICAL INEQUALITIES VIA SUPERTERZATIC FUNCTIONS

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*Dedicated to the memory of the bright mathematician and nice human being
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Abstract. Using superterzaticity of power functions we obtain new refinements of Hölder's, Minkowski's and Beckenbach-Dresher's inequalities with "turning point" $p = 3$. Some new two-sided versions of these inequalities for the intervals $1 < p \leq 2$ and $2 \leq p \leq 3$ are also stated.

1. Introduction

Classical inequalities have been very important for the development of several branches of mathematics as functional analysis, interpolation theory, theory of differential and integral equations, probability, harmonic analysis, function theory, homogenization theory, etc., and thus also for several applications in engineering sciences. Hence, it is not surprising that the area "Inequalities" has been developed to an independent area of increasing interest. A lot of papers are devoted to this area and more than 50 monographs. Of them we want to highlight the latest of them namely the one by L. Nikolova, L.-E. Persson and S. Varošanec [11]. The reason is not that this is the only book devoted to so called continuous inequalities but also in preface and introduction a reader friendly description of the current status can be found. In this paper we even complement this book with some new refinements of some classical inequalities in such more general continuous forms.

It is also known that convexity is one main concept to prove classical inequalities. Here we refer to the books mentioned above and among other publications related to this paper we especially mention the paper [16], the Lecture Notes [14] and the well cited book [13] by C. Niculescu and L.-E. Persson. It is also known that variants of convexity (strong convexity, superquadraticity, weak superquadraticity, γ -convexity,

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etc.) can be used to derive refinements of classical inequalities, see for example [1, 2, 3, 4, 5, 7, 8, 9, 12, 17] and especially the book [11] and the references therein. For example, in [12] (see also [6]) it was proved refinements of some classical inequalities with the “turning point” $p = 2$.

One main novelty of this paper is that we use a new definition of so called superterzaticity to obtain new refinements of some classical inequalities, some even in continuous form, now with “turning point” $p = 3$. Another refinement with this turning point for Hardy’s inequality was derived in [7] but there the idea of proof was different.

The paper is organized as follows: In Section 3, we present our main results concerning new refinements of the Hölder, Minkowski and Beckenbach-Dresher inequalities (see Theorems 6, 7 and 9). The detailed proofs are given in Section 4. By also combining with other known inequalities we even get some new two-sided inequalities of the Hölder, (and even continuous forms) of the Minkowski and Beckenbach-Dresher’s inequalities for the cases $1 < p \leq 2$ and $2 \leq p \leq 3$. Some final remarks and results are given in Section 5. In order not disturb our discussions in these Sections we present some necessary preliminaries in Section 2 with main focus on superterzatic functions.

2. Preliminaries with focus on a new definition of superterzaticity

In this Section we give the following basic definition, remarks and known results (see e.g. [1, 3, 12]) and definition of superterzaticity.

DEFINITION 1. ([3, Definition 2.1]) A function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C_x(y - x)$$

for all $y \geq 0$. We say that f is subquadratic if $-f$ is superquadratic. This inequality reduces to equality when $\varphi(x) = x^2$ with $C_x = 2x$.

THEOREM 1. ([3, Theorem 2.2]) The function φ is superquadratic if and only if

$$\varphi\left(\int_{\Omega} f(s) d\mu(s)\right) \leq \int_{\Omega} \varphi(f(s)) d\mu(s) - \int_{\Omega} \varphi\left(\left|f(s) - \int_{\Omega} f(s) d\mu(s)\right|\right) d\mu(s) \quad (1)$$

holds for all probability measures μ and all non-negative μ -integrable functions f on the measure space (Ω, μ) . Moreover, (1) holds in the reversed direction if and only if φ is subquadratic.

By using this theorem, for example, the following results we need later on can be proved:

THEOREM 2. ([17, Theorem 1.1]) Let (Ω, μ) be a measure space and suppose that $p \geq 2$ and $1/p + 1/p' = 1$. Then

$$\int_{\Omega} fg d\mu \leq \left(\int_{\Omega} (f^p - h_f^p) d\mu\right)^{\frac{1}{p}} \left(\int_{\Omega} g^{p'} d\mu\right)^{\frac{1}{p'}} \quad (2)$$

holds for any two non-negative μ -measurable functions f and g , where

$$h_J = \left| f - \frac{g^{p'-1} \int_{\Omega} fg \, d\mu}{\int_{\Omega} g^{p'} \, d\mu} \right|. \quad (3)$$

Moreover, (2) holds in the reversed direction if $1 < p \leq 2$.

The previous theorem is a result of Hölder type and it was discovered in [17]. The following two theorems are connected with Minkowski's and Beckenbach-Dresher's inequalities and proved in [12].

THEOREM 3. ([12, Theorem 3.3]) *Let f be a non-negative measurable function on $\Omega \times \Psi$ with respect to the measure $\mu \times \nu$ and let $p \geq 2$. Then*

$$\left(\int_{\Omega} \left(\int_{\Psi} f \, d\nu \right)^p \, d\mu \right)^{\frac{1}{p}} \leq \int_{\Psi} \left(\int_{\Omega} (f^p - h^p) \, d\mu \right)^{\frac{1}{p}} \, d\nu, \quad (4)$$

where

$$h = \left| f - \frac{\int_{\Psi} f(x, y) \, d\nu \cdot \int_{\Omega} f \left(\int_{\Psi} f(x, y) \, d\nu \right)^{p-1} \, d\mu}{\int_{\Omega} \left(\int_{\Psi} f(x, y) \, d\nu \right)^p \, d\mu} \right|. \quad (5)$$

If $1 < p \leq 2$, then (4) holds in the reversed direction.

THEOREM 4. ([12, Theorem 3.6]) *Let f and u be non-negative measurable functions on $\Omega \times \Psi$ with respect to the measures $\mu \times \nu$ and $\lambda \times \nu$, respectively, and let $1 < q \leq 2 \leq p$, $s \geq 1$. Then*

$$\frac{\left(\int_{\Omega} \left(\int_{\Psi} f \, d\nu \right)^p \, d\mu \right)^{\frac{s}{p}}}{\left(\int_{\Omega} \left(\int_{\Psi} u \, d\nu \right)^q \, d\lambda \right)^{\frac{s-1}{q}}} \leq \int_{\Psi} \frac{\left(\int_{\Omega} (f^p - h^p) \, d\mu \right)^{\frac{s}{p}}}{\left(\int_{\Omega} (u^q - r^q) \, d\lambda \right)^{\frac{s-1}{q}}} \, d\nu,$$

where h is defined in (5) and

$$r = \left| u - \frac{\int_{\Psi} u(x, y) \, d\nu \cdot \int_{\Omega} u \left(\int_{\Psi} u(x, y) \, d\nu \right)^{q-1} \, d\lambda}{\int_{\Omega} \left(\int_{\Psi} u(x, y) \, d\nu \right)^q \, d\lambda} \right|.$$

The most important definition in this paper is the following definition of superterzaticity (see [7, p. 683]):

DEFINITION 2. A function $\varphi : [0, b) \rightarrow \mathbb{R}$ is superterzatic provided that for all $x \in [0, b)$, there exist constants $C(x) \in \mathbb{R}$ and $D(x) \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) \geq C(x)y(y-x) + D(x)(y-x) + y|y-x|^{-1}\varphi(|y-x|) \quad (6)$$

for all $y \in [0, b)$.

We say that φ is subterzatic if the reverse of inequality (6) holds.

REMARK 1. (See [7, p. 683]) If ψ is a differentiable and superquadratic function such that $\psi(0) = \psi'(0) = 0$, then $\varphi(x) = x\psi(x)$ is superterzatic with $C(x) = \psi'(x)$ and $D(x) = \psi(x)$.

REMARK 2. The definition of superterzaticity was first given in [2]. However, it was completely different but it was showed in [7] that this definition is equivalent to (6) so we prefer to from now use this as the definition.

The name “Superterzatic function” is used because (6) holds for $\varphi(x) = x^p$, $p \geq 3$, $x \geq 0$, with equality for $p = 3$, $C(x) = 2x$ and $D(x) = x^2$. Inequality (6) holds in the reversed direction for $1 < p < 3$.

In particular, it is then possible to prove the following version of Jensen’s inequality (see [7, Theorem 2]).

THEOREM 5. *The function φ is superterzatic if and only if*

$$\begin{aligned} & \int_{\Omega} \varphi(f(s)) \, d\mu(s) - \varphi \left(\int_{\Omega} f(s) \, d\mu(s) \right) \\ & \geq C \left(\int_{\Omega} f(s) \, d\mu(s) \right) \int_{\Omega} f(s) \left(f(s) - \int_{\Omega} f(t) \, d\mu(t) \right) \, d\mu(s) \\ & \quad + \int_{\Omega} f(s) \left| f(s) - \int_{\Omega} f(t) \, d\mu(t) \right|^{-1} \varphi \left(\left| f(s) - \int_{\Omega} f(t) \, d\mu(t) \right| \right) \, d\mu(s) \end{aligned} \quad (7)$$

holds for all probability measures μ and all non-negative μ -integrable functions f on the measure space (Ω, μ) . Moreover, (7) holds in the reversed direction if and only if φ is subterzatic. Furthermore, (7) reduces to equality when $\varphi(x) = x^3$, $x \geq 0$.

3. Main results

In the following discussion, we consider the measure spaces (Ω, μ) , (Ω, λ) and (Ψ, ν) . Moreover, $d\mu$, $d\lambda$ and $d\nu$ are notations for $d\mu(x)$, $d\lambda(x)$ and $d\nu(y)$, respectively. In all statements, we assume that all integrals exist.

Our first main result in this Section is the following refinement of Hölder’s inequality:

THEOREM 6. Let (Ω, μ) be a measure space. Suppose that $p \geq 3$ and $1/p + 1/p' = 1$. Then

$$\int_{\Omega} fg \, d\mu \leq \left\{ \int_{\Omega} f^p \, d\mu - (p-1) \left(\frac{\int_{\Omega} fg \, d\mu}{\int_{\Omega} g^{p'} \, d\mu} \right)^{p-2} \int_{\Omega} fgk \, d\mu - \int_{\Omega} fg|k|^{p-1} \, d\mu \right\}^{1/p} \left(\int_{\Omega} g^{p'} \, d\mu \right)^{1/p'} \quad (8)$$

where

$$k = fg^{1-p'} - \frac{1}{\int_{\Omega} g^{p'} \, d\mu} \int_{\Omega} fg \, d\mu, \quad (9)$$

holds for any two non-negative μ -measurable functions f and g . Moreover, (8) holds in the reversed direction if $1 \leq p \leq 3$. Inequality (8) reduces to equality for $p = 3$.

In particular, by combining Theorem 6 with Theorem 2 we get the following two-sided refinement of the Hölder inequality for $2 \leq p \leq 3$.

COROLLARY 1. Let $2 \leq p \leq 3$ and $1/p + 1/p' = 1$. If f and g are non-negative μ -measurable functions, then

$$\left\{ \int_{\Omega} f^p \, d\mu - (p-1) \left(\frac{\int_{\Omega} fg \, d\mu}{\int_{\Omega} g^{p'} \, d\mu} \right)^{p-2} \int_{\Omega} fgk \, d\mu - \int_{\Omega} fg|k|^{p-1} \, d\mu \right\}^{1/p} \left(\int_{\Omega} g^{p'} \, d\mu \right)^{1/p'} \\ \leq \int_{\Omega} fg \, d\mu \leq \left(\int_{\Omega} (f^p - h_f^p) \, d\mu \right)^{\frac{1}{p}} \left(\int_{\Omega} g^{p'} \, d\mu \right)^{\frac{1}{p'}},$$

where k and h_f are defined in (9) and (3), respectively.

Moreover, by combining Theorem 6 with the classical Hölder inequality we get the two-sided refinement of the Hölder inequality also for $1 < p \leq 2$.

COROLLARY 2. Suppose that $1 < p \leq 2$, $1/p + 1/p' = 1$, and let k be defined in (9). Then

$$\left\{ \int_{\Omega} f^p \, d\mu - (p-1) \left(\frac{\int_{\Omega} fg \, d\mu}{\int_{\Omega} g^{p'} \, d\mu} \right)^{p-2} \int_{\Omega} fgk \, d\mu - \int_{\Omega} fg|k|^{p-1} \, d\mu \right\}^{1/p} \left(\int_{\Omega} g^{p'} \, d\mu \right)^{1/p'} \\ \leq \int_{\Omega} fg \, d\mu \leq \left(\int_{\Omega} f^p \, d\mu \right)^{\frac{1}{p}} \left(\int_{\Omega} g^{p'} \, d\mu \right)^{\frac{1}{p'}}$$

holds for any two non-negative μ -measurable functions f and g .

The next main result is the following refinement of the continuous Minkowski inequality.

THEOREM 7. *Let f be a non-negative measurable function on $\Omega \times \Psi$ with respect to the measure $\mu \times \nu$. If $p \geq 3$, then*

$$\left(\int_{\Omega} \left(\int_{\Psi} f \, d\nu \right)^p \, d\mu \right)^{\frac{1}{p}} \leq \int_{\Psi} \left(\int_{\Omega} (f^p - \eta) \, d\mu \right)^{1/p} \, d\nu, \quad (10)$$

where

$$\begin{aligned} \eta &= (p-1) \left(\frac{\int_{\Omega} f (\int_{\Psi} f \, d\nu)^{p-1} \, d\mu}{\int_{\Omega} (\int_{\Psi} f \, d\nu)^p \, d\mu} \right)^{p-2} \\ &\quad \times f \left(\frac{f}{\int_{\Psi} f \, d\nu} - \frac{\int_{\Omega} f (\int_{\Psi} f \, d\nu)^{p-1} \, d\mu}{\int_{\Omega} (\int_{\Psi} f \, d\nu)^p \, d\mu} \right) \left(\int_{\Psi} f \, d\nu \right)^{p-1} \\ &\quad + f \left| \frac{f}{\int_{\Psi} f \, d\nu} - \frac{\int_{\Omega} f (\int_{\Psi} f \, d\nu)^{p-1} \, d\mu}{\int_{\Omega} (\int_{\Psi} f \, d\nu)^p \, d\mu} \right|^{p-1} \left(\int_{\Psi} f \, d\nu \right)^{p-1}. \end{aligned} \quad (11)$$

If $1 \leq p \leq 3$, then (10) holds in the reversed direction. Inequality (10) reduces to equality for $p = 3$.

In particular, if $\Psi = \bigcup_{i=1}^n \Psi_i$, where $\Psi_i = [i-1, i)$, for all $i = 1, 2, \dots, n$, $d\nu = dy$ is the Lebesgue measure and $f(x, y) = \sum_{i=1}^n f_i(x) \chi_{\Psi_i}(y)$, then

$$\int_{\Psi} f(x, y) \, d\nu = \sum_{i=1}^n f_i(x).$$

Hence, by applying Theorem 7, we obtain the following discrete version:

COROLLARY 3. *Let $p \geq 3$ and let $f_1, f_2, \dots, f_n$ be non-negative μ -measurable functions. Then*

$$\left(\int_{\Omega} \left(\sum_{i=1}^n f_i \right)^p \, d\mu \right)^{\frac{1}{p}} \leq \sum_{i=1}^n \left(\int_{\Omega} (f_i^p - \eta_i) \, d\mu \right)^{1/p}, \quad (12)$$

where

$$\begin{aligned} \eta_i := & (p-1) \left(\frac{\int_{\Omega} f_i \left(\sum_{j=1}^n f_j \right)^{p-1} d\mu}{\int_{\Omega} \left(\sum_{j=1}^n f_j \right)^p d\mu} \right)^{p-2} \\ & \times f_i \left(\frac{\frac{f_i}{\sum_{j=1}^n f_j} - \frac{1}{\int_{\Omega} \left(\sum_{j=1}^n f_j \right)^p d\mu} \int_{\Omega} f_i \left(\sum_{j=1}^n f_j \right)^{p-1} d\mu}{\left(\sum_{j=1}^n f_j \right)^{p-1}} \right) \\ & + f_i \left| \frac{\frac{f_i}{\sum_{j=1}^n f_j} - \frac{1}{\int_{\Omega} \left(\sum_{j=1}^n f_j \right)^p d\mu} \int_{\Omega} f_i \left(\sum_{j=1}^n f_j \right)^{p-1} d\mu}{\left(\sum_{j=1}^n f_j \right)^{p-1}} \right|^{p-1}. \quad (13) \end{aligned}$$

If $1 \leq p \leq 3$, then (12) holds in the reversed direction. Inequality (12) reduces to equality for $p = 3$.

We combine Theorem 7 with Theorem 3 and obtain the following two-sided refinement of continuous Minkowski's inequality for $2 \leq p \leq 3$.

COROLLARY 4. *Let f be a non-negative measurable function on $\Omega \times \Psi$ with respect to the measure $\mu \times \nu$, and let $2 \leq p \leq 3$. Then*

$$\begin{aligned} \int_{\Psi} \left(\int_{\Omega} (f^p - \eta) d\mu \right)^{1/p} d\nu & \leq \left(\int_{\Omega} \left(\int_{\Psi} f d\nu \right)^p d\mu \right)^{\frac{1}{p}} \\ & \leq \int_{\Psi} \left(\int_{\Omega} (f^p - h^p) d\mu \right)^{\frac{1}{p}} d\nu, \end{aligned}$$

where η and h are defined in (11) and (5), respectively.

Similarly, a combination of Theorem 7 and classical Minkowski's inequality implies the corresponding refinement of Minkowski's inequality for $1 \leq p \leq 2$.

COROLLARY 5. *Let f be a non-negative measurable function on $\Omega \times \Psi$ with respect to the measure $\mu \times \nu$, and let $1 \leq p \leq 2$. Then*

$$\begin{aligned} \int_{\Psi} \left(\int_{\Omega} (f^p - \eta) d\mu \right)^{1/p} d\nu & \leq \left(\int_{\Omega} \left(\int_{\Psi} f d\nu \right)^p d\mu \right)^{\frac{1}{p}} \\ & \leq \int_{\Psi} \left(\int_{\Omega} f^p d\mu \right)^{\frac{1}{p}} d\nu, \end{aligned}$$

where η is defined in (11).

The continuous form of the Beckenbach-Dresher inequality was derived in [10, Theorem 3.1] (see also [18]) but was in a different form known even before (see [15]). It has the following form:

THEOREM 8. *Let f and u be non-negative measurable functions on $\Omega \times \Psi$ with respect to the measures $\mu \times \nu$ and $\lambda \times \nu$, respectively, and let*

(i) $s \geq 1$, $q \leq 1 \leq p$, ($q \neq 0$) or

(ii) $s \leq 0$, $p \leq 1 \leq q$, ($p \neq 0$).

Then

$$\frac{\left(\int_{\Omega} \left(\int_{\Psi} f d\nu\right)^p d\mu\right)^{\frac{s}{p}}}{\left(\int_{\Omega} \left(\int_{\Psi} u d\nu\right)^q d\lambda\right)^{\frac{s-1}{q}}} \leq \int_{\Psi} \frac{\left(\int_{\Omega} f^p d\mu\right)^{\frac{s}{p}}}{\left(\int_{\Omega} u^q d\lambda\right)^{\frac{s-1}{q}}} d\nu, \quad (14)$$

provided all occurring integrals exist. If $0 < s \leq 1$, $p \leq 1$, and $q \leq 1$ ($p, q \neq 0$), then inequality (14) is reversed.

Our next main result related to the continuous Beckenbach-Dresher inequality reads as follows:

THEOREM 9. *Let f and u be non-negative measurable functions on $\Omega \times \Psi$ with respect to the measures $\mu \times \nu$ and $\lambda \times \nu$, respectively, and let $1 \leq q \leq 3 \leq p$, $s \geq 1$. Then*

$$\frac{\left(\int_{\Omega} \left(\int_{\Psi} f d\nu\right)^p d\mu\right)^{\frac{s}{p}}}{\left(\int_{\Omega} \left(\int_{\Psi} u d\nu\right)^q d\lambda\right)^{\frac{s-1}{q}}} \leq \int_{\Psi} \frac{\left(\int_{\Omega} (f^p - \eta) d\mu\right)^{\frac{s}{p}}}{\left(\int_{\Omega} (u^q - \gamma) d\lambda\right)^{\frac{s-1}{q}}} d\nu,$$

where η is defined in (11) and γ arises from η by substitutions: $p \rightarrow q$, $f \rightarrow u$, $\mu \rightarrow \lambda$.

By using Theorem 9 and similar arguments as those in the proof of Corollary 3, we can also derive the following discrete version.

COROLLARY 6. *Let $1 \leq q \leq 3 \leq p$, $s \geq 1$, $f_i, u_i : \Omega \rightarrow [0, \infty)$, $f_i^p, u_i^q \in L^1$ for all $i = 1, \dots, n$. Then*

$$\frac{\left(\int_{\Omega} \left(\sum_{i=1}^n f_i\right)^p d\mu\right)^{\frac{s}{p}}}{\left(\int_{\Omega} \left(\sum_{i=1}^n u_i\right)^q d\lambda\right)^{\frac{s-1}{q}}} \leq \sum_{i=1}^n \frac{\left(\int_{\Omega} (f_i^p - \eta_i) d\mu\right)^{\frac{s}{p}}}{\left(\int_{\Omega} (u_i^q - \gamma_i) d\lambda\right)^{\frac{s-1}{q}}},$$

where η_i is defined in (13) and γ_i arises from η_i by substitutions: $p \rightarrow q$, $f_i \rightarrow u_i$, $\mu \rightarrow \lambda$.

As an application of Corollary 6, by making the substitution $s = \frac{p}{p-q}$, $p \neq q$, we obtain the following Beckenbach-Dresher type inequality.

EXAMPLE 1. Let $1 \leq q \leq 3 \leq p$, $p \neq q$, $f_i, u_i : \Omega \rightarrow [0, \infty)$, $f_i^p, u_i^q \in L^1$ for all $i = 1, \dots, n$. Then

$$\left(\frac{\int_{\Omega} \left(\sum_{i=1}^n f_i \right)^p d\mu}{\int_{\Omega} \left(\sum_{i=1}^n u_i \right)^q d\lambda} \right)^{\frac{1}{p-q}} \leq \sum_{i=1}^n \left(\frac{\int_{\Omega} (f_i^p - \eta_i) d\mu}{\int_{\Omega} (u_i^q - \gamma_i) d\lambda} \right)^{\frac{1}{p-q}},$$

where η_i and γ_i are as in Corollary 6.

4. Proofs of the main results

Proof of Theorem 6. Let $p \geq 3$. By applying Remark 1 and Theorem 5 for $\varphi(x) = x^p$, we find that

$$\begin{aligned} \int_{\Omega} f d\lambda &\leq \left\{ \int_{\Omega} f^p d\lambda - (p-1) \left(\int_{\Omega} f d\lambda \right)^{p-2} \int_{\Omega} f \left(f - \int_{\Omega} f d\lambda \right) d\lambda \right. \\ &\quad \left. - \int_{\Omega} f \left| f - \int_{\Omega} f d\lambda \right|^{p-1} d\lambda \right\}^{1/p}, \end{aligned} \quad (15)$$

for all probability measures λ and all non-negative λ -integrable functions f . By just replacing f with $fg^{1-p'}$ and $d\lambda = \frac{g^{p'}}{\int_{\Omega} g^{p'} d\mu} d\mu$ in (15), we obtain (8). Moreover, by using the similar arguments for $1 < p \leq 3$ we obtain the reversed inequality. In particular, (8) reduces to equality for $p = 3$. The proof is complete. $\square$

Proof of Theorem 7. Let $H(x) = \int_{\Psi} f(x, y) dv$. Let $p \geq 3$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Using Theorem 6, by replacing $f(x)$ and $g(x)$ with $f(x, y)$ and $H^{p-1}(x)$, respectively, we get that

$$\begin{aligned} \int_{\Omega} f H^{p-1} d\mu &\leq \left(\int_{\Omega} H^p d\mu \right)^{1/p'} \left\{ \int_{\Omega} f^p d\mu \right. \\ &\quad - (p-1) \left(\frac{\int_{\Omega} f H^{p-1} d\mu}{\int_{\Omega} H^p d\mu} \right)^{p-2} \int_{\Omega} f H^{p-1} \left(f H^{-1} - \frac{1}{\int_{\Omega} H^p d\mu} \int_{\Omega} f H^{p-1} d\mu \right) d\mu \\ &\quad \left. - \int_{\Omega} f H^{p-1} \left| f H^{-1} - \frac{1}{\int_{\Omega} H^p d\mu} \int_{\Omega} f H^{p-1} d\mu \right|^{p-1} d\mu \right\}^{1/p}. \end{aligned} \quad (16)$$

We integrate inequality (16) over Ψ , apply Fubini's theorem on the left-hand side of the inequality and use a definition of η from (11) to find that

$$\int_{\Omega} H^p d\mu \leq \left(\int_{\Omega} H^p d\mu \right)^{1/p'} \int_{\Psi} \left\{ \int_{\Omega} f^p d\mu - \int_{\Omega} \eta d\mu \right\}^{1/p} dv. \quad (17)$$

Since $1 - \frac{1}{p'} = \frac{1}{p}$, by dividing both sides of (17) by $(\int_{\Omega} H^p d\mu)^{1/p'}$, we find that (10) holds. The proof of the case $1 \leq p \leq 3$ is similar so we omit the details and the proof is complete. $\square$

Proof of Theorem 9. In view of Theorem 7, for $p \geq 3$ and $1 \leq q \leq 3$, we have that

$$\begin{aligned} \frac{\left(\int_{\Omega} \left(\int_{\Psi} f dv \right)^p d\mu \right)^{\frac{s}{p}}}{\left(\int_{\Omega} \left(\int_{\Psi} u dv \right)^q d\lambda \right)^{\frac{s-1}{q}}} &\leq \frac{\left(\int_{\Psi} \left(\int_{\Omega} (f^p - \eta) d\mu \right)^{\frac{1}{p}} dv \right)^s}{\left(\int_{\Psi} \left(\int_{\Omega} (u^q - \gamma) d\lambda \right)^{\frac{1}{q}} dv \right)^{s-1}} \\ &= \left(\int_{\Psi} a^{\frac{1}{s}} dv \right)^s \left(\int_{\Psi} b^{\frac{1}{1-s}} dv \right)^{1-s} \\ &\leq \int_{\Psi} ab dv, \end{aligned}$$

where $a^{\frac{1}{s}} = (\int_{\Omega} (f^p - \eta) d\mu)^{\frac{1}{p}}$ and $b^{\frac{1}{1-s}} = (\int_{\Omega} (u^q - \gamma) d\lambda)^{\frac{1}{q}}$. In the last inequality, we used the reverse Hölder inequality for two functions a and b , and conjugate exponents $s > 1$, $1 - s < 0$. The proof is complete. $\square$

5. Concluding remarks and results

REMARK 3. The natural “turning point” in Minkowski and Beckenbach-Dresher type inequalities is 1, but in [12] the authors proved such results with the turning point at $p = 2$. In our versions of these inequalities, we have proved the corresponding results with turning point $p = 3$ (see Theorems 7 and 9).

Next we note that by combining our results with known results, we get other new results. As an example, we present the following:

THEOREM 10. Let p, s , and t be different real numbers such that $s \geq 3, t \geq 3$ and $(s-t)/(p-t) > 1$. Then

$$\int_{\Omega} \left(\sum_{i=1}^n f_i \right)^p d\mu \leq \left(\sum_{i=1}^n \left(\int_{\Omega} (f_i^s - \sigma_i) d\mu \right)^{\frac{1}{s}} \right)^{s \frac{p-t}{s-t}} \left(\sum_{i=1}^n \left(\int_{\Omega} (f_i^t - \tau_i) d\mu \right)^{\frac{1}{t}} \right)^{t \frac{s-p}{s-t}}, \quad (18)$$

holds for any positive μ -measurable functions $f_1, \dots, f_n$, where σ_i and τ_i arises from η_i defined by (13) after substitutions: $p \rightarrow s$ and $p \rightarrow t$, respectively.

Moreover, if $p \neq 0$, $1 \leq s \leq 3$, $t \geq 3$ or $s \geq 3$, $1 \leq t \leq 3$ and $(s-t)/(p-t) < 1$, then (18) holds in the reversed direction.

Proof of Theorem 10. Let $s \geq 3$, $t \geq 3$ such that $\frac{s-t}{p-t} > 1$. Then, by Hölder's inequality,

$$\begin{aligned} \int_{\Omega} \left(\sum_{i=1}^n f_i \right)^p d\mu &= \int_{\Omega} \left[\left(\sum_{i=1}^n f_i \right)^s \right]^{\frac{p-t}{s-t}} \left[\left(\sum_{i=1}^n f_i \right)^t \right]^{\frac{s-p}{s-t}} d\mu \\ &\leq \left[\int_{\Omega} \left(\sum_{i=1}^n f_i \right)^s d\mu \right]^{\frac{p-t}{s-t}} \left[\int_{\Omega} \left(\sum_{i=1}^n f_i \right)^t d\mu \right]^{\frac{s-p}{s-t}}. \end{aligned}$$

In view of Corollary 3, the above inequality becomes

$$\int_{\Omega} \left(\sum_{i=1}^n f_i \right)^p d\mu \leq \left(\sum_{i=1}^n \left(\int_{\Omega} (f_i^s - \sigma_i) d\mu \right)^{\frac{1}{s}} \right)^{s \frac{p-t}{s-t}} \left(\sum_{i=1}^n \left(\int_{\Omega} (f_i^t - \tau_i) d\mu \right)^{\frac{1}{t}} \right)^{t \frac{s-p}{s-t}}.$$

The proof of the other case is similar, so we omit the details and the proof is complete. $\square$

REMARK 4. For a similar result but with turning points $s = 2$ and $t = 2$ see [12, Proposition 5.2].

REMARK 5. If $s > t$, then we can rewrite (18) in Theorem 10 as

$$\left(\int_{\Omega} \left(\sum_{i=1}^n f_i \right)^p d\mu \right)^{s-t} \leq \frac{\left(\sum_{i=1}^n \left(\int_{\Omega} (f_i^s - \sigma_i) d\mu \right)^{\frac{1}{s}} \right)^{s(p-t)}}{\left(\sum_{i=1}^n \left(\int_{\Omega} (f_i^t - \tau_i) d\mu \right)^{\frac{1}{t}} \right)^{t(p-s)}}.$$

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