

ON THE POINT EVALUATIONS ON A CLASS OF SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. We show that it is possible to calculate norms of the point evaluation functionals on a class of spaces of holomorphic functions on the open unit disc in the complex plane that was introduced by us about two decades ago. Our main results, which are incorporated in a theorem, complement some recent results in the literature. We also present some comments and details related to the main result.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, that is,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

The linear space of holomorphic functions on $\mathbb{D}$ we denote by $H(\mathbb{D})$ [1], whereas by $W(\mathbb{D})$ we denote the class of positive continuous functions on $\mathbb{D}$, which are usually called *weights*, and occur in many papers in different situations, for example, in [2, 3, 13, 15, 24, 25]. By $\mathbb{P}_k$, where $k \in \mathbb{N}$, we denote the linear space of all polynomials with degree less than or equal to k .

If $(X, \|\cdot\|_X)$ is a normed space, then by B_X we denote the closed unit ball in X , that is,

$$B_X = \{x \in X : \|x\|_X \leq 1\}.$$

If f is a bounded linear functional on X , then by $\|f\|_{X^*}$ we denote its norm, that is,

$$\|f\|_{X^*} = \sup_{x \in B_X} |f(x)|,$$

(see, for instance, [7, 32]). A set $\Omega \subseteq \mathbb{C}$ is called domain, if it is an open an connected set.

It is a frequent situation that during investigations of concrete linear operators between spaces of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$, the point evaluation functionals

$$\delta_z(f) = f(z),$$

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where $z \in \Omega$, play an important role. Namely, if X is a space of holomorphic functions on the domain, in many cases an estimate of the form

$$|f(z)| \leq w(z)\|f\|_X,$$

holds for every $f \in X$ and $z \in \Omega$, and some nonnegative function $w(z)$. Such type of estimates then can be employed in getting some properties of concrete linear operators acting from the space X to another space, for example, the boundedness, compactness, as well as some other functional theoretic properties (see, e.g., [5, 8, 11, 18, 19, 21, 29] and the related references therein).

In some cases it is possible to use such estimates in calculating the norm of a linear operator $L : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$, that is, to calculate the quantity

$$\|L\|_{X \rightarrow Y} = \sup_{x \in B_X} \|Lx\|_Y,$$

in terms of the symbols and parameters connected to the operator L and spaces X and Y (for some classical examples of such results, see ([7, 17, 32]; for some recent ones, see [12, 16, 20, 23, 27, 30]).

In order to unify some spaces of holomorphic functions with weights, that we had previously studied such as, weighted-type spaces, Bloch-type spaces and Zygmund-type spaces, we introduced a space of holomorphic functions as follows.

Let $\mu \in W(\mathbb{D})$ and $n \in \mathbb{N}$, then the class of functions $\mathcal{S}_\mu^{(n)} = \mathcal{S}_\mu^{(n)}(\mathbb{D})$ consisting of all $f \in H(\mathbb{D})$ such that

$$s_{\mathcal{S}_\mu^{(n)}}(f) := \sup_{z \in \mathbb{D}} \mu(z) |f^{(n)}(z)| < +\infty,$$

are called the n th weighted space (see, e.g., [24]), or Stević-type spaces [9, 36, 37]. The class $\mathcal{S}_\mu^{(n)}$ is a linear space, and the nonlinear functional $s_{\mathcal{S}_\mu^{(n)}}(\cdot)$ is a semi-norm on the space. However, it is a norm on the space $\mathcal{S}_\mu^{(n)}/\mathbb{P}_{n-1}$.

For $n = 2$, it is obtained the Zygmund-type space $\mathcal{Z}_\mu$ ([15]), which for $\mu(z) = 1 - |z|^2$, $z \in \mathbb{D}$, reduces to the standard Zygmund space $\mathcal{Z}$. The notation for the standard Zygmund space was introduced by the present author in [13], and soon after that the notation $\mathcal{Z}_\mu$ for the Zygmund-type space (see, e.g., [14, 15, 28] and the related references therein). The notation is now widely accepted and the linear operators from or to them have been studied a lot (see, e.g., also [4, 6, 10, 33–35] and the related references therein).

Of a special interest are the weights $(1 - |z|^2)^\alpha$ and $(1 - |z|)^\alpha$, with $\alpha > 0$, and for these weights the n th weighted spaces are denoted by $\mathcal{S}_{\alpha,1}^{(n)}$ and $\mathcal{S}_\alpha^{(n)}$, respectively. The first weight naturally occurs in various topics of complex analysis, whereas the second one is more useful in some situations, since it is somewhat simpler, so more suitable for some calculations. We had such a situation during the calculation of the norm of the weighted composition operators from the α -Bloch spaces to the weighted-type space. Namely, in [22], we managed to calculate the norm for the case of the weight $(1 - |z|^2)^\alpha$, only if $\alpha = 1$, whereas, in [26], we managed to calculate the norm

for the case of the weight $(1 - |z|)^\alpha$ for any $\alpha > 0$. It should be noticed that, due to the inequalities

$$(1 - |z|)^\alpha \leq (1 - |z|^2)^\alpha \leq 2^\alpha (1 - |z|)^\alpha, \quad z \in \mathbb{D},$$

the weights produce equivalent norms on the spaces.

We have noticed recently, that this idea can be used in calculating norms of the point evaluation functionals on the α -Zygmund spaces [31]. Among other results, we have proved that

$$\|\delta_z\|_{(\mathcal{Z}^{1,1}/\mathbb{P}_1)^*} = \frac{1}{2} \int_0^{|z|} \ln \frac{1+s}{1-s} ds, \quad (1)$$

where $\mathcal{Z}^{1,1}$ is the Zygmund space with the semi-norm

$$s_{\mathcal{Z}^{1,1}}(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)|.$$

REMARK 1. Note that the integral in (1) can be calculated. Indeed, by integrating in parts, we have

$$\begin{aligned} \int_0^{|z|} \ln \frac{1+s}{1-s} ds &= s \ln \frac{1+s}{1-s} \Big|_0^{|z|} - \int_0^{|z|} \frac{2s ds}{1-s^2} \\ &= |z| \ln \frac{1+|z|}{1-|z|} + \ln(1-s^2) \Big|_0^{|z|} \\ &= (1+|z|) \ln(1+|z|) + (1-|z|) \ln(1-|z|). \end{aligned} \quad (2)$$

By using (2) in (1), we get

$$\|\delta_z\|_{(\mathcal{Z}^{1,1}/\mathbb{P}_1)^*} = \frac{(1+|z|) \ln(1+|z|) + (1-|z|) \ln(1-|z|)}{2}, \quad (3)$$

for $z \in \mathbb{D}$.

REMARK 2. Let

$$f(t) = \frac{(1+t) \ln(1+t) + (1-t) \ln(1-t)}{2},$$

for $t \in [0, 1)$. Then, we have

$$f'(t) = \frac{1}{2} \ln \frac{1+t}{1-t} > 0,$$

for $t \in (0, 1)$. So, the function f is strictly increasing on the interval $[0, 1)$, and consequently, we have

$$f(t) < \lim_{t \rightarrow 1-0} \frac{(1+t) \ln(1+t) + (1-t) \ln(1-t)}{2} = \ln 2, \quad (4)$$

for $t \in (0, 1)$.

Employing (4) in (3), we get

$$\sup_{z \in \mathbb{D}} \|\delta_z\|_{(\mathcal{Z}^{1,1}/\mathbb{P}_1)^*} = \ln 2.$$

It is a natural question if the norms of the point evaluation functionals can be calculated in the case of the n th weighted spaces with $n \geq 3$. Here, we deal with the case $n = 3$, where the corresponding space is equipped with the following semi-norms

$$s_{\mathcal{S}_{\alpha,1}^{(3)}}(f) := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'''(z)|, \quad (5)$$

and

$$s_{\mathcal{S}_\alpha^{(3)}}(f) := \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f'''(z)|. \quad (6)$$

The semi-norms given in (5) and (6) are norms on the quotient spaces $\mathcal{S}_{\alpha,1}^{(3)}/\mathbb{P}_2$ and $\mathcal{S}_\alpha^{(3)}/\mathbb{P}_2$, respectively. So, we will deal with the point evaluation functionals on the spaces of holomorphic functions. More precisely, on the space $\mathcal{S}_{1,1}^{(3)}/\mathbb{P}_2$, that is, only in the case when $\alpha = 1$, and on the spaces $\mathcal{S}_\alpha^{(3)}/\mathbb{P}_2$, for every $\alpha > 0$. These choices of the spaces are connected with the fact that some integrals, occurring during our calculations, can be calculated in quadratures.

In the proof of the main result we need the following known formula

$$f(z) = \sum_{j=0}^2 \frac{f^{(j)}(0)}{j!} z^j + \int_0^z \int_0^\zeta \int_0^\eta f'''(u) du d\eta d\zeta, \quad (7)$$

for $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$, as well as the following relation

$$\int_0^z \int_0^\zeta \int_0^\eta f'''(u) du d\eta d\zeta = z^3 \int_0^1 \int_0^1 \int_0^1 f'''(zrst) dt ds r^2 dr, \quad (8)$$

for $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$, which is easily proved by using some simple changes of variables.

If a function considered in the paper is multi-valued, we regard that we use its principal branch.

2. Main result

The following theorem is the main result in the paper. It calculates the norms of the point evaluation functional on the space $\mathcal{S}_{1,1}^{(3)}/\mathbb{P}_2$, as well as on the space $\mathcal{S}_\alpha^{(3)}/\mathbb{P}_2$, for every $\alpha > 0$. For the case of the space $\mathcal{S}_\alpha^{(3)}/\mathbb{P}_2$, there are four cases that should be considered separately, producing four (different) formulas for the point evaluation functionals in these cases.

THEOREM 1. *Let $\alpha > 0$. Then,*

(a) *if $\alpha = 1$,*

$$\|\delta_z\|_{(\mathcal{S}_{1,1}^{(3)}/\mathbb{P}_2)^*} = \frac{(1+|z|)^2 \ln(1+|z|) - (1-|z|)^2 \ln(1-|z|) - 2|z|}{4}; \quad (9)$$

(b) *if $\alpha \in (0+\infty) \setminus \{1, 2, 3\}$,*

$$\|\delta_z\|_{(\mathcal{S}_\alpha^{(3)}/\mathbb{P}_2)^*} = \frac{2 - 2(3-\alpha)|z| + (2-\alpha)(3-\alpha)|z|^2 - 2(1-|z|)^{3-\alpha}}{2(1-\alpha)(2-\alpha)(3-\alpha)}; \quad (10)$$

(c) *if $\alpha = 3$,*

$$\|\delta_z\|_{(\mathcal{S}_3^{(3)}/\mathbb{P}_2)^*} = \frac{1}{2} \left(\ln \frac{1}{1-|z|} - |z| - \frac{|z|^2}{2} \right); \quad (11)$$

(d) *if $\alpha = 2$,*

$$\|\delta_z\|_{(\mathcal{S}_2^{(3)}/\mathbb{P}_2)^*} = (1-|z|) \ln(1-|z|) + |z| - \frac{|z|^2}{2}; \quad (12)$$

(e) *if $\alpha = 1$,*

$$\|\delta_z\|_{(\mathcal{S}_1^{(3)}/\mathbb{P}_2)^*} = \frac{3|z|^2 - 2|z| - 2(1-|z|)^2 \ln(1-|z|)}{4}, \quad (13)$$

for $z \in \mathbb{D}$.

Proof. Without loss of generality, we suppose

$$f(0) = f'(0) = f''(0) = 0,$$

from which, together with (7) and (8), the following inequality is easily obtained

$$|f(z)| \leq |z|^3 \int_0^1 \int_0^1 \int_0^1 |f'''(zrst)| dt s dr^2, \quad (14)$$

for $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

(a) Employing (14), and some calculations, we have

$$\begin{aligned}
 |f(z)| &\leq |z|^3 \int_0^1 \int_0^1 \int_0^1 |f'''(zrst)| dt ds r^2 dr \\
 &\leq s_{\mathcal{S}_{1,1}^{(3)}}(f) \int_0^1 \int_0^1 \int_0^1 \frac{|z|^3}{1-|zrst|^2} dt ds r^2 dr \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) |z|^2 \int_0^1 \int_0^1 \int_0^{|z|rs} \frac{du}{1-u^2} ds r dr \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{|z|^2}{2} \int_0^1 \int_0^1 \ln \frac{1+|z|rs}{1-|z|rs} ds r dr \quad (15) \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{|z|}{2} \int_0^1 \int_0^{|z|r} \ln \frac{1+v}{1-v} dv dr \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{|z|}{2} \int_0^1 \left(|z|r \ln \frac{1+|z|r}{1-|z|r} - \int_0^{|z|r} \frac{2v dv}{1-v^2} \right) dr \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{|z|}{2} \int_0^1 \left(|z|r \ln \frac{1+|z|r}{1-|z|r} + \ln(1-(|z|r)^2) \right) dr \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{1}{2} \int_0^{|z|} \left(w \ln \frac{1+w}{1-w} + \ln(1-w^2) \right) dw \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{1}{2} \int_0^{|z|} ((1+w) \ln(1+w) + (1-w) \ln(1-w)) dw \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{1}{2} |z| ((1+|z|) \ln(1+|z|) + (1-|z|) \ln(1-|z|)) \\
 &\quad - s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{1}{2} \int_0^{|z|} w \ln \frac{1+w}{1-w} dw \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{1}{2} |z| ((1+|z|) \ln(1+|z|) + (1-|z|) \ln(1-|z|)) \\
 &\quad - s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{1}{2} \left(\frac{|z|^2}{2} \ln \frac{1+|z|}{1-|z|} - \int_0^{|z|} \frac{w^2}{1-w^2} dw \right) \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{1}{2} |z| ((1+|z|) \ln(1+|z|) + (1-|z|) \ln(1-|z|)) \\
 &\quad - s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{1}{2} \left(\frac{|z|^2}{2} \ln \frac{1+|z|}{1-|z|} + |z| - \frac{1}{2} \ln \frac{1+|z|}{1-|z|} \right) \\
 &= s_{\mathcal{S}_{1,1}^{(3)}}(f) \frac{(1+|z|)^2 \ln(1+|z|) - (1-|z|)^2 \ln(1-|z|) - 2|z|}{4}, \quad (16)
 \end{aligned}$$

from which, it follows that

$$\|\delta_z\|_{(\mathcal{S}_{1,1}^{(3)}/\mathbb{P}_2)^*} \leq \frac{(1+|z|)^2 \ln(1+|z|) - (1-|z|)^2 \ln(1-|z|) - 2|z|}{4}. \quad (17)$$

Now we define the following family of test functions

$$f_t(z) = \frac{1}{2} \int_0^z \int_0^\zeta \ln \frac{1+t\eta}{1-t\eta} d\eta d\zeta, \quad z \in \mathbb{D}, \quad (18)$$

where $t \in \mathbb{D}$.

Note that $f_t(0) = 0$ and

$$f'_t(z) = \frac{1}{2} \int_0^z \ln \frac{1+t\eta}{1-t\eta} d\eta, \quad (19)$$

and consequently, $f'_t(0) = 0$ and

$$f''_t(z) = \frac{1}{2} \ln \frac{1+tz}{1-tz},$$

which further implies

$$f''_t(0) = 0 \quad \text{and} \quad f'''_t(z) = \frac{t}{1-t^2z^2}.$$

Hence

$$s_{S_{1,1}^{(3)}}(f_t) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \frac{|t|}{|1 - t^2z^2|} \leq \sup_{z \in \mathbb{D}} \frac{|t|(1 - |z|^2)}{(1 - |t^2z^2|)} \leq |t|. \quad (20)$$

On the other hand, we have

$$s_{S_{1,1}^{(3)}}(f_t) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \frac{|t|}{|1 - t^2z^2|} \geq (1 - |z|^2) \frac{|t|}{|1 - t^2z^2|} \Big|_{z=0} = |t|. \quad (21)$$

From (20) and (21), it follows that

$$s_{S_{1,1}^{(3)}}(f_t) = |t|. \quad (22)$$

Note that

$$\delta_z(f_t) = \frac{z^2}{2} \int_0^1 \int_0^1 \ln \frac{1+tsrz}{1-ts rz} dsr dr. \quad (23)$$

Take t such that $\arg t = -\arg z$. Then, from (23), it follows that

$$|\delta_z(f_t)| = \frac{|z|^2}{2} \int_0^1 \int_0^1 \ln \frac{1+|t|sr|z|}{1-|t|sr|z|} dsr dr. \quad (24)$$

Employing the monotone convergence theorem, and the calculations producing (16) from (15), we obtain

$$\begin{aligned} \lim_{|t| \rightarrow 1-0} |\delta_z(f_t)| &= \lim_{|t| \rightarrow 1-0} \frac{|z|^2}{2} \int_0^1 \int_0^1 \ln \frac{1+|t|sr|z|}{1-|t|sr|z|} dsr dr \\ &= \frac{|z|^2}{2} \int_0^1 \int_0^1 \ln \frac{1+sr|z|}{1-sr|z|} dsr dr \\ &= \frac{(1+|z|)^2 \ln(1+|z|) - (1-|z|)^2 \ln(1-|z|) - 2|z|}{4}. \end{aligned} \quad (25)$$

Using the relation

$$|\delta_z(f_t)| \leq \|\delta_z\|_{(S_{1,1}^{(3)}/\mathbb{P}_2)^*} s_{S_{1,1}^{(3)}}(f_t),$$

(22), and (25), we obtain

$$\|\delta_z\|_{(S_{1,1}^{(3)}/\mathbb{P}_2)^*} \geq \frac{(1+|z|)^2 \ln(1+|z|) - (1-|z|)^2 \ln(1-|z|) - 2|z|}{4}. \quad (26)$$

Combining (17) and (26), we get the formula in (9).

(b) Using (14), and some calculations, we have

$$\begin{aligned} |f(z)| &\leq |z|^3 \int_0^1 \int_0^1 \int_0^1 |f'''(zrst)| dt ds r^2 dr \\ &\leq s_{S_\alpha^{(3)}}(f) \int_0^1 \int_0^1 \int_0^1 \frac{|z|^3}{(1-tsr|z|)^\alpha} dt ds r^2 dr \\ &= s_{S_\alpha^{(3)}}(f) |z|^2 \int_0^1 \int_0^1 \int_0^{sr|z|} \frac{du}{(1-u)^\alpha} ds r dr \end{aligned} \quad (27)$$

$$\begin{aligned} &= s_{S_\alpha^{(3)}}(f) \frac{|z|^2}{1-\alpha} \int_0^1 \int_0^1 (1 - (1-sr|z|)^{1-\alpha}) ds r dr \\ &= \frac{s_{S_\alpha^{(3)}}(f)}{1-\alpha} |z| \int_0^1 \int_0^{r|z|} (1 - (1-\hat{u})^{1-\alpha}) d\hat{u} dr \end{aligned} \quad (28)$$

$$\begin{aligned} &= \frac{s_{S_\alpha^{(3)}}(f)}{1-\alpha} |z| \int_0^1 \left(\hat{u} + \frac{(1-\hat{u})^{2-\alpha}}{2-\alpha} \right) \Big|_0^{r|z|} dr \\ &= \frac{s_{S_\alpha^{(3)}}(f)}{1-\alpha} |z| \int_0^1 \left(r|z| + \frac{(1-r|z|)^{2-\alpha} - 1}{2-\alpha} \right) dr \\ &= \frac{s_{S_\alpha^{(3)}}(f)}{(1-\alpha)(2-\alpha)} \int_0^{|z|} ((2-\alpha)v + (1-v)^{2-\alpha} - 1) dv \\ &= \frac{s_{S_\alpha^{(3)}}(f)}{(1-\alpha)(2-\alpha)} \left((2-\alpha) \frac{v^2}{2} - \frac{(1-v)^{3-\alpha}}{3-\alpha} - v \right) \Big|_0^{|z|} \\ &= \frac{s_{S_\alpha^{(3)}}(f)}{(1-\alpha)(2-\alpha)} \left((2-\alpha) \frac{|z|^2}{2} + \frac{1 - (1-|z|)^{3-\alpha}}{3-\alpha} - |z| \right) \\ &= s_{S_\alpha^{(3)}}(f) \frac{2 - 2(3-\alpha)|z| + (2-\alpha)(3-\alpha)|z|^2 - 2(1-|z|)^{3-\alpha}}{2(1-\alpha)(2-\alpha)(3-\alpha)}, \end{aligned} \quad (29)$$

from which, it follows that

$$\|\delta_z\|_{(S_\alpha^{(3)}/\mathbb{P}_2)^*} \leq \frac{2 - 2(3-\alpha)|z| + (2-\alpha)(3-\alpha)|z|^2 - 2(1-|z|)^{3-\alpha}}{2(1-\alpha)(2-\alpha)(3-\alpha)}. \quad (30)$$

Now we define the following family of test functions

$$\widehat{f_t}(z) = \frac{1}{1-\alpha} \int_0^z \int_0^\eta (1 - (1-t\zeta)^{1-\alpha}) d\zeta d\eta, \quad z \in \mathbb{D}, \quad (31)$$

where $t \in \mathbb{D}$. Then, $\widehat{f}_t(0) = 0$ and

$$\widehat{f}'_t(z) = \frac{1}{1-\alpha} \int_0^z (1 - (1-t\zeta)^{1-\alpha}) d\zeta, \quad z \in \mathbb{D}, \quad (32)$$

from which, we have $\widehat{f}'_t(0) = 0$ and

$$\widehat{f}''_t(z) = \frac{1 - (1-tz)^{1-\alpha}}{1-\alpha},$$

which further implies that

$$\widehat{f}''_t(0) = 0 \quad \text{and} \quad \widehat{f}'''_t(z) = \frac{t}{(1-tz)^\alpha}.$$

Hence

$$s_{S_\alpha^{(3)}}(\widehat{f}_t) = \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha \frac{|t|}{|1-tz|^\alpha} \leq \sup_{z \in \mathbb{D}} \frac{|t|(1-|z|)^\alpha}{(1-|tz|)^\alpha} \leq |t|. \quad (33)$$

On the other hand, we have

$$s_{S_\alpha^{(3)}}(\widehat{f}_t) = \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha \frac{|t|}{|1-tz|^\alpha} \geq (1 - |z|)^\alpha \frac{|t|}{(1-|t||z|)^\alpha} \Big|_{z=0} = |t|. \quad (34)$$

From (33) and (34), it follows that

$$s_{S_\alpha^{(3)}}(\widehat{f}_t) = |t|. \quad (35)$$

Further, we have

$$\delta_z(\widehat{f}_t) = \frac{z^2}{1-\alpha} \int_0^1 \int_0^1 (1 - (1-ts rz)^{1-\alpha}) ds r dr. \quad (36)$$

Take t such that $\arg t = -\arg z$. Then, from (36), it follows that

$$|\delta_z(\widehat{f}_t)| = \frac{|z|^2}{|1-\alpha|} \int_0^1 \int_0^1 (1 - (1-|t|sr|z|)^{1-\alpha}) ds r dr. \quad (37)$$

Employing the monotone convergence theorem, it follows that

$$\begin{aligned} \lim_{|t| \rightarrow 1-0} |\delta_z(\widehat{f}_t)| &= \lim_{|t| \rightarrow 1-0} \frac{|z|^2}{1-\alpha} \int_0^1 \int_0^1 (1 - (1-|t|sr|z|)^{1-\alpha}) ds r dr \\ &= \frac{|z|^2}{1-\alpha} \int_0^1 \int_0^1 (1 - (1-sr|z|)^{1-\alpha}) ds r dr. \end{aligned} \quad (38)$$

Using the relation

$$|\delta_z(\widehat{f}_t)| \leq \|\delta_z\|_{(S_\alpha^{(3)}/\mathbb{P}_2)^*} s_{S_\alpha^{(3)}}(\widehat{f}_t),$$

(29), (35), (38), we get

$$\|\delta_z\|_{(\mathcal{S}_\alpha^{(3)}/\mathbb{P}_2)^*} \geq \frac{2 - 2(3 - \alpha)|z| + (2 - \alpha)(3 - \alpha)|z|^2 - 2(1 - |z|)^{3-\alpha}}{2(1 - \alpha)(2 - \alpha)(3 - \alpha)}, \quad (39)$$

when $\alpha \in (0 + \infty) \setminus \{1, 2, 3\}$.

From (30) and (39), we obtain (10).

(c) From (28), in this case, we have

$$\begin{aligned} |f(z)| &\leq |z|^3 \int_0^1 \int_0^1 \int_0^1 |f'''(zrst)| dt ds r^2 dr \\ &\leq \frac{s_{\mathcal{S}_3^{(3)}}(f)}{2} |z| \int_0^1 \int_0^{|z|} \left(\frac{1}{(1 - \hat{u})^2} - 1 \right) d\hat{u} dr \end{aligned} \quad (40)$$

$$\begin{aligned} &= \frac{s_{\mathcal{S}_3^{(3)}}(f)}{2} |z| \int_0^1 \left(\frac{1}{1 - \hat{u}} - \hat{u} \right) \Big|_0^{|z|} dr \\ &= \frac{s_{\mathcal{S}_3^{(3)}}(f)}{2} |z| \int_0^1 \left(\frac{1}{1 - r|z|} - r|z| - 1 \right) dr \\ &= \frac{s_{\mathcal{S}_3^{(3)}}(f)}{2} \int_0^{|z|} \left(\frac{1}{1 - v} - v - 1 \right) dv \\ &\leq \frac{s_{\mathcal{S}_3^{(3)}}(f)}{2} \left(\ln \frac{1}{1 - |z|} - |z| - \frac{|z|^2}{2} \right), \end{aligned} \quad (41)$$

from which, it follows that

$$\|\delta_z\|_{(s_{\mathcal{S}_3^{(3)}}/\mathbb{P}_2)^*} \leq \frac{1}{2} \left(\ln \frac{1}{1 - |z|} - |z| - \frac{|z|^2}{2} \right). \quad (42)$$

Now we use the test functions in (31) with $\alpha = 3$. Then, (35), (36), (37) and (38), hold for $\alpha = 3$. Hence, by using calculations which lead to (41) from (40), we have

$$\begin{aligned} \lim_{|t| \rightarrow 1-0} |\delta_z(\hat{f}_t)| &= \frac{|z|^2}{2} \int_0^1 \int_0^1 \left(\frac{1}{(1 - sr|z|)^2} - 1 \right) ds r dr \\ &= \frac{|z|}{2} \int_0^1 \int_0^{|z|} \left(\frac{1}{(1 - u)^2} - 1 \right) du dr \\ &= \frac{1}{2} \left(\ln \frac{1}{1 - |z|} - |z| - \frac{|z|^2}{2} \right). \end{aligned} \quad (43)$$

Using the relation

$$|\delta_z(\hat{f}_t)| \leq \|\delta_z\|_{(\mathcal{S}_3^{(3)}/\mathbb{P}_2)^*} s_{\mathcal{S}_3^{(3)}}(\hat{f}_t),$$

(35) and (43), we obtain

$$\|\delta_z\|_{(s_{\mathcal{S}_3^{(3)}}/\mathbb{P}_2)^*} \geq \frac{1}{2} \left(\ln \frac{1}{1 - |z|} - |z| - \frac{|z|^2}{2} \right). \quad (44)$$

Inequalities (42) and (44), directly imply (11).

(d) From (28), in this case, we have

$$\begin{aligned}
 |f(z)| &\leq |z|^3 \int_0^1 \int_0^1 \int_0^1 |f'''(zrst)| dt ds r^2 dr \\
 &\leq s_{S_2^{(3)}}(f) |z|^3 \int_0^1 \int_0^1 \int_0^1 \frac{dt}{(1-|zrst|)^2} ds r^2 dr \\
 &= s_{S_2^{(3)}}(f) |z|^2 \int_0^1 \int_0^1 \int_0^{rs|z|} \frac{du}{(1-u)^2} ds r dr \\
 &= s_{S_2^{(3)}}(f) |z|^2 \int_0^1 \int_0^1 \left(\frac{1}{1-rs|z|} - 1 \right) ds r dr \quad (45) \\
 &= s_{S_2^{(3)}}(f) |z| \int_0^1 \int_0^{r|z|} \left(\frac{1}{1-u} - 1 \right) du dr \\
 &= s_{S_2^{(3)}}(f) |z| \int_0^1 \left(\ln \frac{1}{1-u} - u \right) \Big|_0^{r|z|} dr \\
 &= s_{S_2^{(3)}}(f) |z| \int_0^1 \left(\ln \frac{1}{1-r|z|} - r|z| \right) dr \\
 &= s_{S_2^{(3)}}(f) \int_0^{|z|} \left(\ln \frac{1}{1-u} - u \right) du \\
 &= s_{S_2^{(3)}}(f) \left(\left(u \ln \frac{1}{1-u} - \frac{u^2}{2} \right) \Big|_0^{|z|} - \int_0^{|z|} \frac{u}{1-u} du \right) \\
 &= s_{S_2^{(3)}}(f) \left((1-|z|) \ln(1-|z|) + |z| - \frac{|z|^2}{2} \right), \quad (46)
 \end{aligned}$$

from which it follows that

$$\|\delta_z\|_{(s_{S_2^{(3)}}/\mathbb{P}_2)^*} \leq (1-|z|) \ln(1-|z|) + |z| - \frac{|z|^2}{2}. \quad (47)$$

Now we use the test functions in (31) with $\alpha = 2$. Then, (35), (36), (37) and (38), hold for $\alpha = 2$. Hence, we have

$$|\delta_z(\widehat{f_t})| = |z|^2 \int_0^1 \int_0^1 \left(\frac{1}{1-|t|sr|z|} - 1 \right) ds r dr. \quad (48)$$

Employing the monotone convergence theorem, and the calculations which lead to (46) from (45), we have

$$\begin{aligned}
 \lim_{|t| \rightarrow 1-0} |\delta_z(\widehat{f_t})| &= \lim_{|t| \rightarrow 1-0} |z|^2 \int_0^1 \int_0^1 \left(\frac{1}{1-|t|sr|z|} - 1 \right) ds r dr \\
 &= |z|^2 \int_0^1 \int_0^1 \left(\frac{1}{1-sr|z|} - 1 \right) ds r dr \\
 &= (1-|z|) \ln(1-|z|) + |z| - \frac{|z|^2}{2}. \quad (49)
 \end{aligned}$$

Using the relation

$$|\delta_z(\widehat{f_t})| \leq \|\delta_z\|_{(S_2^{(3)}/\mathbb{P}_2)^*} s_{S_2^{(3)}}(\widehat{f_t}),$$

(35), and (49), we obtain

$$\|\delta_z\|_{(S_2^{(3)}/\mathbb{P}_2)^*} \geq (1 - |z|) \ln(1 - |z|) + |z| - \frac{|z|^2}{2}. \quad (50)$$

Inequalities (47) and (50), directly imply (12).

(e) From (28), in this case, we have

$$\begin{aligned} |f(z)| &\leq |z|^3 \int_0^1 \int_0^1 \int_0^1 |f'''(zrst)| dt s ds r^2 dr \\ &\leq s_{S_1^{(3)}}(f) |z|^3 \int_0^1 \int_0^1 \int_0^1 \frac{dt}{1 - |zrst|} s ds r^2 dr \\ &= s_{S_1^{(3)}}(f) |z|^2 \int_0^1 \int_0^1 \int_0^{rs|z|} \frac{du}{1 - u} ds r dr \\ &= s_{S_1^{(3)}}(f) |z|^2 \int_0^1 \int_0^1 \ln \frac{1}{1 - rs|z|} ds r dr \\ &= s_{S_1^{(3)}}(f) |z| \int_0^1 \int_0^{|z|} \ln \frac{1}{1 - ru} du r dr \\ &= s_{S_1^{(3)}}(f) |z| \int_0^1 \left(u \ln \frac{1}{1 - ru} + u - \frac{1}{r} \ln \frac{1}{1 - ru} \right) \Big|_0^{|z|} r dr \\ &= s_{S_1^{(3)}}(f) |z| \int_0^1 \left(r|z| \ln \frac{1}{1 - r|z|} + r|z| - \ln \frac{1}{1 - r|z|} \right) dr \\ &= s_{S_1^{(3)}}(f) \int_0^{|z|} ((1 - u) \ln(1 - u) + u) du \\ &= s_{S_1^{(3)}}(f) \left(\left(u - \frac{u^2}{2} \right) \ln(1 - u) \Big|_0^{|z|} + \frac{|z|^2}{2} + \int_0^{|z|} \frac{2u - u^2}{2(1 - u)} du \right) \\ &= s_{S_1^{(3)}}(f) \left(\frac{3|z|^2 - 2|z| - 2(1 - |z|)^2 \ln(1 - |z|)}{4} \right), \end{aligned} \quad (51)$$

from which, it follows that

$$\|\delta_z\|_{(S_1^{(3)}/\mathbb{P}_2)^*} \leq \frac{3|z|^2 - 2|z| - 2(1 - |z|)^2 \ln(1 - |z|)}{4}. \quad (53)$$

Now we use the following test functions

$$\widetilde{f_t}(z) = \int_0^z \int_0^\zeta \ln \frac{1}{1 - t\eta} d\eta d\zeta, \quad z \in \mathbb{D}, \quad (54)$$

where $t \in \mathbb{D}$. Then, $\widetilde{f_t}(0) = 0$ and

$$\widetilde{f_t}'(z) = \int_0^z \ln \frac{1}{1 - t\eta} d\eta, \quad (55)$$

which implies that $\tilde{f}_t'(0) = 0$ and

$$\tilde{f}_t''(z) = \ln \frac{1}{1-tz},$$

and consequently

$$\tilde{f}_t''(0) = 0 \quad \text{and} \quad \tilde{f}_t'''(z) = \frac{t}{1-tz}.$$

Hence

$$s_{S_1^{(3)}}(\tilde{f}_t) = \sup_{z \in \mathbb{D}} (1-|z|) \frac{|t|}{|1-tz|} \leq \sup_{z \in \mathbb{D}} (1-|z|) \frac{|t|}{(1-|t||z|)} \leq |t|. \quad (56)$$

On the other hand, we have

$$s_{S_1^{(3)}}(\tilde{f}_t) = \sup_{z \in \mathbb{D}} (1-|z|) \frac{|t|}{|1-tz|} \geq \frac{|t|(1-|z|)}{|1-tz|} \Big|_{z=0} = |t|. \quad (57)$$

From (56) and (57), it follows that

$$s_{S_1^{(3)}}(\tilde{f}_t) = |t|. \quad (58)$$

It is easy to see that

$$\delta_z(\tilde{f}_t) = z^2 \int_0^1 \int_0^1 \ln \frac{1}{1-ts rz} ds r dr. \quad (59)$$

Take t such that $\arg t = -\arg z$. Then, from (59), it follows that

$$\delta_z(\tilde{f}_t) = z^2 \int_0^1 \int_0^1 \ln \frac{1}{1-|t|sr|z|} ds r dr. \quad (60)$$

Employing the monotone convergence theorem, and the calculations which lead to (52) from (51), we have

$$\begin{aligned} \lim_{|t| \rightarrow 1-0} |\delta_z(\tilde{f}_t)| &= \lim_{|t| \rightarrow 1-0} |z|^2 \int_0^1 \int_0^1 \ln \frac{1}{1-|t|sr|z|} ds r dr \\ &= |z|^2 \int_0^1 \int_0^1 \ln \frac{1}{1-sr|z|} ds r dr \\ &= \frac{3|z|^2 - 2|z| - 2(1-|z|)^2 \ln(1-|z|)}{4}. \end{aligned} \quad (61)$$

Using the relation

$$|\delta_z(\tilde{f}_t)| \leq \|\delta_z\|_{(S_1^{(3)})/\mathbb{P}_2} s_{S_1^{(3)}}(\tilde{f}_t),$$

(58), and (61), we obtain

$$\|\delta_z\|_{(S_1^{(3)})/\mathbb{P}_2} \geq \frac{3|z|^2 - 2|z| - 2(1-|z|)^2 \ln(1-|z|)}{4}. \quad (62)$$

Inequalities (53) and (62), imply (13), finishing the proof of the theorem. $\square$

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