

TWO-EXPONENT WEIGHTED BEESACK-OPIAL INEQUALITIES ON TIME SCALES AND APPLICATIONS

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Abstract. In this paper, we establish a class of weighted Beesack–Opial inequalities with two independent exponents on arbitrary time scales. The resulting estimates extend the weighted Opial framework by allowing the powers of the function and its delta derivative to vary independently, while an auxiliary nondecreasing function contributes through both its value and its delta derivative. The main result is a product-type inequality obtained by combining a weighted Hölder estimate with the exact triangular change-of-order formula on time scales. Corresponding additive and balanced forms are also derived by means of Young’s inequality.

The general results yield several dynamic Opial-type consequences, including composite weighted inequalities, power-type estimates, and classical discrete inequalities of Wong–Lasota type. In the continuous case, the main product estimate includes the $p = q = 2$ case of the two-weight Beesack–Opial inequality of Chesneau as a special case. Discrete and quantum versions are also obtained, with the forward-jump structure preserved explicitly.

1. Introduction

The classical continuous Opial inequality, due to Opial [24], states that if $x \in C^1([\alpha, \beta])$ satisfies $x(\alpha) = x(\beta) = 0$, then

$$\int_{\alpha}^{\beta} |x(t)x'(t)| dt \leq \frac{\beta - \alpha}{4} \int_{\alpha}^{\beta} |x'(t)|^2 dt. \quad (1)$$

Because the left-hand side couples a function to its derivative while the right-hand side involves the derivative alone, (1) has become a fundamental tool in the qualitative analysis of differential equations, in particular in disconjugacy and uniqueness criteria. Opial’s result was simplified by Olech [23] and refined in several directions shortly afterwards, notably by Yang [35] and Hua [17], who replaced the pair of powers (1, 2) by more general ones.

A separate and, for the present paper, decisive step was taken by Beesack [7], who established the one-sided inequality

$$\int_{\alpha}^{\beta} |x(t)x'(t)| dt \leq \frac{\beta - \alpha}{2} \int_{\alpha}^{\beta} |x'(t)|^2 dt,$$

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for functions satisfying $x(\alpha) = 0$. This result replaced the two-endpoint condition in the classical Opial inequality by a condition at only one endpoint. The theory was subsequently developed by Beesack and Das [8] and Das [15], who introduced auxiliary weight functions and obtained a broad class of weighted extensions now commonly referred to as Beesack–Opial inequalities. A systematic account of the resulting body of continuous results is given by Agarwal and Pang [6].

In the discrete framework, the corresponding development was initiated by Wong [33] and Lasota [18]. In particular, Lasota proved that, for a finite sequence $\{x_i\}_{i=0}^h$ satisfying $x_0 = x_h = 0$,

$$\sum_{i=1}^{h-1} |x_i \Delta x_i| \leq \frac{1}{2} \left\lfloor \frac{h+1}{2} \right\rfloor \sum_{i=0}^{h-1} |\Delta x_i|^2.$$

This result helped establish the theory of discrete analogues of (1) and their applications to difference equations.

The two lines of development were unified by the time-scale calculus introduced by Hilger and presented systematically in the monograph of Bohner and Peterson [10]. In this setting, Bohner and Kaymakçalan [11] obtained the dynamic inequality

$$\int_0^h |x(t) + x^\sigma(t)| |x^\Delta(t)| \Delta t \leq h \int_0^h |x^\Delta(t)|^2 \Delta t,$$

which contains the continuous and discrete forms simultaneously and thereby opened a still active line of research. Improved dynamic Opial-type inequalities with applications were established in [12], a unified treatment of the discrete, continuous, delta, nabla, and diamond- α cases was given in [13], and weighted dynamic forms were obtained in [27, 28, 31, 34].

Among the more recent contributions, the ones closest to the present work are the weighted dynamic estimates of the last few years, and it is worth recording precisely which structural features they carry, since it is exactly the combination of these features that the present paper addresses. To make the comparison transparent, the cited formulas are restated in the notation used throughout this paper, that is, with interval $[\alpha, \beta]_{\mathbb{T}}$, dependent function u , delta derivative u^Δ , and weight w .

Saker, Alzabut, Sayed, and O'Regan [29] obtained convexity-based Opial-type inequalities on time scales, among them

$$\begin{aligned} \int_\alpha^\beta w(t) |u(\sigma(t))|^v |u^\Delta(t)| \Delta t &\leq \frac{1}{v+1} \left(\int_\alpha^\beta w(t) \varphi(t) \Delta t \right)^v \\ &\quad \times \int_\alpha^\beta \varphi(t) |u^\Delta(t)|^{v+1} \Delta t. \end{aligned} \quad (2)$$

Two weights, w and the auxiliary function φ , are available here, but the power of $|u^\Delta(t)|$ on the left is fixed and equal to 1, and the remaining two powers, v and $v+1$, are governed by the single parameter v . The same coupling of the powers persists in the very recent refinement of Al-Shehri, El-Sheikh, Zakarya, Ali, Rezk, and Khamis [1], whose stated aim is to relax the hypotheses in a different direction, namely to permit nonzero endpoint values of the dependent function and nonincreasing weights.

A second group of results enlarges the differential rather than the algebraic structure. Osman, Saker, and Anderson [25] proved higher-order two-function inequalities bounding

$$\begin{aligned} & \int_{\alpha}^{\beta} w_1(t) \left(|u_2^{\Delta'}(t)|^{\lambda} |u_1^{\Delta''}(t)|^{\delta} + |u_1^{\Delta'}(t)|^{\lambda} |u_2^{\Delta''}(t)|^{\delta} \right) \Delta t \\ & \leq K_2(\lambda, \delta, k) \left(\int_{\alpha}^{\beta} w_2(t) (|u_1^{\Delta''}(t)|^k + |u_2^{\Delta''}(t)|^k) \Delta t \right)^{\frac{\lambda+\delta}{k}}, \end{aligned}$$

with $\lambda, \delta > 0$, $k > 1$, $k > \delta$, and K_2 explicit; here two exponents do occur, but they are attached to delta derivatives of two different orders, and the bounding side is a *single* integral raised to an outer power. Almarri, Makarish, and El-Deeb [3] established weighted dynamic inequalities with higher-order delta derivatives together with applications to initial value problems; in the first-order case their estimate takes the form

$$\int_{\alpha}^{\beta} \frac{|(u + u^{\sigma})u^{\Delta}|(t)}{\delta^{\sigma}(t)} \Delta t \leq \left(\int_{\alpha}^{\beta} \zeta(t) \Delta t \right) \left(\int_{\alpha}^{\beta} \frac{|u^{\Delta}(t)|^2}{\zeta(t)\delta(t)} \Delta t \right), \quad (3)$$

for $\zeta, \delta \geq 0$ rd-continuous with δ nondecreasing and $u(\alpha) = 0$. The bound in (3) is a product of two integrals; its first factor, however, is a pure weight integral in which neither u nor u^{Δ} occurs, neither factor carries an outer exponent, and both powers on the left are equal to 1.

Closest to the present paper in spirit are the weighted higher-order Beesack–Opial inequalities of Mahmoud, Saker, Anderson, and Abdo [19]. For functions that are n times delta differentiable and whose delta derivatives of orders $0, 1, \dots, n-1$ vanish at the left endpoint, that paper bounds the mixed functional

$$\int_{\alpha}^{\beta} w(t) |u(t)|^p |u^{\Delta^n}(t)|^v \Delta t$$

by expressions involving the highest-order delta derivative alone, with a weight structure generated by a nondecreasing auxiliary function and its delta derivative.

A third group of results decouples the two powers but dispenses with the weights. Elsaid, Zaghrout, and El-Deeb [16] proved monotonicity-based dynamic inequalities, including for $\lambda_1, \lambda_2 > 0$

$$\int_{\alpha}^{\beta} |u(t)|^{\lambda_1} |u^{\Delta}(t)|^{\lambda_2} \Delta t \leq \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}}}{\lambda_1 + \lambda_2} (\beta - \alpha)^{\lambda_1} \int_{\alpha}^{\beta} |u^{\Delta}(t)|^{\lambda_1+\lambda_2} \Delta t. \quad (4)$$

The two left-hand exponents are independent, and the derivative is of first order; the price is that (4) is unweighted, its right-hand side being a single integral in which no auxiliary function appears and the whole geometry of the interval is compressed into the factor $(\beta - \alpha)^{\lambda_1}$.

In the purely continuous setting, on the other hand, the weighted product structure has recently been pushed considerably further. Chesneau [14] established a Beesack–Opial-type inequality involving two auxiliary weights g and h interacting through h' :

if $h(\alpha) = 0$, $h'(t) > 0$, and $G(s) = \int_s^\beta g(t) dt$, then, for $p > 1$, $q = p/(p-1)$, and $u(\alpha) = 0$,

$$\int_\alpha^\beta g(t) |u(t) u'(t)| dt \leq \left(\int_\alpha^\beta g(t) h(t) |u'(t)|^p dt \right)^{1/p} \left(\int_\alpha^\beta G(t) \frac{|u'(t)|^q}{[h'(t)]^{p-1}} dt \right)^{1/q}. \quad (5)$$

This unifies several classical results of Beesack and Das [8], Olech [23], Yang [35], and related authors, and it is structurally the richest of the estimates recalled so far: the bound is a genuine product of two integrals, each containing the derivative, raised to conjugate outer exponents, and the second factor involves the tail G of the outer weight together with the negative power $[h']^{-(p-1)}$ of the derivative of the auxiliary weight. Its left-hand side, however, is the product $|uu'|$, so that both left-hand powers equal 1, and it is stated only for $\mathbb{T} = \mathbb{R}$. The same two limitations appear in the closely related product-type bound of Alazman and Dragomir [2], who proved Hölder-type generalizations of Opial's inequality for two absolutely continuous functions, with a right-hand side again consisting of two integrals raised to the conjugate powers $1/p$ and $1/q$, but with explicit kernel weights in place of free auxiliary functions. Parallel to these developments, Opial-type inequalities have been extended to calculi other than the delta calculus, namely to generalized fractional integral operators [32, 20], to the conformable calculus on time scales [21], and to the quantum setting [4]; these frameworks are adjacent to, but structurally different from, the weighted delta setting considered here.

Reading these results side by side reveals a definite gap. Three structural features have each appeared, but never together. The first is a free pair of powers on $|u|$ and $|u^\Delta|$ in the first-order dynamic setting: present in (4), absent from (2), (3), and (5). The second is a bounding side that is a genuine product of two integrals, both carrying the derivative and raised to outer exponents dictated by the powers on the left: present in (5) and in [2], but only for $\mathbb{T} = \mathbb{R}$ and only for the powers $(1, 1)$, and absent from (4) and from (3), whose product degenerates in the sense described above. The third is an auxiliary monotone function that acts on the bound simultaneously through itself, through a power of itself, and through its derivative: present in (5) through h , G , and h' , and reduced in (2) to φ alone. The natural question, and the one that motivates this paper, is therefore whether the two-weight Beesack–Opial mechanism of (5) survives on an arbitrary time scale once the powers of $|u|$ and $|u^\Delta|$ are decoupled. The answer turns out to be affirmative, but not by a term-by-term transcription: decoupling the powers creates an additional power of the auxiliary function inside the tail weight, which is invisible when both powers equal 1, and passing to an arbitrary time scale forces the change of order of integration to be carried out in its exact, σ -shifted form.

Accordingly, we establish in Theorem 3.2 the dynamic estimate

$$\begin{aligned} \int_\alpha^\beta w(\zeta) |u(\zeta)|^p |u^\Delta(\zeta)|^v \Delta\zeta &\leq \left(\int_\alpha^\beta w(\zeta) \varphi(\zeta) |u^\Delta(\zeta)|^p \Delta\zeta \right)^{v/p} \\ &\times \left(\int_\alpha^\beta \mathscr{W}(\zeta) \frac{|u^\Delta(\zeta)|^p}{(\varphi^\Delta(\zeta))^{p-1}} \Delta\zeta \right)^{p/p}, \end{aligned} \quad (6)$$

where $p = \rho + \nu$, the tail weight is $\mathscr{W}(\omega) = \int_{\omega}^{\beta} w(\zeta) \varphi(\zeta)^e \Delta \zeta$, and $e = (\rho - 1)(\rho + \nu)/\rho$. Four features distinguish (6) from (2)–(5). First, the exponents $\rho \geq 1$ and $\nu > 0$ of $|u|$ and $|u^{\Delta}|$ are independent parameters, whereas the derivative exponent is fixed at 1 in (2) and in (3), both left-hand powers are fixed at 1 in (5) and in [2], and the two exponents of [25] belong to derivatives of two different orders; among the dynamic estimates recalled above, only (4) carries a free pair of powers of $|u|$ and $|u^{\Delta}|$, and it does so without any weight. Second, the bounding side is a product of two integrals *each of which contains* $|u^{\Delta}|^p$, raised to the outer powers ν/p and ρ/p ; these are the reciprocals of the conjugate Hölder indices p/ν and p/ρ , so that the outer exponents are not free but are dictated by the powers on the left through $\nu/p + \rho/p = 1$, and the two factors carry different weights, the first φ and the second $\mathscr{W}$ together with $(\varphi^{\Delta})^{-(p-1)}$. Third, the auxiliary function enters simultaneously through φ , through the tail power φ^e , and through φ^{Δ} ; the tail exponent e is produced by the two-exponent structure itself and vanishes exactly when $\rho = 1$. In particular, for $\mathbb{T} = \mathbb{R}$ with $\rho = \nu = 1$, the estimate reduces term by term to (5) with $p = q = 2$ (Remark 3.3). Fourth, the statement is dynamic, and its discrete and quantum consequences are genuinely σ -sensitive rather than formal transcriptions of a continuous statement, since the exact change-of-order formula uses the σ -shifted head integral $\int_{\alpha}^{\sigma(r)}$, which leads to the unshifted tail integral $\int_r^{\beta}$.

We also state the additive form (Theorem 3.4) and the form with the balanced constant obtained by optimizing the Young parameter (Theorem 3.5). The scope of these results is limited in four respects, which we state explicitly: the derivative is of first order, only one function is involved, the first exponent is restricted by $\rho \geq 1$, which is exactly the condition making e nonnegative and hence $\mathscr{W}$ finite at $\varphi(\alpha) = 0$ on scattered time scales, and no claim of optimality is made for the constants. On the last point we recall that best constants in Opial-type inequalities form a separate line of research; sharp unifying forms of (1), including improvements of the constants for higher-order derivatives, were obtained by Klaassen [22].

The consequences established in Section 4 recover or extend several classical and recent Opial- and Beesack-type inequalities. These include the discrete inequalities of Wong [33] and Lasota [18], the composite weighted dynamic inequalities of Saker [27], and the power-type dynamic estimate (4). In the continuous case, the resulting inequalities are connected with the weighted Opial-type results of Agarwal and Pang [6], Beesack [7], Das [15], Pachpatte [26], and Salem, Agwo, and Khallaf [30].

The paper is organized as follows. Section 2 recalls the required notions from time-scale calculus and records the triangular change-of-order formula in the form actually used in the proofs. Section 3 contains the main dynamic two-exponent inequalities, and Section 4 is devoted to their dynamic, continuous, discrete, and quantum consequences.

2. Preliminaries

In this section we recall the notation, the analytic tools, and the auxiliary lemmas from time-scale calculus that will be needed in the proofs of our main results. Standard references are [5, 10]. Throughout, $\mathbb{T}$ denotes an arbitrary time scale, $\alpha, \beta \in \mathbb{T}$ with

$\alpha < \beta$, and $[\alpha, \beta]_{\mathbb{T}} = [\alpha, \beta] \cap \mathbb{T}$. As usual, $[\alpha, \beta]_{\mathbb{T}}^{\kappa}$ denotes $[\alpha, \beta]_{\mathbb{T}}$ with the left-scattered maximum removed, when such a maximum exists.

A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous (written $f \in C_{\text{rd}}$) if it is continuous at right-dense points and has finite left-sided limits at left-dense points. The forward jump operator is $\sigma(t)$, and the graininess is $\mu(t) = \sigma(t) - t$; the symbol μ is reserved for the graininess throughout, the exponents of the main results being denoted by ρ and ν . If f is delta differentiable, its delta derivative is denoted by $f^{\Delta}(t)$, and we write $f \in C_{\text{rd}}^1([\alpha, \beta]_{\mathbb{T}})$ if f is delta differentiable on $[\alpha, \beta]_{\mathbb{T}}^{\kappa}$ with $f^{\Delta} \in C_{\text{rd}}([\alpha, \beta]_{\mathbb{T}}^{\kappa})$. When $\mathbb{T} = \mathbb{R}$, one recovers $f^{\Delta} = f'$; when $\mathbb{T} = \mathbb{Z}$, one has $f^{\Delta}(k) = f(k+1) - f(k)$. The delta integral is defined through antiderivatives: if $F^{\Delta} = f$, then $\int_{\alpha}^{\beta} f(t) \Delta t = F(\beta) - F(\alpha)$. On $\mathbb{R}$ this reduces to the usual Riemann integral, and on $\mathbb{Z}$ one has $\int_{\alpha}^{\beta} f(t) \Delta t = \sum_{t=\alpha}^{\beta-1} f(t)$. The following lemmas are used repeatedly.

LEMMA 2.1. ([10]) *Let $f \in C_{\text{rd}}([\alpha, \beta]_{\mathbb{T}})$ and $F(t) = \int_{\alpha}^t f(\tau) \Delta \tau$. Then F is delta differentiable and $F^{\Delta}(t) = f(t)$ for all $t \in [\alpha, \beta]_{\mathbb{T}}^{\kappa}$. Moreover, if $f, g \in C_{\text{rd}}^1([\alpha, \beta]_{\mathbb{T}})$, then the integration-by-parts formula*

$$\int_{\alpha}^{\beta} f g^{\Delta} \Delta t = f(\beta)g(\beta) - f(\alpha)g(\alpha) - \int_{\alpha}^{\beta} f^{\Delta}(t)g(\sigma(t)) \Delta t \quad (7)$$

holds.

LEMMA 2.2. (Hölder inequality [10]) *Let $s > 1$ and $s' = s/(s-1)$. If $v_1, v_2 \in C_{\text{rd}}([\alpha, \beta]_{\mathbb{T}})$ and the integrals on the right-hand side are finite, then*

$$\int_{\alpha}^{\beta} |v_1 v_2| \Delta t \leq \left(\int_{\alpha}^{\beta} |v_1|^s \Delta t \right)^{1/s} \left(\int_{\alpha}^{\beta} |v_2|^{s'} \Delta t \right)^{1/s'}.$$

We use the established triangular Fubini theorem on time scales in the head-integral form recorded in [9]: if F is rd-continuous on the relevant triangular region and the displayed integrals are finite, then

$$\int_{\alpha}^{\beta} \left(\int_{\alpha}^{\sigma(t)} F(t, s) \Delta s \right) \Delta t = \int_{\alpha}^{\beta} \left(\int_s^{\beta} F(t, s) \Delta t \right) \Delta s. \quad (8)$$

3. Main results

Throughout this section, $\rho \geq 1$ and $\nu > 0$ are fixed, and we write

$$p = \rho + \nu, \quad r = \frac{p}{\nu}, \quad r' = \frac{p}{\rho}, \quad e = p - 1 - \frac{\nu}{\rho} = \frac{(\rho - 1)(\rho + \nu)}{\rho}. \quad (9)$$

Since $\rho \geq 1$ and $\nu > 0$, we have $p > 1$, $r > 1$, $r' > 1$, $e \geq 0$, and $1/r + 1/r' = (\nu + \rho)/p = 1$. The constraint $\rho \geq 1$ is used only through the sign of e : it guarantees that the tail weight φ^e of (12) remains bounded at the left endpoint, where $\varphi(\alpha) = 0$.

We use the convention $\varphi^0 \equiv 1$, so that $e = 0$ causes no ambiguity. The auxiliary function φ is always assumed to satisfy

$$\varphi \in C_{\text{rd}}^1([\alpha, \beta]_{\mathbb{T}}), \quad \varphi(\alpha) = 0, \quad \varphi^\Delta(\zeta) > 0 \quad \text{for all } \zeta \in [\alpha, \beta]_{\mathbb{T}}^{\kappa}. \quad (10)$$

In particular, φ is increasing and $\varphi(\zeta) > 0$ for $\zeta \in (\alpha, \beta]_{\mathbb{T}}$, which makes φ suitable as a monotone weight below.

We begin with the control estimate on which everything rests. Its exponent is the full derivative power $p = \rho + \nu$ occurring in the main theorem; this is what makes the inner integral appear with outer power exactly 1 and thus makes the head-integral formula (8) applicable.

LEMMA 3.1. *Let $p > 1$ and let $u \in C_{\text{rd}}^1([\alpha, \beta]_{\mathbb{T}})$ satisfy $u(\alpha) = 0$, and let φ satisfy (10). If*

$$\Lambda(\zeta) := \int_{\alpha}^{\zeta} \frac{|u^\Delta(\eta)|^p}{(\varphi^\Delta(\eta))^{p-1}} \Delta\eta$$

is finite for $\zeta \in [\alpha, \beta]_{\mathbb{T}}$, then

$$|u(\zeta)|^p \leq \varphi(\zeta)^{p-1} \Lambda(\zeta), \quad \zeta \in [\alpha, \beta]_{\mathbb{T}}. \quad (11)$$

Proof. For $\zeta = \alpha$ both sides vanish. Let $\zeta \in (\alpha, \beta]_{\mathbb{T}}$. Since $u(\alpha) = 0$, Lemma 2.1 gives $u(\zeta) = \int_{\alpha}^{\zeta} u^\Delta(\eta) \Delta\eta$, and therefore

$$|u(\zeta)| \leq \int_{\alpha}^{\zeta} (\varphi^\Delta(\eta))^{\frac{p-1}{p}} \frac{|u^\Delta(\eta)|}{(\varphi^\Delta(\eta))^{\frac{p-1}{p}}} \Delta\eta,$$

which is legitimate because $\varphi^\Delta > 0$ on $[\alpha, \beta]_{\mathbb{T}}^{\kappa}$. Applying Lemma 2.2 with the conjugate exponents $s = p/(p-1)$ and $s' = p$ yields

$$|u(\zeta)| \leq \left(\int_{\alpha}^{\zeta} \varphi^\Delta(\eta) \Delta\eta \right)^{\frac{p-1}{p}} \left(\int_{\alpha}^{\zeta} \frac{|u^\Delta(\eta)|^p}{(\varphi^\Delta(\eta))^{p-1}} \Delta\eta \right)^{\frac{1}{p}} = \varphi(\zeta)^{\frac{p-1}{p}} \Lambda(\zeta)^{\frac{1}{p}},$$

where we used $\varphi(\alpha) = 0$. Raising this inequality to the power p gives (11). $\square$

We are now ready to state the first main two-exponent Beesack–Opial inequality.

THEOREM 3.2. *Let $\alpha < \beta$ in $\mathbb{T}$, let $\rho \geq 1$ and $\nu > 0$, and let p, r, r', e be given by (9). Assume that*

$$u, \varphi \in C_{\text{rd}}^1([\alpha, \beta]_{\mathbb{T}}), \quad w \in C_{\text{rd}}([\alpha, \beta]_{\mathbb{T}}) \quad \text{with } w \geq 0,$$

that $u(\alpha) = 0$, and that φ satisfies (10). Define the tail weight

$$\mathscr{W}(\omega) = \int_{\omega}^{\beta} w(\zeta) \varphi(\zeta)^e \Delta\zeta, \quad \omega \in [\alpha, \beta]_{\mathbb{T}}, \quad (12)$$

and assume that $\mathcal{W}(\alpha) < \infty$ and that the two integrals on the right-hand side of (13) are finite. Then

$$\begin{aligned} \int_{\alpha}^{\beta} w(\zeta) |u(\zeta)|^{\rho} |u^{\Delta}(\zeta)|^{\nu} \Delta\zeta &\leq \left(\int_{\alpha}^{\beta} w(\zeta) \varphi(\zeta) |u^{\Delta}(\zeta)|^p \Delta\zeta \right)^{1/r} \\ &\times \left(\int_{\alpha}^{\beta} \mathcal{W}(\zeta) \frac{|u^{\Delta}(\zeta)|^p}{(\varphi^{\Delta}(\zeta))^{p-1}} \Delta\zeta \right)^{1/r'}. \end{aligned} \quad (13)$$

Proof. Let Λ be as in Lemma 3.1. Raising (11) to the power ρ/p gives

$$|u(\zeta)|^{\rho} \leq \varphi(\zeta)^{\frac{\rho(p-1)}{p}} \Lambda(\zeta)^{\frac{\rho}{p}}, \quad \zeta \in [\alpha, \beta]_{\mathbb{T}},$$

and therefore, since $w \geq 0$,

$$w(\zeta) |u(\zeta)|^{\rho} |u^{\Delta}(\zeta)|^{\nu} \leq \left(w(\zeta) \varphi(\zeta) |u^{\Delta}(\zeta)|^p \right)^{1/r} \left(w(\zeta) \varphi(\zeta)^e \Lambda(\zeta) \right)^{1/r'}. \quad (14)$$

Indeed, on the right of (14) the factor w carries the power $1/r + 1/r' = 1$, the factor $|u^{\Delta}|$ the power $p/r = \nu$, the factor Λ the power $1/r' = \rho/p$, and φ the power

$$\frac{1}{r} + \frac{e}{r'} = \frac{\nu}{p} + \frac{\rho}{p} \left(p - 1 - \frac{\nu}{p} \right) = \frac{\rho(p-1)}{p},$$

which is exactly the power appearing in the bound for $|u|^{\rho}$; all powers of φ in (14) are nonnegative because $e \geq 0$. Integrating (14) over $[\alpha, \beta]_{\mathbb{T}}$ and applying Lemma 2.2 with the conjugate exponents (r, r') gives

$$\int_{\alpha}^{\beta} w(\zeta) |u(\zeta)|^{\rho} |u^{\Delta}(\zeta)|^{\nu} \Delta\zeta \leq A^{1/r} B^{1/r'}, \quad (15)$$

where

$$A = \int_{\alpha}^{\beta} w(\zeta) \varphi(\zeta) |u^{\Delta}(\zeta)|^p \Delta\zeta, \quad B = \int_{\alpha}^{\beta} w(\zeta) \varphi(\zeta)^e \Lambda(\zeta) \Delta\zeta.$$

It remains to estimate B . The functions $g(\zeta) = w(\zeta) \varphi(\zeta)^e$ and $h(\eta) = |u^{\Delta}(\eta)|^p (\varphi^{\Delta}(\eta))^{-(p-1)}$ are nonnegative and rd-continuous. Since $\Lambda(\zeta) \leq \Lambda(\sigma(\zeta))$ and $\Lambda(\sigma(\zeta)) = \int_{\alpha}^{\sigma(\zeta)} h(\eta) \Delta\eta$, the head-integral Fubini formula (8), applied to $F(\zeta, \eta) = g(\zeta) h(\eta)$, yields

$$B \leq \int_{\alpha}^{\beta} g(\zeta) \Lambda(\sigma(\zeta)) \Delta\zeta = \int_{\alpha}^{\beta} \mathcal{W}(\eta) \frac{|u^{\Delta}(\eta)|^p}{(\varphi^{\Delta}(\eta))^{p-1}} \Delta\eta.$$

Substituting this bound into (15) completes the proof. $\square$

REMARK 3.3. Let $\mathbb{T} = \mathbb{R}$ and $\rho = \nu = 1$. Then $p = 2$, $r = r' = 2$, $e = 0$, and $\mathscr{W}(\zeta) = \int_{\zeta}^{\beta} w(\eta) d\eta$, so that (13) becomes

$$\begin{aligned} \int_{\alpha}^{\beta} w(\zeta) |u(\zeta) u'(\zeta)| d\zeta &\leq \left(\int_{\alpha}^{\beta} w(\zeta) \varphi(\zeta) |u'(\zeta)|^2 d\zeta \right)^{1/2} \\ &\quad \times \left(\int_{\alpha}^{\beta} \mathscr{W}(\zeta) \frac{|u'(\zeta)|^2}{\varphi'(\zeta)} d\zeta \right)^{1/2}. \end{aligned}$$

This is precisely the case $p = q = 2$ of the continuous Beesack–Opial inequality (5) of Chesneau [14], with $g = w$, $h = \varphi$, and $G = \mathscr{W}$. For $\rho > 1$ the tail exponent e is strictly positive, and we are not aware of a continuous counterpart of (13) in this form.

An equivalent additive formulation of Theorem 3.2, with a free balancing parameter, is given next.

THEOREM 3.4. *Let the assumptions of Theorem 3.2 hold. Then, for every $\kappa > 0$,*

$$\begin{aligned} \int_{\alpha}^{\beta} w(\zeta) |u(\zeta)|^{\rho} |u^{\Delta}(\zeta)|^{\nu} \Delta\zeta &\leq \frac{\kappa^r}{r} \int_{\alpha}^{\beta} w(\zeta) \varphi(\zeta) |u^{\Delta}(\zeta)|^p \Delta\zeta \\ &\quad + \frac{1}{r' \kappa^{r'}} \int_{\alpha}^{\beta} \mathscr{W}(\zeta) \frac{|u^{\Delta}(\zeta)|^p}{(\varphi^{\Delta}(\zeta))^{p-1}} \Delta\zeta. \end{aligned} \quad (16)$$

Proof. Let A and B denote the two integrals on the right-hand side of (13), so that Theorem 3.2 reads $\int_{\alpha}^{\beta} w(\zeta) |u(\zeta)|^{\rho} |u^{\Delta}(\zeta)|^{\nu} \Delta\zeta \leq A^{1/r} B^{1/r'}$. Young's inequality

$$xy \leq \frac{x^r}{r} + \frac{y^{r'}}{r'}, \quad x, y \geq 0,$$

applied to $x = \kappa A^{1/r}$ and $y = \kappa^{-1} B^{1/r'}$, gives

$$A^{1/r} B^{1/r'} \leq \frac{\kappa^r}{r} A + \frac{1}{r' \kappa^{r'}} B,$$

which is (16). $\square$

THEOREM 3.5. *Let the assumptions of Theorem 3.2 hold. Then the choice*

$$\kappa = \left(\frac{\rho}{\nu} \right)^{\frac{\rho\nu}{(\rho+\nu)^2}}$$

in (16) equalizes the two coefficients and yields

$$\begin{aligned} \int_{\alpha}^{\beta} w(\zeta) |u(\zeta)|^{\rho} |u^{\Delta}(\zeta)|^{\nu} \Delta\zeta &\leq K \int_{\alpha}^{\beta} [w(\zeta) \varphi(\zeta) |u^{\Delta}(\zeta)|^p \\ &\quad + \mathscr{W}(\zeta) \frac{|u^{\Delta}(\zeta)|^p}{(\varphi^{\Delta}(\zeta))^{p-1}}] \Delta\zeta, \end{aligned} \quad (17)$$

where

$$K = \frac{1}{r} \left(\frac{\rho}{v} \right)^{\rho/p} = \frac{(r-1)^{1-1/r}}{r}.$$

Moreover, K is the smallest constant that can be obtained from (16) by a choice of $\kappa > 0$.

Proof. Write A and B as in the proof of Theorem 3.4. The two coefficients in (16) coincide when

$$\frac{\kappa^r}{r} = \frac{1}{r' \kappa^{r'}}, \quad \text{that is,} \quad \kappa^{r+r'} = \frac{r}{r'}$$

By (9),

$$r + r' = \frac{p}{v} + \frac{p}{\rho} = \frac{p^2}{\rho v}, \quad \frac{r}{r'} = \frac{\rho}{v},$$

so that $\kappa = (\rho/v)^{\rho v/p^2}$. For this value,

$$\frac{\kappa^r}{r} = \frac{1}{r} \left(\frac{\rho}{v} \right)^{\frac{\rho v}{p^2} \frac{p}{v}} = \frac{1}{r} \left(\frac{\rho}{v} \right)^{\rho/p} = K,$$

and, since $\kappa^{r'} = (\rho/v)^{v/p}$ and $1/r' = \rho/p$,

$$\frac{1}{r' \kappa^{r'}} = \frac{\rho}{p} \left(\frac{v}{\rho} \right)^{v/p} = \frac{v}{p} \left(\frac{\rho}{v} \right)^{\rho/p} = K,$$

where the middle equality follows from $\rho(\rho/v)^{-v/p} = v(\rho/v)^{\rho/p}$. Hence the right-hand side of (16) equals $K(A+B)$, which is (17). Finally, $\rho/v = r-1$ and $\rho/p = (r-1)/r$, whence $K = r^{-1}(r-1)^{1-1/r}$.

For the last assertion, let $\Phi(\kappa) = \max\{\kappa^r/r, (r'\kappa^{r'})^{-1}\}$. The first entry is increasing and the second is decreasing in κ , so Φ attains its minimum where the two agree, namely at the above κ , with value K . $\square$

REMARK 3.6. For $\mathbb{T} = \mathbb{R}$ and $\rho = v = 1$ one has $K = 1/2$, and (17) becomes the additive Beesack–Opial inequality of [14] for $p = q = 2$, with $\mathscr{W}(\zeta) = \int_{\zeta}^{\beta} w(\eta) d\eta$.

4. Applications

The dynamic inequalities of Section 3 provide a unified tool for recovering and extending a number of Opial-type estimates. We organize the consequences as follows: general dynamic consequences in Section 4.1, a polynomial-weight consequence on arbitrary time scales in Section 4.2, the continuous case in Section 4.3, the discrete case in Section 4.4, and the quantum case in Section 4.5. Consequences that differ only by a renaming of parameters are collected in remarks rather than stated separately, and routine verifications are indicated in one line. Throughout, $p = \rho + v$, and $r, r', e, \mathscr{W}$ are as in (9) and (12).

4.1. Dynamic consequences

The simplest admissible auxiliary function on an arbitrary time scale is $\varphi(\zeta) = \zeta - \alpha$, for which $\varphi(\alpha) = 0$ and $\varphi^\Delta \equiv 1 > 0$. It gives the following two-exponent weighted inequality.

COROLLARY 4.1. *Let $\alpha < \beta$ in $\mathbb{T}$, let $q : [\alpha, \beta]_{\mathbb{T}} \rightarrow [0, \infty)$ be rd-continuous, and let $y \in C_{\text{rd}}^1([\alpha, \beta]_{\mathbb{T}})$ satisfy $y(\alpha) = 0$. Let $\lambda \geq 1$ and $\gamma > 0$, and put $p = \lambda + \gamma$ and $e = p - 1 - \gamma/\lambda$. If the integrals below are finite, then*

$$\begin{aligned} \int_{\alpha}^{\beta} q(\zeta) |y(\zeta)|^{\lambda} |y^{\Delta}(\zeta)|^{\gamma} \Delta\zeta &\leq \left(\int_{\alpha}^{\beta} q(\zeta) (\zeta - \alpha) |y^{\Delta}(\zeta)|^p \Delta\zeta \right)^{\frac{\gamma}{p}} \\ &\quad \times \left(\int_{\alpha}^{\beta} Q(\zeta) |y^{\Delta}(\zeta)|^p \Delta\zeta \right)^{\frac{\lambda}{p}}, \end{aligned} \quad (18)$$

where

$$Q(\omega) = \int_{\omega}^{\beta} q(\eta) (\eta - \alpha)^e \Delta\eta, \quad \omega \in [\alpha, \beta]_{\mathbb{T}}.$$

Proof. This follows from Theorem 3.2 by taking $\rho = \lambda$, $v = \gamma$, $w = q$, and $\varphi(\zeta) = \zeta - \alpha$; then $\varphi^\Delta \equiv 1$, so $(\varphi^\Delta)^{p-1} \equiv 1$, and $\mathscr{W} = Q$. $\square$

The product estimate in (18) can be reduced to a one-term inequality whenever its tail weight is controlled by the original weight. The following proposition records the unweighted case, a general tail condition, and the important special case of a non-increasing weight. The unweighted estimate is also a consequence of the power-type dynamic inequality of Elsaid, Zaghrout, and El-Deeb [16], while the nonincreasing-weight case applies, in particular, to composite weights of the type considered by Saker [27].

PROPOSITION 4.2. *Let $\alpha < \beta$ in $\mathbb{T}$, let $\lambda \geq 1$ and $\gamma > 0$, and put*

$$p = \lambda + \gamma, \quad e = p - 1 - \frac{\gamma}{\lambda}.$$

Let $y \in C_{\text{rd}}^1([\alpha, \beta]_{\mathbb{T}})$ satisfy $y(\alpha) = 0$. Then the following assertions hold.

(i) Unweighted case. *One has*

$$\int_{\alpha}^{\beta} |y(\zeta)|^{\lambda} |y^{\Delta}(\zeta)|^{\gamma} \Delta\zeta \leq (\beta - \alpha)^{\lambda} \int_{\alpha}^{\beta} |y^{\Delta}(\zeta)|^p \Delta\zeta.$$

(ii) Weights satisfying a tail condition. *Let*

$$\omega : [\alpha, \beta]_{\mathbb{T}} \longrightarrow [0, \infty)$$

be rd-continuous, and suppose that there exists a constant $C_0 > 0$ such that

$$\int_{\zeta}^{\beta} \omega(\eta)(\eta - \alpha)^e \Delta\eta \leq C_0 \omega(\zeta), \quad \zeta \in [\alpha, \beta]_{\mathbb{T}}. \quad (19)$$

Then

$$\begin{aligned} \int_{\alpha}^{\beta} \omega(\zeta) |y(\zeta)|^{\lambda} |y^{\Delta}(\zeta)|^{\gamma} \Delta\zeta &\leq (\beta - \alpha)^{\frac{\gamma}{p}} C_0^{\frac{\lambda}{p}} \\ &\times \int_{\alpha}^{\beta} \omega(\zeta) |y^{\Delta}(\zeta)|^p \Delta\zeta. \end{aligned}$$

(iii) Nonincreasing weights. Let $\omega : [\alpha, \beta]_{\mathbb{T}} \rightarrow [0, \infty)$ be rd-continuous and nonincreasing. Then (19) holds with

$$C_0 = (\beta - \alpha)^{e+1},$$

and consequently

$$\int_{\alpha}^{\beta} \omega(\zeta) |y(\zeta)|^{\lambda} |y^{\Delta}(\zeta)|^{\gamma} \Delta\zeta \leq (\beta - \alpha)^{\lambda} \int_{\alpha}^{\beta} \omega(\zeta) |y^{\Delta}(\zeta)|^p \Delta\zeta.$$

Proof. We first prove part (ii). Apply Corollary 4.1 with $q = \omega$. Set

$$I = \int_{\alpha}^{\beta} \omega(\zeta) |y^{\Delta}(\zeta)|^p \Delta\zeta.$$

Since

$$0 \leq \zeta - \alpha \leq \beta - \alpha,$$

the first factor on the right-hand side of (18) satisfies

$$\left(\int_{\alpha}^{\beta} \omega(\zeta) (\zeta - \alpha) |y^{\Delta}(\zeta)|^p \Delta\zeta \right)^{\frac{\gamma}{p}} \leq (\beta - \alpha)^{\frac{\gamma}{p}} I^{\frac{\gamma}{p}}.$$

The corresponding tail weight is

$$Q(\zeta) = \int_{\zeta}^{\beta} \omega(\eta)(\eta - \alpha)^e \Delta\eta.$$

By (19), $Q(\zeta) \leq C_0 \omega(\zeta)$, and hence

$$\left(\int_{\alpha}^{\beta} Q(\zeta) |y^{\Delta}(\zeta)|^p \Delta\zeta \right)^{\frac{\lambda}{p}} \leq C_0^{\frac{\lambda}{p}} I^{\frac{\lambda}{p}}.$$

Multiplying the last two estimates and using

$$\frac{\gamma}{p} + \frac{\lambda}{p} = 1$$

gives part (ii).

To prove part (iii), first observe that

$$e = p - 1 - \frac{\gamma}{\lambda} = \frac{(\lambda - 1)(\lambda + \gamma)}{\lambda} \geq 0.$$

If ω is nonincreasing, then $\omega(\eta) \leq \omega(\zeta)$ whenever $\eta \geq \zeta$. Therefore,

$$\begin{aligned} Q(\zeta) &= \int_{\zeta}^{\beta} \omega(\eta)(\eta - \alpha)^e \Delta\eta \\ &\leq \omega(\zeta)(\beta - \alpha)^e \int_{\zeta}^{\beta} 1 \Delta\eta \\ &= \omega(\zeta)(\beta - \alpha)^e(\beta - \zeta) \\ &\leq \omega(\zeta)(\beta - \alpha)^{e+1}. \end{aligned}$$

Thus, (19) holds with $C_0 = (\beta - \alpha)^{e+1}$. Substituting this value into part (ii) gives

$$\begin{aligned} (\beta - \alpha)^{\frac{\gamma}{p}} C_0^{\frac{\lambda}{p}} &= (\beta - \alpha)^{\frac{\gamma + \lambda(e+1)}{p}} \\ &= (\beta - \alpha)^{\lambda}, \end{aligned}$$

because

$$e + 1 = p - \frac{\gamma}{\lambda}.$$

This proves part (iii). Finally, part (i) follows from part (iii) by taking $\omega \equiv 1$. $\square$

4.2. A polynomial-weight consequence on arbitrary time scales

The additive inequality also yields a useful polynomial-weight consequence that remains valid on an arbitrary time scale.

COROLLARY 4.3. *Let $0 \leq \alpha < \beta$ in $\mathbb{T}$, let $m \in \mathbb{N}_0$, and let $u \in C_{\text{rd}}^1([\alpha, \beta]_{\mathbb{T}})$ satisfy $u(\alpha) = 0$. Define*

$$\mathcal{W}_m(\zeta) = \int_{\zeta}^{\beta} \eta^m \Delta\eta, \quad \zeta \in [\alpha, \beta]_{\mathbb{T}}.$$

Then, for every $\kappa > 0$,

$$\begin{aligned} \int_{\alpha}^{\beta} \zeta^m |u(\zeta) u^{\Delta}(\zeta)| \Delta\zeta &\leq \frac{\kappa^2}{2} \int_{\alpha}^{\beta} \zeta^m (\zeta - \alpha) |u^{\Delta}(\zeta)|^2 \Delta\zeta \\ &\quad + \frac{1}{2\kappa^2} \int_{\alpha}^{\beta} \mathcal{W}_m(\zeta) |u^{\Delta}(\zeta)|^2 \Delta\zeta. \end{aligned} \tag{20}$$

In particular, the choice $\kappa = 1$ gives the balanced estimate

$$\begin{aligned} \int_{\alpha}^{\beta} \zeta^m |u(\zeta) u^{\Delta}(\zeta)| \Delta\zeta &\leq \frac{1}{2} \int_{\alpha}^{\beta} \zeta^m (\zeta - \alpha) |u^{\Delta}(\zeta)|^2 \Delta\zeta \\ &\quad + \frac{1}{2} \int_{\alpha}^{\beta} \mathcal{W}_m(\zeta) |u^{\Delta}(\zeta)|^2 \Delta\zeta. \end{aligned}$$

Proof. Apply Theorem 3.4 with

$$\rho = \nu = 1, \quad w(\zeta) = \zeta^m, \quad \varphi(\zeta) = \zeta - \alpha.$$

Then

$$p = 2, \quad r = r' = 2, \quad \varphi^\Delta \equiv 1, \quad e = 0,$$

and the corresponding tail weight is

$$\mathcal{W}(\zeta) = \int_{\zeta}^{\beta} \eta^m \Delta \eta = \mathcal{W}_m(\zeta).$$

The weight $w(\zeta) = \zeta^m$ is nonnegative because $\alpha \geq 0$. Consequently, Theorem 3.4 gives (20), while Theorem 3.5 yields the balanced case with constant $1/2$ when $\kappa = 1$. $\square$

REMARK 4.4. When $\mathbb{T} = \mathbb{R}$, the tail weight has the explicit form

$$\mathcal{W}_m(\zeta) = \int_{\zeta}^{\beta} \eta^m d\eta = \frac{\beta^{m+1} - \zeta^{m+1}}{m+1}.$$

Hence, (20) becomes the corresponding continuous polynomial-weight additive Opial-type inequality studied by Salem, Agwo, and Khallaf [30]; see also the related discussion in [14]. On a general time scale, the delta integral defining $\mathcal{W}_m$ need not admit the same elementary closed form.

4.3. The continuous case

We next specialize the general dynamic inequality to the continuous time scale $\mathbb{T} = \mathbb{R}$. In this case,

$$\sigma(\zeta) = \zeta, \quad u^\Delta(\zeta) = u'(\zeta), \quad \Delta\zeta = d\zeta.$$

COROLLARY 4.5. Let $\alpha < \beta$ be real numbers, let $\rho \geq 1$ and $\nu > 0$, and let p, r, r' , and e be defined as in (9). Suppose that $u \in C^1([\alpha, \beta])$ satisfies $u(\alpha) = 0$, that $w : [\alpha, \beta] \rightarrow [0, \infty)$ is continuous, and that $\varphi \in C^1([\alpha, \beta])$ satisfies

$$\varphi(\alpha) = 0 \quad \text{and} \quad \varphi'(\zeta) > 0 \quad \text{for } \zeta \in [\alpha, \beta].$$

Define

$$\mathcal{W}(\omega) = \int_{\omega}^{\beta} w(\eta) \varphi(\eta)^e d\eta, \quad \omega \in [\alpha, \beta].$$

Then

$$\begin{aligned} \int_{\alpha}^{\beta} w(\zeta) |u(\zeta)|^{\rho} |u'(\zeta)|^{\nu} d\zeta &\leq \left(\int_{\alpha}^{\beta} w(\zeta) \varphi(\zeta) |u'(\zeta)|^p d\zeta \right)^{1/r} \\ &\quad \times \left(\int_{\alpha}^{\beta} \mathcal{W}(\zeta) \frac{|u'(\zeta)|^p}{(\varphi'(\zeta))^{p-1}} d\zeta \right)^{1/r'}. \end{aligned} \quad (21)$$

Proof. Set $\mathbb{T} = \mathbb{R}$ in Theorem 3.2. Then $\sigma(\zeta) = \zeta$, $u^\Delta = u'$, and the delta integral reduces to the ordinary integral. Moreover, the triangular change-of-order formula (8) becomes an equality without a forward shift. Consequently, Theorem 3.2 reduces directly to (21). $\square$

REMARK 4.6. The continuous inequality (21) belongs to the line of weighted Opial- and Beesack-type inequalities developed by Beesack [7], Das [15], Pachpatte [26], and other authors; a systematic account of many of these continuous results is given by Agarwal and Pang [6]. The distinguishing feature of (21) is that the powers ρ and ν are independent and that the resulting bound retains a two-factor weighted structure.

The connection with the recent two-weight inequality of Chesneau [14] is especially direct. Indeed, when

$$\rho = \nu = 1,$$

we have

$$p = 2, \quad r = r' = 2, \quad e = 0,$$

and therefore

$$\mathcal{W}(\zeta) = \int_{\zeta}^{\beta} w(\eta) d\eta.$$

In this case, (21) becomes

$$\begin{aligned} \int_{\alpha}^{\beta} w(\zeta) |u(\zeta) u'(\zeta)| d\zeta &\leq \left(\int_{\alpha}^{\beta} w(\zeta) \varphi(\zeta) |u'(\zeta)|^2 d\zeta \right)^{1/2} \\ &\quad \times \left(\int_{\alpha}^{\beta} \mathcal{W}(\zeta) \frac{|u'(\zeta)|^2}{\varphi'(\zeta)} d\zeta \right)^{1/2}. \end{aligned}$$

Thus, after identifying

$$g = w, \quad h = \varphi, \quad G = \mathcal{W},$$

the preceding estimate coincides with the case $p = q = 2$ of the two-weight continuous inequality in [14]. Hence, (21) extends that structure by allowing the independent powers $\rho \geq 1$ and $\nu > 0$.

4.4. The discrete case

Let $\mathbb{T} = \mathbb{Z}$, so that $u^\Delta(k) = \Delta u(k) = u(k+1) - u(k)$ and $\int_{\alpha}^{\beta} f(\zeta) \Delta \zeta = \sum_{k=\alpha}^{\beta-1} f(k)$.

COROLLARY 4.7. Let $\alpha < \beta$ be integers, let $\rho \geq 1$, $\nu > 0$, and let p, r, r', e be as in (9). Let $u, w, \varphi : \{\alpha, \dots, \beta\} \rightarrow \mathbb{R}$ satisfy $u(\alpha) = 0$, $w(k) \geq 0$, $\varphi(\alpha) = 0$, and

$\Delta\varphi(k) > 0$ for $k = \alpha, \dots, \beta - 1$, and put $\mathscr{W}(j) = \sum_{k=j}^{\beta-1} w(k)\varphi(k)^e$. Then

$$\begin{aligned} \sum_{k=\alpha}^{\beta-1} w(k) |u(k)|^\rho |\Delta u(k)|^\nu &\leq \left(\sum_{k=\alpha}^{\beta-1} w(k) \varphi(k) |\Delta u(k)|^p \right)^{1/r} \\ &\quad \times \left(\sum_{k=\alpha}^{\beta-1} \mathscr{W}(k) \frac{|\Delta u(k)|^p}{(\Delta\varphi(k))^{p-1}} \right)^{1/r'}. \end{aligned} \quad (22)$$

Proof. This is Theorem 3.2 on $\mathbb{T} = \mathbb{Z}$. $\square$

REMARK 4.8. (Wong–Lasota type) Let $\alpha < \beta$ be integers and let $u(\alpha) = 0$. Taking $\rho = \nu = 1$, $w \equiv 1$, and $\varphi(k) = k - \alpha$ in Corollary 4.7, we have

$$p = 2, \quad r = r' = 2, \quad \Delta\varphi \equiv 1, \quad e = 0,$$

and hence $\mathscr{W}(j) = \beta - j$. Therefore, (22) yields

$$\sum_{k=\alpha}^{\beta-1} |u(k) \Delta u(k)| \leq \left(\sum_{k=\alpha}^{\beta-1} (k - \alpha) |\Delta u(k)|^2 \right)^{1/2} \left(\sum_{k=\alpha}^{\beta-1} (\beta - k) |\Delta u(k)|^2 \right)^{1/2}.$$

Using $\sqrt{XY} \leq (X + Y)/2$ for $X, Y \geq 0$, we obtain

$$\begin{aligned} \sum_{k=\alpha}^{\beta-1} |u(k) \Delta u(k)| &\leq \frac{1}{2} \sum_{k=\alpha}^{\beta-1} [(k - \alpha) + (\beta - k)] |\Delta u(k)|^2 \\ &= \frac{\beta - \alpha}{2} \sum_{k=\alpha}^{\beta-1} |\Delta u(k)|^2. \end{aligned}$$

Thus, the corresponding Wong–Lasota-type discrete inequality is recovered.

4.5. The quantum case

Finally, let

$$\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}, \quad q > 1,$$

so that $\sigma(t) = qt$, $\mu(t) = (q - 1)t$, and the delta derivative is the q -derivative

$$u^\Delta(t) = D_q u(t) = \frac{u(qt) - u(t)}{(q - 1)t}, \quad t \in \mathbb{T}.$$

Delta integration becomes the Jackson q -integral: if $\alpha = q^{k_0} < \beta = q^{k_1}$ with integers $k_0 < k_1$, then

$$\int_\alpha^\beta f(t) d_q t = \sum_{k=k_0}^{k_1-1} (q - 1) q^k f(q^k).$$

COROLLARY 4.9. *Let $\rho \geq 1$, $v > 0$, and let p, r, r', e be as in (9). Let $u, w, \varphi : [\alpha, \beta]_{\mathbb{T}} \rightarrow \mathbb{R}$ satisfy $u(\alpha) = 0$, $w \geq 0$, $\varphi(\alpha) = 0$, and $D_q \varphi(t) > 0$ for $t \in \{q^{k_0}, \dots, q^{k_1-1}\}$, and put $\mathscr{W}(\omega) = \int_{\omega}^{\beta} w \varphi^e d_q \zeta$. Then*

$$\int_{\alpha}^{\beta} w(\zeta) |u(\zeta)|^{\rho} |D_q u(\zeta)|^v d_q \zeta \leq \left(\int_{\alpha}^{\beta} w(\zeta) \varphi(\zeta) |D_q u(\zeta)|^p d_q \zeta \right)^{1/r} \\ \times \left(\int_{\alpha}^{\beta} \mathscr{W}(\zeta) \frac{|D_q u(\zeta)|^p}{(D_q \varphi(\zeta))^{p-1}} d_q \zeta \right)^{1/r'}.$$

Proof. This is Theorem 3.2 on $\mathbb{T} = q^{\mathbb{N}_0}$, all sums being finite. $\square$

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