

SOME RESULTS FOR A FAMILY OF INTEGRAL OPERATORS OF SINGULAR TYPE IN WEIGHTED VARIABLE LEBESGUE SPACES

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Abstract. In this study, our purpose is to prove inequalities concerning the boundedness of a family of singular type integral operators in weighted variable Lebesgue spaces $L_{p(\cdot), \omega, v}(\mathbb{R}_{k,+}^n)$ with general weight functions. As an application, we finally provide an inequality for the Riesz-Bessel transform of high order in weighted variable Lebesgue space.

1. Introduction

Variable Lebesgue spaces which have first been taken into consideration by Orlicz [17] and date back a long time, play an essential role in harmonic analysis. In recent years, there is a considerable interest on variable function spaces since they give us the opportunity to solve PDEs. Therefore, many results have been presented in existing literature of harmonic analysis, however, there are still gaps. For example, in a recent study of Bandaliev [3], it has been obtained that the certain sublinear operators are bounded in weighted variable Lebesgue spaces. In harmonic analysis, it also worths for studying convolution type singular integral operators. As applications of this theory, the boundedness of these operators and the boundedness of their versions which are connected with the Laplace-Bessel differential operator briefly called the B -singular integral operator have great importance. Such problems have been considered by several mathematicians [1, 2, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15]. In [6], the boundedness of the Riesz-Bessel transform of high order, which is a convolution type singular integral operator, has been obtained in weighted Lebesgue spaces with general weight functions.

In the present paper, from the motivation of these results, we will investigate the B -singular integral operator acting from weighted variable Lebesgue space into itself. In these spaces, the boundedness of the B -singular integral operator will be obtained with the use of well-known inequalities for the Hardy type operators.

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2. Preliminaries

As it is well-known, $\mathbb{R}^n$ denotes the n -dimensional Euclidean space. Let $\mathbb{R}_{k,+}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0, 1 \leq k \leq n\}$ where $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$. Denote $v = (v_1, \dots, v_k)$, $v_1 > 0, \dots, v_k > 0$, $|v| = v_1 + \dots + v_k$. Also $B(x, r) = \{y \in \mathbb{R}_{k,+}^n : |x - y| < r\}$. A weight function $\omega : \mathbb{R}_{k,+}^n \rightarrow \mathbb{R}$ is defined as locally integrable and an almost everywhere positive function. Let $B(0, r) \subset \mathbb{R}_{k,+}^n$ be measurable. Then

$$|B(0, r)|_v = \int_{B(0, r)} (x')^v dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{v_i+1}{2}\right)}{\Gamma\left(\frac{v_i}{2}\right)} r^Q,$$

where $(x')^v = x_1^{v_1} \dots x_k^{v_k}$, $Q = n + |v|$ and $\Gamma(v) = \int_0^\infty e^{-x} x^{v-1} dx$.

Let $\mathcal{P}(\mathbb{R}_{k,+}^n) = \{p(\cdot) : \mathbb{R}_{k,+}^n \rightarrow [1, \infty) : p(\cdot) \text{ is measurable}\}$. Thus, every function in $\mathcal{P}(\mathbb{R}_{k,+}^n)$ is called a variable exponent function. If the following conditions

$$|p(x) - p(y)| \leq \frac{C_0}{-\log|x - y|},$$

and

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}$$

holds for all $x, y \in \mathbb{R}_{k,+}^n$ satisfying $|x - y| \leq \frac{1}{2}$, then we denote it as $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}_{k,+}^n)$ and $p(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{R}_{k,+}^n)$, respectively. Furthermore, if $p(\cdot)$ satisfies both inequalities, then it is denoted by $p(\cdot) \in LH(\mathbb{R}_{k,+}^n)$. Here $p_\infty = \lim_{x \rightarrow \infty} p(x) > 1$.

Given a variable exponent $p(\cdot) : \mathbb{R}_{k,+}^n \rightarrow [1, \infty)$, the space $L_{p(\cdot), \omega, v}(\mathbb{R}_{k,+}^n)$ consists of all measurable functions f such that

$$\|f\|_{L_{p(\cdot), \omega, v}(\mathbb{R}_{k,+}^n)} = \inf \{\mu > 0 : \rho_{p(\cdot), \omega, v}(f/\mu) \leq 1\} < \infty,$$

where

$$\rho_{p(\cdot), \omega, v} = \int_{\mathbb{R}_{k,+}^n} |f(x)\omega(x)|^{p(x)} (x')^v dx.$$

In the case of $\omega = 1$, it is obvious that $L_{p(x), \omega, v}(\mathbb{R}_{k,+}^n)$ reduces $L_{p(x), v}(\mathbb{R}_{k,+}^n)$.

The following theorem is the analogue of Theorem 3 in [4] and its proof follows by similar manner.

THEOREM 1. Assume that $1 \leq p_- < p(x) \leq q_- \leq q(y) \leq q_+ < \infty$ for all $x, y \in \mathbb{R}_{k,+}^n$. If $p(x)$ is a continuous function on $\mathbb{R}_{k,+}^n$, then

$$\left\| \|f\|_{L_{p(\cdot), v}(\mathbb{R}_{k,+}^n)} \right\|_{L_{q(\cdot), v}(\mathbb{R}_{k,+}^n)} \leq \left(\frac{p_+}{q_-} + \frac{q_+ - p_-}{q_+} \right)^{2/p_-} \left\| \|f\|_{L_{q(\cdot), v}(\mathbb{R}_{k,+}^n)} \right\|_{L_{p(\cdot), v}(\mathbb{R}_{k,+}^n)}$$

holds, where f denotes a measurable function on $\mathbb{R}_{k,+}^n \times \mathbb{R}_{k,+}^n$.

LEMMA 1. [10] Let $p(\cdot)$ and $q(\cdot)$ be bounded variable exponents on $\mathbb{R}_{k,+}^n$. Then $L_{p(\cdot), V}(\mathbb{R}_{k,+}^n) \hookrightarrow L_{q(\cdot), V}(\mathbb{R}_{k,+}^n)$ if and only if $q(\cdot) \leq p(\cdot)$ and $\limsup_{\lambda \rightarrow 0^+} \int_{\mathbb{R}_{k,+}^n} \lambda^{\frac{p(x)q(x)}{p(x)-q(x)}} (x')^V dx = 0$. Here $\lambda^{\frac{p(x)q(x)}{p(x)-q(x)}} := 0$ for $0 \leq \lambda < 1$ and $p(x) = q(x)$.

For $1 \leq i \leq k$, $1 \leq k \leq n$, we define the generalized translation operator by

$$T^y f(x) := C_{V,k} \int_0^\pi \dots \int_0^\pi f[(x_1, y_1)_{\varphi_1}, \dots, (x_k, y_k)_{\varphi_k}, x'' - y''] \prod_{i=1}^k (\sin \varphi_i)^{v_i-1} d\varphi_i,$$

where $C_{V,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^k \frac{\Gamma(\frac{v_i+1}{2})}{\Gamma(\frac{v_i}{2})}$, $(x_i, y_i)_{\varphi_i} = (x_i^2 - 2x_i y_i \cos \varphi_i + y_i^2)^{\frac{1}{2}}$ [15, 16]. Observe that the generalized translation operator is constructed based on the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^k \frac{\partial^2}{\partial x_i^2} + \frac{v_i}{x_i} \frac{\partial}{\partial x_i} + \sum_{i=k+1}^n \frac{\partial^2}{\partial x_i^2}.$$

Respectively, the definitions of B -convolution and B -singular integral operator are as follows:

$$(f \otimes g)(x) = \int_{\mathbb{R}_{k,+}^n} f(y) T^y g(x) (y')^V dy.$$

$$\begin{aligned} T_V f(x) &= p.v. \int_{\mathbb{R}_{k,+}^n} \frac{\Theta(\xi)}{|y|Q} [T^y f(x)] (y')^V dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\{y \in \mathbb{R}_{k,+}^n : |y| > \varepsilon\}} \frac{\Theta(\xi)}{|y|Q} [T^y f(x)] (y')^V dy = \lim_{\varepsilon \rightarrow 0} T_{V,\varepsilon} f(x), \end{aligned} \quad (1)$$

where $\xi = y/|y|$, and $\Theta(\xi)$ belongs to some function spaces on the semisphere $S_{k,+} = \{x \in \mathbb{R}_{k,+}^n : |x| = 1\}$ and also satisfies

$$\int_{S_{k,+}} \Theta(\xi) (\xi')^V d\sigma(\xi) = 0, \quad (2)$$

where $d\sigma(\theta)$ denotes the surface measure on the unit sphere $|\theta| = 1$. B -singular integral operator is of convolution type with the kernel $K(y) = \frac{\Theta(\xi)}{|y|Q}$ and so it can be written as $T_V f(x) = (K \otimes f)(x)$. This kernel satisfies the followings:

- $\sup_{\xi \in S_{k,+}} |K(\xi)| < \infty$,
- $\int_{B_+(0,2|x|)} |K(y) - T^y K(x)| (y')^V dy \leq C$, $y \neq 0$, where C denotes an absolute constant,

- There exist constants $\alpha > 0$ and $C > 0$ such that $|K(x)| \leq C|x|^{-Q}$ and

$$|T^y K(x) - K(y)| \leq C|x|^\alpha |y|^{-Q-\alpha},$$

for all $x, y \in \mathbb{R}_{k,+}^n$ with $2|x| < |y|$.

The following theorem states the boundedness of the generalized translation operator. For more details, one can see [10, Theorem 4.1].

THEOREM 2. Let $p(\cdot) : \mathbb{R}_{k,+}^n \rightarrow [1, \infty)$ with

$$1 < p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}_{k,+}^n} p(x) \leq p_+ = \operatorname{ess\,sup}_{x \in \mathbb{R}_{k,+}^n} p(x) < \infty.$$

Then

$$\|T^y f(x)\|_{p(\cdot),v} \leq A \exp((2 + 2^{\alpha n} |y|) A_0(p(\cdot))) \|f\|_{p(\cdot),v},$$

for all $f \in L_{p(\cdot),v} \cap \mathcal{S}'_+(\mathbb{R}_{k,+}^n)$ with $\operatorname{supp} \mathcal{F}_v f \subset \{\theta \in \mathbb{R}_{k,+}^n : |\theta| \leq 2^{\alpha+1}\}$, $\alpha \in \mathbb{N}_0$, where $\mathcal{F}_v f$, $\mathcal{S}'_+(\mathbb{R}_{k,+}^n)$, respectively denote the Fourier-Bessel transform and the dual space of tempered distributions on $\mathbb{R}_{k,+}^n$, and $A = A(n, p(\cdot), v) > 0$ is a constant.

3. $L_{p(\cdot),\omega,v}$ estimates for B -singular integral operator

This section is devoted to the B -singular integral operator defined on weighted variable Lebesgue space.

The generalized Hardy type operator is main tool of proofs of our main results. The following theorems are analogues of Theorem 3 and Theorem 4 in [3] and their proofs can be obtained in a similar way.

THEOREM 3. Let $p(x) \in \mathcal{P}(\mathbb{R}_{k,+}^n)$ satisfying $1 < p_- \leq p(x) \leq p_+ < \infty$ and such that $\int_{\mathbb{R}_{k,+}^n} \lambda \frac{p(x)q(x)}{p(x)-q(x)} (x')^v dx < \infty$ holds for $\lambda \in [0, 1)$. Assume that $\omega(x)$ and $v(x)$ are weight functions on $\mathbb{R}_{k,+}^n$ and there exists a constant $\eta < 1$ such that

$$H = \sup_{r>0} \left(\int_{|y|<r} |v(y)|^{-p'_+(y')^v} dy \right)^{\frac{\eta}{p_+}} \left\| \omega(\cdot) \left(\int_{|y|<|\cdot|} |v(y)|^{-p'_+(y')^v} dy \right)^{\frac{1-\eta}{p_+}} \right\|_{L_{p(\cdot),v}(|x|>r)} < \infty \quad (3)$$

holds. Then we have

$$\left\| \int_{|y|<|\cdot|} f(y)(y')^v dy \right\|_{L_{p(\cdot),\omega,v}(\mathbb{R}_{k,+}^n)} \leq C \|f\|_{L_{p(\cdot),v}(\mathbb{R}_{k,+}^n)}$$

for every $f \in L_{p(x),v(x),v}(\mathbb{R}_{k,+}^n)$, where $C = \left(1 + \frac{p_+-p_-}{p_+}\right)^{1/p_-} (1-\eta)^{-1/p'_+} C_1 H$.

THEOREM 4. Let $p(x) \in \mathcal{P}(\mathbb{R}_{k,+}^n)$ satisfying $1 < p_- \leq p(x) \leq p_+ < \infty$ and such that $\int_{\mathbb{R}_{k,+}^n} \lambda \frac{p(x)q(x)}{p(x)-q(x)} (x')^v dx < \infty$ holds for $\lambda \in [0, 1)$. Assume that $\omega(x)$ and $v(x)$ are weight functions on $\mathbb{R}_{k,+}^n$ and there exists a constant $\zeta < 1$ such that

$$\mathcal{H} = \sup_{r>0} \left(\int_{|y|>r} |v(y)|^{-p'_+} (y')^v dy \right)^{\frac{\zeta}{p_+}} \left\| \omega(\cdot) \left(\int_{|y|<|\cdot|} |v(y)|^{-p'_+} (y')^v dy \right)^{\frac{1-\zeta}{p_+}} \right\|_{L_{p(\cdot), v}(|x|<r)} < \infty \quad (4)$$

holds. Then we have

$$\left\| \int_{|y|<|\cdot|} f(y)(y')^v dy \right\|_{L_{p(\cdot), \omega, v}(\mathbb{R}_{k,+}^n)} \leq C \|f\|_{L_{p(\cdot), v, v}(\mathbb{R}_{k,+}^n)}$$

for every $f \in L_{p(x), v(x), v}(\mathbb{R}_{k,+}^n)$ where $C = \left(1 + \frac{p_+ - p_-}{p_+}\right)^{1/p_-} (1 - \zeta)^{-1/p'_+} C_1 \mathcal{H}$.

We are now ready to present the key result of the study.

THEOREM 5. Let $p(x) \in \mathcal{P}(\mathbb{R}_{k,+}^n)$ satisfying $1 < p_- \leq p(x) \leq p_+ < \infty$ for $x \in \mathbb{R}_{k,+}^n$ and $\int_{\mathbb{R}_{k,+}^n} \lambda \frac{p(x)q(x)}{q(x)-p(x)} (x')^v dx < \infty$ for $\lambda \in [0, 1)$. Suppose that $\Theta(\xi)$ satisfies (2). Furthermore, let $\omega(x)$, $v(x)$ be weight functions on $\mathbb{R}_{k,+}^n$ satisfying the followings:

$$H' \equiv \sup_{r>0} \left(\int_{|y|<r} [v(y)]^{-p'_+} (y')^v dy \right)^{\frac{\eta}{p_+}} \left\| \frac{\omega(\cdot)}{|\cdot|^Q} \left(\int_{|y|<|\cdot|} [v(y)]^{-p'_+} (y')^v dy \right)^{\frac{1-\eta}{p_+}} \right\|_{L_{p(\cdot), v}(|\cdot|>r)} < \infty, \quad (5)$$

$$\begin{aligned} \mathcal{H}' &\equiv \sup_{r>0} \left(\int_{|y|>r} [v(y)|y|^Q]^{-p'_+} (y')^v dy \right)^{\frac{\zeta}{p_+}} \\ &\quad \times \left\| \omega(\cdot) \left(\int_{|y|>|\cdot|} [v(y)|y|^Q]^{-p'_+} (y')^v dy \right)^{\frac{1-\zeta}{p_+}} \right\|_{L_{p(\cdot), v}(|\cdot|<r)} \\ &< \infty, \end{aligned} \quad (6)$$

there exists $M > 0$ such that

$$\sup_{\frac{|x|}{4} < |y| \leq 4|x|} \omega(y) \leq M \operatorname{ess\,inf}_{x \in \mathbb{R}_{k,+}^n} v(x) \quad (7)$$

holds. Then, we have the followings:

(i) There exists a constant $C > 0$ which is independent of f and ε such that

$$\|T_{v,\varepsilon}f\|_{L_{p(x),\omega,v}(\mathbb{R}_{k,+}^n)} \leq C \|f\|_{L_{p(x),v,v}(\mathbb{R}_{k,+}^n)} \quad (8)$$

holds for all $f \in L_{p(x),v,v}(\mathbb{R}_{k,+}^n)$.

(ii) The limit $\lim_{\varepsilon \rightarrow 0} T_{v,\varepsilon}f$ exists for $f \in L_{p(x),\omega,v}(\mathbb{R}_{k,+}^n)$, which will also be denoted by $T_v f$ and

$$\|T_v f\|_{L_{p(x),\omega,v}(\mathbb{R}_{k,+}^n)} \leq C \|f\|_{L_{p(x),v,v}(\mathbb{R}_{k,+}^n)} \quad (9)$$

holds.

Moreover, condition (7) can be replaced by the following:

$$\omega(x) \left(\sup_{\frac{|x|}{4} \leq |y| \leq 4|x|} \frac{1}{v(y)} \right) \leq M, \quad (10)$$

where $M > 0$ is a constant.

Proof. First define $B_k = \{x \in \mathbb{R}_{k,+}^n : 2^k < |x| \leq 2^{k+1}\}$, $B_{k,1} = \{x \in \mathbb{R}_{k,+}^n : |x| \leq 2^{k-1}\}$, $B_{k,2} = \{x \in \mathbb{R}_{k,+}^n : 2^{k-1} < |x| \leq 2^{k+2}\}$, $B_{k,3} = \{x \in \mathbb{R}_{k,+}^n : |x| > 2^{k-1}\}$ for $k \in \mathbb{Z}$. Then one can easily observe that $B_{k,2} = B_{k-1} \cup B_k \cup B_{k+1}$ and the multiplicity of the covering $\{B_{k,2}\}_{k \in \mathbb{Z}}$ is equal to 3.

For $f \in L_{p(x),v,v}(\mathbb{R}_{k,+}^n)$, one can write

$$\begin{aligned} |T_{v,\varepsilon}f(x)| &= \sum_{k \in \mathbb{Z}} |T_{v,\varepsilon}f(x)| \chi_{B_k}(x) \\ &\leq \sum_{k \in \mathbb{Z}} |T_{v,\varepsilon}f_{k,1}(x)| \chi_{B_k}(x) + \sum_{k \in \mathbb{Z}} |T_{v,\varepsilon}f_{k,2}(x)| \chi_{B_k}(x) + \sum_{k \in \mathbb{Z}} |T_{v,\varepsilon}f_{k,3}(x)| \chi_{B_k}(x) \\ &\equiv T_{v,1,\varepsilon}f(x) + T_{v,2,\varepsilon}f(x) + T_{v,3,\varepsilon}f(x). \end{aligned} \quad (11)$$

Here χ_B is the characteristic function of B and $f \chi_{B_{k,i}} = f_{k,i}$ for $i = 1, 2, 3$.

First, we consider $\|T_{v,1,\varepsilon}f\|_{L_{p(x),v,v}}$. For $x \in B_k$, $y \in B_{k,1}$, we get $|y| \leq 2^{k-1} \leq |x|/2$. Furthermore, $B_k \cap \text{supp } f_{k,1} = \emptyset$ and $|x - y| \geq |x|/2$. By (2), for any $x \in B_k$,

$$\begin{aligned} |T_{v,1,\varepsilon}f(x)| &\leq C \sum_{k \in \mathbb{Z}} \left(\int_{\mathbb{R}_{k,+}^n} T^y |f_{k,1}(x)| |y|^{-Q} (y')^v dy \right) \chi_{B_k} \\ &\leq C \sum_{k \in \mathbb{Z}} \left(\int_{\mathbb{R}_{k,+}^n} T^y |x|^{-Q} |f_{k,1}(y)| (y')^v dy \right) \chi_{B_k} \\ &\leq C \int_{|y| < |x|/2} |x - y|^{-Q} |f(y)| (y')^v dy \\ &\leq C \int_{|y| < |x|} |x - y|^{-Q} |f(y)| (y')^v dy \\ &\leq 2^Q C |x|^{-Q} \int_{|y| < |x|} |f(y)| (y')^v dy, \end{aligned}$$

where $C = \sup_{\xi \in S_{k,+}} |K(\xi)|$ is a constant. Thus, we get

$$\begin{aligned} \int_{\mathbb{R}_{k,+}^n} |T_{V,1,\varepsilon} f(x)|^{p(x)} \omega(x) (x')^v dx \\ \leq (2^Q C)^{p(x)} \int_{\mathbb{R}_{k,+}^n} \left(\int_{|y| < |\cdot|} |f(y)| (y')^v dy \right)^{p(x)} |x|^{-Qp(x)} \omega(x) (x')^v dx. \end{aligned}$$

By Theorem 1 and (5), we get

$$\int_{\mathbb{R}_{k,+}^n} \omega(x) |x|^{-Qp(x)} \left(\int_{|y| < |\cdot|} |f(y)| (y')^v dy \right)^{p(x)} (x')^v dx \leq C \int_{\mathbb{R}_{k,+}^n} |f(x)|^{p(x)} v(x) (x')^v dx.$$

Therefore,

$$\int_{\mathbb{R}_{k,+}^n} |T_{V,1,\varepsilon} f(x)|^{p(x)} \omega(x) (x')^v dx \leq C_1 \int_{\mathbb{R}_{k,+}^n} |f(x)|^{p(x)} v(x) (x')^v dx, \quad (12)$$

where C_1 is a constant which is independent of f and ε .

We now derive an estimate for $\|T_{V,3,\varepsilon} f\|_{L_{p(\cdot),\omega,v}}$. For $x \in B_k$ and $y \in B_{k,3}$, we get $|y| > 2|x|$, and $|x - y| \geq |y|/2$. Moreover, by virtue of $B_k \cap \text{supp } f_{k,3} = \emptyset$ for $x \in B_k$ and by (2), we have

$$\begin{aligned} T_{V,3,\varepsilon} f(x) &\leq C \int_{|y| > 2|x|} T^v |x|^{-Q} |f(y)| (y')^v dy \\ &\leq C \int_{|y| > 2|x|} |f(y)| |x - y|^{-Q} (y')^v dy \\ &\leq 2^Q C \int_{|y| > 2|x|} |f(y)| |y|^{-Q} (y')^v dy. \end{aligned}$$

Then,

$$\begin{aligned} \int_{\mathbb{R}_{k,+}^n} |T_{V,3,\varepsilon} f(x)|^{p(x)} \omega(x) (x')^v dx \\ \leq (2^Q C)^{p(x)} \int_{\mathbb{R}_{k,+}^n} \left(\int_{|y| > 2|x|} |f(y)| |y|^{-Q} (y')^v dy \right)^{p(x)} v(x) (x')^v dx. \end{aligned}$$

By (6), we get

$$\int_{\mathbb{R}_{k,+}^n} \omega(x) \left(\int_{|y| > 2|x|} |f(y)| |y|^{-Q} (y')^v dy \right)^{p(x)} (x')^v dx \leq C \int_{\mathbb{R}_{k,+}^n} |f(x)|^{p(x)} \omega(x) (x')^v dx,$$

where $C > 0$ is a constant. Hence,

$$\int_{\mathbb{R}_{k,+}^n} |T_{V,3,\varepsilon} f(x)|^{p(x)} \omega(x) (x')^v dx \leq C_2 \int_{\mathbb{R}_{k,+}^n} |f(x)|^{p(x)} v(x) (x')^v dx, \quad (13)$$

where C_2 is a constant which is independent of f and ε .

Finally, we obtain an estimate for $\|T_{v,2,\varepsilon}f\|_{L_{p(x),v,v}}$. By virtue of the $L_{p(x),v}$ -boundedness of $T_{v,\varepsilon}$ and by (7), we obtain

$$\begin{aligned} \int_{\mathbb{R}_{k,+}^n} |T_{v,2,\varepsilon}f(x)|^{p(x)} \omega(x)(x')^v dx &= \int_{\mathbb{R}_{k,+}^n} \left(\sum_{k \in \mathbb{Z}} |T_{v,\varepsilon}f_{k,2}(x)| \chi_{B_k}(x) \right)^{p(x)} \omega(x)(x')^v dx \\ &= \int_{\mathbb{R}_{k,+}^n} \left(\sum_{k \in \mathbb{Z}} |T_{v,\varepsilon}f_{k,2}(x)|^{p(x)} \chi_{B_k}(x) \right) \omega(x)(x')^v dx \\ &= \sum_{k \in \mathbb{Z}} \int_{B_k} |T_{v,\varepsilon}f_{k,2}(x)|^{p(x)} \omega(x)(x')^v dx \\ &\leq \sum_{k \in \mathbb{Z}} \sup_{x \in B_k} \omega(x) \int_{\mathbb{R}_{k,+}^n} |T_{v,\varepsilon}f_{k,2}(x)|^{p(x)} (x')^v dx \\ &\leq C_2 \sum_{k \in \mathbb{Z}} \sup_{x \in B_k} \omega(x) \int_{\mathbb{R}_{k,+}^n} |f_{k,2}(x)|^{p(x)} (x')^v dx \\ &\leq C_2 \sum_{k \in \mathbb{Z}} \sup_{y \in B_k} \omega(y) \int_{B_{k,2}} |f(x)|^{p(x)} (x')^v dx. \end{aligned}$$

For $x \in B_{k,2}$ and $2^{k-1} < |x| \leq 2^{k+2}$, by (7),

$$\sup_{y \in B_k} \omega(y) = \sup_{2^{k-1} < |y| \leq 2^{k+2}} \omega(y) \leq \sup_{|x|/4 < |y| \leq 4|x|} \omega(y) \leq M v(x).$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}_{k,+}^n} |T_{v,2,\varepsilon}f(x)|^{p(x)} \omega(x)(x')^v dx &\leq C_2 M \sum_{k \in \mathbb{Z}} \int_{B_{k,2}} |f(x)|^{p(x)} v(x)(x')^v dx \\ &\leq C_3 \int_{\mathbb{R}_{k,+}^n} |f(x)|^{p(x)} v(x)(x')^v dx, \end{aligned} \quad (14)$$

where $C_3 = 3C_2M$, by the virtue of the fact that multiplicity of covering $\{B_{k,2}\}_{k \in \mathbb{Z}}$ is equal to 3.

Inequalities (12), (13) and (14) yield the proof of first part of the theorem.

To prove that $\lim_{\varepsilon \rightarrow 0} T_{v,\varepsilon}f = T_v f$ exists in $L_{p(x),\omega,v}(\mathbb{R}_{k,+}^n)$ for any $f \in L_{p(x),v,v}(\mathbb{R}_{k,+}^n)$, we employ a density argument. Since smooth, compactly supported functions that are even with respect to $x_1, \dots, x_k$ are dense, it suffices to establish the existence of the limit for such functions. To this end, we can represent any function $f \in L_{p(x),v,v}(\mathbb{R}_{k,+}^n)$ as $f = f_1 + f_2$, where f_1 is a smooth, compactly supported function even with respect to $x_1, \dots, x_k$, and f_2 is chosen such that $\|f_2\|_{L_{p(x),v,v}}$ is sufficiently small. By writing $T_{v,\varepsilon}f = T_{v,\varepsilon}f_1 + T_{v,\varepsilon}f_2$ and using (8), we obtain

$$\|T_{v,\varepsilon}f - T_v f\|_{L_{p(x),\omega,v}} \leq C \|f_2\|_{L_{p(x),v,v}} \leq \delta,$$

for a sufficiently small $\delta > 0$. Now, let f be a smooth, compactly supported function that is even with respect to $x_1, \dots, x_k$. For $0 < \varepsilon_1 < \varepsilon_2$, using (2), we get

$$\begin{aligned} T_{v,\varepsilon_2}f(x) - T_{v,\varepsilon_1}f(x) &= \int_{B(0,\varepsilon_2) \setminus B(0,\varepsilon_1)} \frac{\Theta(\xi)}{|y|^Q} [T^y f(x)] (y')^v dy \\ &= \int_{B(0,\varepsilon_2) \setminus B(0,\varepsilon_1)} \frac{\Theta(\xi)}{|y|^Q} [T^y f(x) - f(x)] (y')^v dy, \end{aligned}$$

where $x \in \mathbb{R}_{k,+}^n$. Recalling the Taylor-Delsarte formula [16] for $T^y f$, we have

$$\|T^y f(x) - f(x)\|_{L_{p(x),v}} \leq C|y|.$$

Consequently, it follows that

$$\|T_{v,\varepsilon_2}f - T_{v,\varepsilon_1}f\|_{L_{p(x),\omega,v}} \leq \int_{B(0,\varepsilon_2) \setminus B(0,\varepsilon_1)} \frac{C|y|}{|y|^Q} (y')^v dy \leq C(\varepsilon_2 - \varepsilon_1).$$

This shows that $L_{p(x),\omega,v}(\mathbb{R}_{k,+}^n)$ is complete. Due to the completeness of the space $L_{p(x),\omega,v}(\mathbb{R}_{k,+}^n)$, the limit $\lim_{\varepsilon \rightarrow 0} T_{v,\varepsilon}f = T_v f$ exists in this space. The proof is completed. $\square$

4. Concluding Remarks

One of a special case of B -singular integral operators is Riesz-Bessel transform of high order. Therefore, we can present the following concluding remark for this transform in this section. First, let us recall its definition:

$$\begin{aligned} R_v^{(k)} f(x) &= C_{v,k} \left[p.v. \left(\frac{P_k(y)}{|y|^{k+Q}} \otimes f \right) \right] (x) \\ &= p.v. C_{v,k} \int_{\mathbb{R}_{k,+}^n} \frac{P_k(y)}{|y|^{k+Q}} [T^y f(x)] (y')^v dy \\ &\equiv C_{v,k} \lim_{\varepsilon \rightarrow 0} \int_{\{y \in \mathbb{R}_{k,+}^n : |y| > \varepsilon\}} \frac{P_k(y)}{|y|^{k+Q}} [T^y f(x)] (y')^v dy \\ &\equiv \lim_{\varepsilon \rightarrow 0} R_{v,\varepsilon}^{(k)} f(x), \end{aligned}$$

where $C_{v,k} = 2^{Q/2} \frac{\Gamma(Q/2)}{\Gamma(k/2)}$, $k = 1, 2, \dots, n$. Note that $P_k(y) = P_k(y_1^2, \dots, y_k^2, y_{k+1}, \dots, y_n)$ is a homogeneous polynomial of degree k satisfying $\Delta_B P_k = 0$ and the followings:

$$\int_{S_{k,+}} P_k(\xi) (\xi')^v d\xi = 0 \quad \text{and} \quad \sup_{\xi \in S_+} |P_k(\xi)| = M < \infty. \quad (15)$$

Here the characteristic $P_k(\xi) \in S_{k,+}$ and $\xi = \frac{y}{|y|}$. We denote the kernel of Riesz-Bessel transforms of high order by

$$K(y) = \frac{\Theta_k(y)}{|y|^Q} \quad \text{with} \quad \Theta_k(y) = C_{k,v} \frac{P_k(y)}{|y|^k},$$

where $\Theta_k(y)$ belongs to certain function spaces on the semisphere $S_{k,+} = \{x \in \mathbb{R}_{k,+}^n : |x| = 1\}$. One can easily observe that $|T^\gamma f(x) - f(x)| \leq C|y|$ is valid for Schwarz test functions $f(x)$ with the use of standart technique [6, 8].

Now, we can give the following as a consequence of our main theorem.

COROLLARY 1. *Let $p(x) \in \mathcal{P}(\mathbb{R}_{k,+}^n)$ satisfying $1 < p_- \leq p(x) \leq p_+ < \infty$ and such that $\int_{\mathbb{R}_{k,+}^n} \lambda^{\frac{p(x)q(x)}{q(x)-p(x)}} (x')^\nu dx < \infty$ for $\lambda \in [0, 1)$. Suppose that $P_k(\xi)$ satisfies (15). Furthermore, let $\omega(x)$, $v(x)$ be weight functions on $\mathbb{R}_{k,+}^n$ and satisfying (5) and (6). Suppose also that there exists $M > 0$ such that*

$$\sup_{\frac{|x|}{4} < |y| \leq 4|x|} \omega(y) \leq M \inf_{x \in \mathbb{R}_{k,+}^n} v(x)$$

holds. Then, we have the followings:

(i) *There exists a constant $C > 0$ which is independent of f and ε such that*

$$\left\| R_{v,\varepsilon}^{(k)} f \right\|_{L_{p(x),\omega,\gamma}(\mathbb{R}_{k,+}^n)} \leq C \|f\|_{L_{p(x),v,\gamma}(\mathbb{R}_{k,+}^n)}$$

holds for all $f \in L_{p(x),v,\gamma}(\mathbb{R}_{k,+}^n)$.

(ii) *The limit $\lim_{\varepsilon \rightarrow 0} R_{v,\varepsilon}^{(k)} f = R_v^{(k)} f$ exists for a function $f \in L_{p(x),\omega,\gamma}(\mathbb{R}_{k,+}^n)$ and*

$$\left\| R_v^{(k)} f \right\|_{L_{p(x),\omega,\gamma}(\mathbb{R}_{k,+}^n)} \leq C \|f\|_{L_{p(x),v,\gamma}(\mathbb{R}_{k,+}^n)}.$$

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