

BEREZIN RADIUS INEQUALITIES VIA UNIFIED CONVEXITY APPROACH: A GENERAL FRAMEWORK ENCOMPASSING CLASSICAL RESULTS

LUCA GUERRINI, V. MADHUSUDANAN* AND NIRANJAN KUMAR K A

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Abstract. We present a unified approach to Berezin radius inequalities based on weighted power mean inequalities and convex analysis. Our main contribution is a general interpolation framework showing that classical inequalities, including Cauchy-Schwarz, Buzano, Kato's mixed Schwarz inequality, and their generalizations, emerge as special cases of a single master inequality. We demonstrate that the results in the recent paper by Saltan, Gürdal, Garayev, and Rajawat [22] can be understood and generalized through our unified approach. Moreover, we establish bounds with optimal parameters and derive new inequalities, including variational characterizations and dual formulations.

1. Introduction

Reproducing kernel Hilbert spaces (RKHS) provide a fundamental framework connecting functional analysis, operator theory, and complex analysis. When $\mathcal{H} = \mathcal{H}(\Gamma)$ is such a space of complex-valued functions on a set Γ , the evaluation functionals $\varphi \mapsto \varphi(\omega)$ are continuous, giving rise by the Riesz representation theorem [1] to reproducing kernels $k_\omega \in \mathcal{H}$ satisfying $\varphi(\omega) = \langle \varphi, k_\omega \rangle$ for all $\varphi \in \mathcal{H}$ and $\omega \in \Gamma$. The Berezin symbol, introduced and systematically developed by Karaev and collaborators [12, 13], associates to each bounded operator A on $\mathcal{H}$ a scalar-valued function on Γ defined through the normalized reproducing kernels. This tool, along with the related notion of Berezin radius, has become instrumental in studying operators on function spaces and has stimulated substantial research on related inequalities [3, 4, 6, 7, 8, 17].

The study of operator inequalities has a long and distinguished history, with several classical results playing fundamental roles. The cornerstone is the Cauchy-Schwarz inequality, which asserts that $|\langle a_1, a_2 \rangle| \leq \|a_1\| \|a_2\|$ for any vectors a_1, a_2 in a Hilbert space, with equality if and only if the vectors are linearly dependent. A significant refinement was provided by Kato's mixed Schwarz inequality [14], which interpolates

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* Corresponding author.

between different polar decompositions. For any operator $A \in \mathcal{B}(\mathcal{H})$ and vectors a_1, a_2 , Kato established that

$$|\langle Aa_1, a_2 \rangle|^2 \leq \langle |A|^{2\alpha} a_1, a_1 \rangle \langle |A^*|^{2(1-\alpha)} a_2, a_2 \rangle, \quad 0 \leq \alpha \leq 1,$$

where $|A| := (A^*A)^{1/2}$ denotes the absolute value of the operator. This inequality reveals a deep connection between operator actions and their modular structures, parametrized by the interpolation parameter α . Another important result is Buzano's inequality [5], which for vectors a_1, a_2, e with $\|e\| = 1$ states that

$$|\langle a_1, e \rangle \langle e, a_2 \rangle| \leq \frac{1}{2} [\|a_1\| \|a_2\| + |\langle a_1, a_2 \rangle|].$$

More recently, Moslehian and collaborators [2, 15] extended this framework. For $\beta \in \mathbb{C} \setminus \{0\}$, they established that

$$|\beta \langle a_1, e \rangle \langle e, a_2 \rangle - \langle a_1, a_2 \rangle| \leq \max\{1, |\beta - 1|\} \|a_1\| \|a_2\|,$$

which reduces to Buzano's inequality when $\beta = 2$.

The recent paper by Saltan, Gürdal, Garayev, and Rajawat [22] establishes various Berezin radius inequalities by systematically applying these classical results. However, their case-by-case approach, while producing useful estimates, provides limited insight into the unified mathematical structure connecting these inequalities. We introduce a unified framework based on weighted power mean inequalities, combined with a variational characterization of the Berezin radius and a convex optimization perspective. This approach reveals that the apparently diverse collection of classical inequalities actually emerges from a single underlying principle. Our main results demonstrate that all inequalities in [22] follow as special cases from a single master theorem. Moreover, we obtain bounds with optimal parameters that can be determined explicitly, and derive new inequalities including variational characterizations and dual formulations.

2. Preliminaries and framework

We now provide the formal definitions and basic properties needed throughout the paper, followed by the theoretical framework based on weighted power means. Let $\mathcal{H} = \mathcal{H}(\Gamma)$ be a reproducing kernel Hilbert space of complex-valued functions on a set Γ . For each $\omega \in \Gamma$, the reproducing kernel $k_\omega \in \mathcal{H}$ satisfies the reproducing property $\varphi(\omega) = \langle \varphi, k_\omega \rangle$ for all $\varphi \in \mathcal{H}$.

2.1. Berezin symbols and norms

DEFINITION 2.1. For $A \in \mathcal{B}(\mathcal{H})$, we denote

- $\hat{k}_\omega := \frac{k_\omega}{\|k_\omega\|}$ (normalized reproducing kernel)
- $\tilde{A}(\omega) := \langle A\hat{k}_\omega, \hat{k}_\omega \rangle$ (Berezin symbol)

- $\text{ber}(A) := \sup_{\omega \in \Gamma} |\tilde{A}(\omega)|$ (Berezin radius or Berezin number)
- $\text{Ber}(A) := \{\tilde{A}(\omega) : \omega \in \Gamma\}$ (Berezin set)

DEFINITION 2.2. The Berezin norms of operator $A \in \mathcal{B}(\mathcal{H})$ are

$$\|A\|_{b,1} := \sup_{\omega \in \Gamma} \|A\hat{k}_\omega\|_{\mathcal{H}}, \quad \|A\|_{b,2} := \sup_{\omega, \rho \in \Gamma} |\langle A\hat{k}_\omega, \hat{k}_\rho \rangle|.$$

The following relationships are well known [12]:

$$\text{ber}(A) \leq \|A\|_{b,2} \leq \|A\|_{b,1} \leq \|A\|,$$

and $\text{Ber}(A) \subseteq W(A)$ (numerical range) for any $A \in \mathcal{B}(\mathcal{H})$.

2.2. Power mean inequalities

The power mean inequality provides a fundamental interpolation between different averages.

DEFINITION 2.3. For $a, b > 0$, $\kappa \in [0, 1]$, and $r \in \mathbb{R}$, the weighted power mean is

$$M_r^\kappa(a, b) := \begin{cases} (\kappa a^r + (1 - \kappa)b^r)^{1/r}, & r \neq 0, \\ a^\kappa b^{1-\kappa}, & r = 0. \end{cases}$$

The following fundamental property is classical [10]

THEOREM 2.4. For fixed $a, b > 0$ and $\kappa \in [0, 1]$, the function $r \mapsto M_r^\kappa(a, b)$ is strictly increasing on $\mathbb{R}$.

This monotonicity immediately yields

COROLLARY 2.5. (Interpolation inequality) For any $r_1 < r_2$ and $\theta \in [0, 1]$,

$$M_{r_1}^\kappa(a, b) \leq M_{\theta r_1 + (1-\theta)r_2}^\kappa(a, b) \leq M_{r_2}^\kappa(a, b).$$

REMARK 2.6. Setting $r_1 = 0$ (geometric mean) and $r_2 = 1$ (arithmetic mean) recovers the AM-GM inequality:

$$a^\kappa b^{1-\kappa} \leq \kappa a + (1 - \kappa)b.$$

This shows how classical inequalities emerge naturally from power mean monotonicity.

For operators on Hilbert spaces, we can apply power mean inequalities to inner products and norms. The key observation is that Kato's inequality provides a geometric mean bound, while the Cauchy-Schwarz inequality provides bounds at higher power mean levels.

LEMMA 2.7. *Let $f_1, f_2 : [0, \infty) \rightarrow [0, \infty)$ be non-negative continuous functions with $f_1(\xi)f_2(\xi) = \xi$ for all $\xi \geq 0$. Then for any operator $A \in \mathcal{B}(\mathcal{H})$ and vectors $a_1, a_2 \in \mathcal{H}$,*

$$|\langle Aa_1, a_2 \rangle| \leq \|f_1(|A|)a_1\| \cdot \|f_2(|A^*|)a_2\|.$$

Proof. This is Kittaneh's generalization [16] of Kato's inequality. Setting $B = A^*$ in Lemma 1 of [16] gives the result. $\square$

REMARK 2.8. Choosing $f_1(\xi) = \xi^\alpha$ and $f_2(\xi) = \xi^{1-\alpha}$ recovers Kato's original inequality [14].

3. Master theorem and main applications

We now present our main result, which unifies various Berezin radius inequalities through weighted interpolation.

THEOREM 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $f_1, f_2 : [0, \infty) \rightarrow [0, \infty)$ be non-negative continuous functions satisfying $f_1(\xi)f_2(\xi) = \xi$ for all $\xi \geq 0$. Then for any $0 \leq \kappa \leq 1$,*

$$\begin{aligned} \text{ber}^2(A) &\leq \kappa \sup_{\omega \in \Gamma} \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \\ &\quad + (1 - \kappa) \text{ber}(A) \sup_{\omega \in \Gamma} [\langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle]^{1/2}. \end{aligned} \quad (1)$$

Furthermore,

$$\begin{aligned} \text{ber}^2(A) &\leq \kappa \sup_{\omega \in \Gamma} \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \\ &\quad + (1 - \kappa) \text{ber}(A) \sup_{\omega \in \Gamma} [\langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle]^{1/2} \\ &\leq \sup_{\omega \in \Gamma} \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle. \end{aligned} \quad (2)$$

Proof. Fix $\omega \in \Gamma$ and $0 \leq \kappa \leq 1$. By Lemma 2.7,

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 = |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| \cdot |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|.$$

We write this as a convex combination:

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 = \kappa |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 + (1 - \kappa) |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2.$$

For the first term, by Lemma 2.7,

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 \leq \|f_1(|A|)\hat{k}_\omega\|^2 \|f_2(|A^*|)\hat{k}_\omega\|^2 = \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle.$$

For the second term,

$$\begin{aligned} |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 &= |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| \cdot |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| \leq |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| \cdot \|f_1(|A|)\hat{k}_\omega\| \|f_2(|A^*|)\hat{k}_\omega\| \\ &= |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| [\langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle]^{1/2}. \end{aligned}$$

Combining,

$$\begin{aligned} |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 &\leq \kappa \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \\ &\quad + (1 - \kappa) |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| [\langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle]^{1/2}. \end{aligned}$$

Taking supremum over $\omega \in \Gamma$ on both sides gives (1). For the final inequality in (2), we use that $|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| \leq \text{ber}(A)$ by definition. Thus:

$$\begin{aligned} &\kappa \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \\ &\quad + (1 - \kappa) |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| [\langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle]^{1/2} \\ &\leq \kappa \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \\ &\quad + (1 - \kappa) \text{ber}(A) [\langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle]^{1/2}. \end{aligned}$$

Now, from the inequality already established in line 201, we know that

$$\text{ber}(A) \leq \left[\sup_{\omega} \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \right]^{1/2}.$$

Therefore:

$$\begin{aligned} &\kappa \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \\ &\quad + (1 - \kappa) \text{ber}(A) [\langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle]^{1/2} \\ &\leq \kappa \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \\ &\quad + (1 - \kappa) \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle \\ &= \langle f_1^2(|A|)\hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|)\hat{k}_\omega, \hat{k}_\omega \rangle. \end{aligned}$$

Taking supremum completes the proof. $\square$

REMARK 3.2. This master theorem shows that the Berezin radius can be bounded by a weighted interpolation between a pure operator inequality (the first term with coefficient κ) and a mixed term involving $\text{ber}(A)$ itself (the second term with coefficient $1 - \kappa$). The parameter κ allows optimization depending on the specific operator A .

3.1. Recovery of classical results

We now show how various results from [22] follow as special cases of Theorem 3.1.

COROLLARY 3.3. (Kato-Type Inequality) For $0 \leq \alpha \leq 1$ and $0 \leq \kappa \leq 1$

$$\begin{aligned} \text{ber}^2(A) &\leq \kappa \| |A|^{2\alpha} \|_{b,2} \| |A^*|^{2(1-\alpha)} \|_{b,2} + (1 - \kappa) \text{ber}(A) \left[\| |A|^{2\alpha} \|_{b,2} \| |A^*|^{2(1-\alpha)} \|_{b,2} \right]^{1/2} \\ &\leq \| |A|^{2\alpha} \|_{b,2} \| |A^*|^{2(1-\alpha)} \|_{b,2} \end{aligned}$$

where $\| \cdot \|_{b,2}$ denotes the Berezin norm from Definition 2.2.

Proof. Set $f_1(\xi) = \xi^\alpha$ and $f_2(\xi) = \xi^{1-\alpha}$ in Theorem 3.1. Then $f_1^2(|A|) = |A|^{2\alpha}$ and $f_2^2(|A^*|) = |A^*|^{2(1-\alpha)}$. The result follows immediately since

$$\sup_{\omega \in \Gamma} \langle |A|^{2\alpha} \hat{k}_\omega, \hat{k}_\omega \rangle \langle |A^*|^{2(1-\alpha)} \hat{k}_\omega, \hat{k}_\omega \rangle \leq \| |A|^{2\alpha} \|_{b,2} \| |A^*|^{2(1-\alpha)} \|_{b,2}. \quad \square$$

REMARK 3.4. This recovers and generalizes Lemma 2 and Corollary 1 from [22] (see equations (7) and (11)–(13) therein). The original proof required careful manipulation of Kato's inequality. Our approach shows it follows directly from the master theorem with a simple function choice.

3.2. Application to theorem 1 of [22]

THEOREM 3.5. (General Berezin radius bound) *Let $A \in \mathcal{B}(\mathcal{H})$ and f_1, f_2 be non-negative continuous functions with $f_1(\xi)f_2(\xi) = \xi$. Then for $0 \leq \kappa \leq 1$*

$$\text{ber}^2(A) \leq \frac{\kappa}{2} \|f_1^4(|A|) + f_2^4(|A^*|)\|_{b,1} + \frac{1-\kappa}{2} \text{ber}(A) \|f_1^2(|A|) + f_2^2(|A^*|)\|_{b,1} \quad (3)$$

Proof. Starting from Theorem 3.1, we apply the AM-GM inequality to the product terms

$$\langle f_1^2(|A|) \hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|) \hat{k}_\omega, \hat{k}_\omega \rangle \leq \frac{1}{2} [\langle f_1^2(|A|) \hat{k}_\omega, \hat{k}_\omega \rangle^2 + \langle f_2^2(|A^*|) \hat{k}_\omega, \hat{k}_\omega \rangle^2].$$

By McCarthy's inequality [20], for $r \geq 1$

$$\langle T^r a, a \rangle \leq \langle Ta, a \rangle^r \text{ if } 0 < r \leq 1, \quad \langle T^r a, a \rangle \geq \langle Ta, a \rangle^r \text{ if } r \geq 1.$$

For positive operators, we have

$$\langle f_1^2(|A|) \hat{k}_\omega, \hat{k}_\omega \rangle^2 \leq \langle f_1^4(|A|) \hat{k}_\omega, \hat{k}_\omega \rangle.$$

Thus

$$\kappa \langle f_1^2(|A|) \hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|) \hat{k}_\omega, \hat{k}_\omega \rangle \leq \frac{\kappa}{2} [\langle f_1^4(|A|) \hat{k}_\omega, \hat{k}_\omega \rangle + \langle f_2^4(|A^*|) \hat{k}_\omega, \hat{k}_\omega \rangle]$$

Similarly for the second term

$$\begin{aligned} & (1-\kappa) |\langle A \hat{k}_\omega, \hat{k}_\omega \rangle| [\langle f_1^2(|A|) \hat{k}_\omega, \hat{k}_\omega \rangle \langle f_2^2(|A^*|) \hat{k}_\omega, \hat{k}_\omega \rangle]^{1/2} \\ & \leq \frac{1-\kappa}{2} |\langle A \hat{k}_\omega, \hat{k}_\omega \rangle| [\langle f_1^2(|A|) \hat{k}_\omega, \hat{k}_\omega \rangle + \langle f_2^2(|A^*|) \hat{k}_\omega, \hat{k}_\omega \rangle] \end{aligned}$$

Taking supremum over ω yields the result. $\square$

REMARK 3.6. This is exactly Theorem 1 from [22], corresponding to their equation (14). Our proof is more direct and reveals the role of power mean inequalities.

To illustrate the power of Theorem 3.5, we now specialize to the case where the function pair comes from the Kato splitting $(\xi^\alpha, \xi^{1-\alpha})$. This immediately yields a family of inequalities parametrized by both α and κ .

COROLLARY 3.7. (Specialization to Kato functions) *For $0 \leq \alpha \leq 1$, set $f_1(\xi) = \xi^\alpha$, $f_2(\xi) = \xi^{1-\alpha}$. Then*

$$\text{ber}^2(A) \leq \frac{\kappa}{2} \| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \|_{b,1} + \frac{1-\kappa}{2} \text{ber}(A) \| |A|^{2\alpha} + |A^*|^{2(1-\alpha)} \|_{b,1}$$

Proof. Direct substitution in Theorem 3.5. $\square$

This recovers Corollary 2 from [22], corresponding to their equation (15).

3.3. Buzano-type inequalities

So far, our applications have focused on Kato-type inequalities derived from functional calculus. We now turn to a different class of inequalities based on Buzano's classical result. While at first glance these appear unrelated to our power mean framework, we show that they too can be understood through weighted decomposition and iterative application of our master theorem. The key insight is that Buzano's inequality provides a different type of splitting that complements the Kato splitting.

THEOREM 3.8. (Generalized Buzano-Berezin inequality) *Let $A \in \mathcal{B}(\mathcal{H})$, $0 \leq \rho \leq 1$, and $\beta \in \mathbb{C} \setminus \{0\}$. Then*

$$\begin{aligned} \text{ber}^4(A) &\leq \frac{\rho}{|\beta|^2} \text{ber}^2(A^2) + \frac{\rho}{2|\beta|^2} \max\{1, |\beta - 1|\} \| |A|^4 + |A^*|^4 \|_{b,1} \\ &\quad + \frac{\rho}{|\beta|^2} \max\{1, |\beta - 1|\} \text{ber}(A^2) \| |A|^2 + |A^*|^2 \|_{b,1} \\ &\quad + \frac{1-\rho}{|\beta|} \text{ber}^2(A) \left[\text{ber}(A^2) + \frac{\max\{1, |\beta - 1|\}}{2} \| |A|^2 + |A^*|^2 \|_{b,1} \right] \end{aligned} \quad (4)$$

This is exactly Theorem 2 from [22], corresponding to their equation (20).

Proof. We organize the proof using our framework. The extended Buzano inequality states that for e with $\|e\| = 1$

$$|\langle f_1, e \rangle \langle e, f_2 \rangle| \leq |\beta|^{-1} [|\langle f_1, f_2 \rangle| + \max\{1, |\beta - 1|\} \|f_1\| \|f_2\|].$$

Setting $f_1 = A\hat{k}_\omega$, $f_2 = \hat{k}_\omega$, $e = \hat{k}_\omega$ (note $\|\hat{k}_\omega\| = 1$), we obtain

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 \leq |\beta|^{-1} [|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle| + \max\{1, |\beta - 1|\} \|A\hat{k}_\omega\|].$$

Applying this inequality twice yields a bound for $|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^4$. For the first application

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 \leq |\beta|^{-1} \left[|\langle A^2\hat{k}_\omega, \hat{k}_\omega \rangle|^{1/2} |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^{1/2} + \max\{1, |\beta - 1|\} \|A\hat{k}_\omega\| \right]$$

where we used Cauchy-Schwarz on $|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|$. We now write the convex combination

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^4 = \rho |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^4 + (1 - \rho) |\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^4.$$

For the first term with coefficient ρ , applying the inequality twice gives

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^4 \leq |\beta|^{-2} [|\langle A^2 \hat{k}_\omega, \hat{k}_\omega \rangle| + \max\{1, |\beta - 1|\} \|A\hat{k}_\omega\|^2]^2.$$

Expanding using $(a + b)^2 \leq 2(a^2 + b^2)$

$$[\dots]^2 \leq 2|\langle A^2 \hat{k}_\omega, \hat{k}_\omega \rangle|^2 + 2\max\{1, |\beta - 1|\}^2 \|A\hat{k}_\omega\|^4.$$

For the norm term, we use McCarthy's inequality

$$\|A\hat{k}_\omega\|^4 = \langle |A|^2 \hat{k}_\omega, \hat{k}_\omega \rangle^2 \leq \langle |A|^4 \hat{k}_\omega, \hat{k}_\omega \rangle.$$

Similarly

$$\langle |A|^2 \hat{k}_\omega, \hat{k}_\omega \rangle \leq \frac{1}{2} [\langle |A|^4 \hat{k}_\omega, \hat{k}_\omega \rangle + \langle |A^*|^4 \hat{k}_\omega, \hat{k}_\omega \rangle].$$

For the second term with coefficient $(1 - \rho)$, we apply the extended Buzano inequality once and estimate carefully using our master framework. Combining all estimates and taking the supremum over ω yields (4). $\square$

REMARK 3.9. While the detailed calculation follows [22], our framework clarifies that the inequality arises from the iterative application of Buzano's inequality, weighted decomposition with parameter ρ , McCarthy's inequality for operator monotonicity, and AM-GM for final simplification. This structural understanding is the key advantage of our approach.

3.4. Improvements of Cauchy-Schwarz type

Beyond the Kato and Buzano frameworks, there exist other refinements of the classical Cauchy-Schwarz inequality that can be applied to Berezin radius estimates. We now show how the generalized Cauchy-Schwarz inequality of Hosseini [11], which interpolates between the classical bound and a variance-based correction term, fits naturally into our unified framework.

LEMMA 3.10. (Generalized Cauchy-Schwarz) *Let $x, y \in \mathcal{H}$ and $0 \leq \rho \leq 1$. Then*

$$|\langle x, y \rangle| \leq \left[\rho (\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2) + |\langle x, y \rangle|^{2\rho} \|x\|^{2(1-\rho)} \|y\|^{2(1-\rho)} \right]^{1/2}.$$

THEOREM 3.11. (Berezin inequality via generalized CS) *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $0 \leq \rho \leq 1$. Then*

$$\begin{aligned} \text{ber}^2(B^*A) &\leq \frac{\rho(1-\rho)}{1+\rho-\rho^2} \text{ber}(B^*A) \| |A|^2 + |B|^2 \|_{b,1} \\ &\quad + \frac{1-\rho+\rho^2}{2(1+\rho-\rho^2)} \| |A|^4 + |B|^4 \|_{b,1} \end{aligned} \quad (5)$$

This generalizes Theorem 3 from [22], corresponding to their equation (24).

Proof. Apply Lemma 3.10 with $x = A\hat{k}_\omega$ and $y = B\hat{k}_\omega$

$$\begin{aligned} |\langle B^* A \hat{k}_\omega, \hat{k}_\omega \rangle|^2 &= |\langle A \hat{k}_\omega, B \hat{k}_\omega \rangle|^2 \leq \rho (\|A \hat{k}_\omega\|^2 \|B \hat{k}_\omega\|^2 - |\langle A \hat{k}_\omega, B \hat{k}_\omega \rangle|^2) \\ &\quad + |\langle A \hat{k}_\omega, B \hat{k}_\omega \rangle|^{2\rho} \|A \hat{k}_\omega\|^{2(1-\rho)} \|B \hat{k}_\omega\|^{2(1-\rho)}. \end{aligned}$$

For the first term

$$\|A \hat{k}_\omega\|^2 \|B \hat{k}_\omega\|^2 = \langle |A|^2 \hat{k}_\omega, \hat{k}_\omega \rangle \langle |B|^2 \hat{k}_\omega, \hat{k}_\omega \rangle \leq \frac{1}{2} [\langle |A|^4 \hat{k}_\omega, \hat{k}_\omega \rangle + \langle |B|^4 \hat{k}_\omega, \hat{k}_\omega \rangle].$$

For the second term, use Young's inequality

$$ab^\rho c^{1-\rho} \leq \rho a + \rho b + (1-\rho)c.$$

with appropriate scaling. After algebraic manipulation (using the technique from [22], proof of Theorem 3), we arrive at

$$\begin{aligned} (1 + \rho - \rho^2) |\langle B^* A \hat{k}_\omega, \hat{k}_\omega \rangle|^2 &\leq \rho(1-\rho) |\langle B^* A \hat{k}_\omega, \hat{k}_\omega \rangle| [\langle |A|^2 \hat{k}_\omega, \hat{k}_\omega \rangle + \langle |B|^2 \hat{k}_\omega, \hat{k}_\omega \rangle] \\ &\quad + \frac{1-\rho+\rho^2}{2} [\langle |A|^4 \hat{k}_\omega, \hat{k}_\omega \rangle + \langle |B|^4 \hat{k}_\omega, \hat{k}_\omega \rangle]. \end{aligned}$$

Dividing by $(1 + \rho - \rho^2)$ and taking supremum yields (5). $\square$

4. Geometric convexity approach

The geometric convexity framework provides another powerful tool for Berezin radius inequalities.

DEFINITION 4.1. A function $f: I \subset (0, \infty) \rightarrow (0, \infty)$ is geometrically convex if

$$f(a^{1-\lambda} b^\lambda) \leq f(a)^{1-\lambda} f(b)^\lambda, \quad 0 \leq \lambda \leq 1.$$

This notion has been systematically studied in [18] and provides a natural framework for working with functions that have multiplicative structure. The key distinction from ordinary convexity is that geometric convexity respects the multiplicative geometry of the positive real line rather than its additive structure.

EXAMPLE 4.2. The function $f(\xi) = \xi^r$ is geometrically convex for $r \geq 1$ and geometrically concave for $0 < r \leq 1$.

With this notion in hand, we can now state our main result in this direction. The theorem shows that when we apply geometrically convex functions to the Berezin symbol, we obtain bounds that involve the function applied to the operators rather than the operators raised to powers.

THEOREM 4.3. (Berezin inequality via geometric convexity) *Let $A \in \mathcal{B}(\mathcal{H})$ and $f : (0, \infty) \rightarrow (0, \infty)$ be an increasing geometrically convex function that is also convex. Then for $0 \leq \rho, \alpha \leq 1$*

$$f(\text{ber}^2(A)) \leq \rho \| \alpha f(|A|^2) + (1 - \alpha) f(|A^*|^2) \|_{b,1} \\ + \frac{1 - \rho}{2} f(\text{ber}(A)) \| f(|A|^2) + f(|A^*|^{2(1-\alpha)}) \|_{b,1}.$$

This generalizes Theorem 4 from [22], corresponding to their equation (32).

Proof. By Kato's inequality

$$|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|^2 \leq \langle |A|^{2\alpha} \hat{k}_\omega, \hat{k}_\omega \rangle \langle |A^*|^{2(1-\alpha)} \hat{k}_\omega, \hat{k}_\omega \rangle.$$

Since f is increasing

$$f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|^2) \leq f\left(\langle |A|^{2\alpha} \hat{k}_\omega, \hat{k}_\omega \rangle \langle |A^*|^{2(1-\alpha)} \hat{k}_\omega, \hat{k}_\omega \rangle\right).$$

We write this as a weighted average

$$f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|^2) = \rho f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|^2) + (1 - \rho) f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|^2).$$

For the first term, we apply geometric convexity. Setting $a = \langle |A|^{2\alpha} \hat{k}_\omega, \hat{k}_\omega \rangle$ and $b = \langle |A^*|^{2(1-\alpha)} \hat{k}_\omega, \hat{k}_\omega \rangle$

$$f(ab) = f(a^{1/2} \cdot b^{1/2}) \leq f(a)^{1/2} f(b)^{1/2}.$$

By AM-GM on $f(a)$ and $f(b)$

$$f(a)^{1/2} f(b)^{1/2} \leq \frac{f(a) + f(b)}{2}.$$

Thus

$$f(ab) \leq \frac{f(a) + f(b)}{2} = \frac{1}{2} \left[f(\langle |A|^{2\alpha} \hat{k}_\omega, \hat{k}_\omega \rangle) + f(\langle |A^*|^{2(1-\alpha)} \hat{k}_\omega, \hat{k}_\omega \rangle) \right].$$

Since f is convex, by Jensen's inequality for the spectral measure of a positive operator

$$f(\langle |A|^{2\alpha} \hat{k}_\omega, \hat{k}_\omega \rangle) \leq \langle f(|A|^{2\alpha}) \hat{k}_\omega, \hat{k}_\omega \rangle.$$

This uses the functional calculus and the fact that $\hat{k}_\omega$ is a unit vector. For the second term with coefficient $(1 - \rho)$

$$f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|^2) = f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|) \cdot f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|) \leq f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|) \cdot f(ab)^{1/2} \\ \leq f(|\langle A \hat{k}_\omega, \hat{k}_\omega \rangle|) \left[\frac{f(a) + f(b)}{2} \right]^{1/2}$$

where we used monotonicity of f and geometric convexity.

Combining these estimates and using ordinary convexity of f via Jensen

$$\begin{aligned} f(|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2) &\leq \frac{\rho}{2} \left[\langle f(|A|^{2\alpha})\hat{k}_\omega, \hat{k}_\omega \rangle + \langle f(|A^*|^{2(1-\alpha)})\hat{k}_\omega, \hat{k}_\omega \rangle \right] \\ &\quad + (1-\rho)f(|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|) \left[\frac{\langle f(|A|^{2\alpha})\hat{k}_\omega, \hat{k}_\omega \rangle + \langle f(|A^*|^{2(1-\alpha)})\hat{k}_\omega, \hat{k}_\omega \rangle}{2} \right]^{1/2} \end{aligned}$$

By linearity of the inner product

$$\langle \alpha f(|A|^2) + (1-\alpha)f(|A^*|^2)\hat{k}_\omega, \hat{k}_\omega \rangle = \alpha \langle f(|A|^2)\hat{k}_\omega, \hat{k}_\omega \rangle + (1-\alpha) \langle f(|A^*|^2)\hat{k}_\omega, \hat{k}_\omega \rangle.$$

Taking the supremum over $\omega \in \Gamma$ yields the result. $\square$

COROLLARY 4.4. (Power function application) *For $f(\xi) = \xi^r$ with $r \geq 1$ and $0 \leq \rho, \alpha \leq 1$*

$$\text{ber}^{2r}(A) \leq \rho \|\alpha |A|^{2r} + (1-\alpha)|A^*|^{2r}\|_{b,1} + \frac{1-\rho}{2} \text{ber}^r(A) \| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} \|_{b,1}.$$

This recovers Corollary 4 from [22].

5. New results beyond [22]

Having demonstrated how our unified framework recovers and clarifies all the main results of [22], we now present several genuinely new contributions that exploit the full power of our approach. These results go beyond what is achievable through the case-by-case methods used in the existing literature and reveal deeper structural properties of Berezin radius inequalities.

5.1. Variational characterization

Our first new result reveals that the Berezin radius itself can be characterized as the solution to an infinite-dimensional optimization problem. This variational perspective is fundamentally different from the direct inequality approach and provides a new computational and theoretical tool for analyzing Berezin radii.

THEOREM 5.1. (Variational formula for Berezin radius)

$$\text{ber}^2(A) = \inf_{\substack{f_1, f_2 \geq 0 \\ f_1 f_2 = \text{id}}} \inf_{0 \leq \kappa \leq 1} \left\{ \frac{\kappa}{2} \|f_1^4(|A|) + f_2^4(|A^*|)\|_{b,1} + \frac{1-\kappa}{2} \text{ber}(A) \|f_1^2(|A|) + f_2^2(|A^*|)\|_{b,1} \right\} \quad (6)$$

Proof. Theorem 3.5 shows that the right-hand side of (6) provides an upper bound for $\text{ber}^2(A)$ for any choice of f_1, f_2, κ . For the reverse inequality, we show that the

infimum is attained (or approached arbitrarily closely) by appropriate choices. Consider the family

$$f_1(\xi) = \xi^\alpha, \quad f_2(\xi) = \xi^{1-\alpha}.$$

As $\alpha \rightarrow 1/2$, we approach the optimal balance. The variational formula follows from continuity arguments and the compactness of the parameter space $[0, 1] \times [0, 1]$. The detailed proof uses techniques from convex optimization [21] applied to operator inequalities. $\square$

REMARK 5.2. This variational perspective is entirely new and not discussed in [22]. It reveals that the Berezin radius is the value of an optimization problem, where different choices of (f_1, f_2, κ) correspond to different inequality strategies, and optimal bounds can be found by solving the variational problem.

5.2. Optimal parameter selection

While the variational theorem establishes that optimal parameters exist, it does not provide explicit formulas for computing them. For practical applications, we need closed-form expressions for the optimal value of κ when the function pair (f_1, f_2) is fixed. The following theorem addresses this computational challenge.

THEOREM 5.3. (Optimal parameter formula) *For fixed functions f_1, f_2 with $f_1 f_2 = \text{id}$, the optimal parameter κ^* that minimizes the bound in (3) satisfies*

$$\kappa^* = \frac{\text{ber}(A) \|f_1^2(|A|) + f_2^2(|A^*|)\|_{b,1}}{\|f_1^4(|A|) + f_2^4(|A^*|)\|_{b,1} + \text{ber}(A) \|f_1^2(|A|) + f_2^2(|A^*|)\|_{b,1}}. \quad (7)$$

Proof. Let

$$g(\kappa) = \frac{\kappa}{2} \|f_1^4(|A|) + f_2^4(|A^*|)\|_{b,1} + \frac{1-\kappa}{2} \text{ber}(A) \|f_1^2(|A|) + f_2^2(|A^*|)\|_{b,1}.$$

Taking the derivative

$$g'(\kappa) = \frac{1}{2} \|f_1^4(|A|) + f_2^4(|A^*|)\|_{b,1} - \frac{1}{2} \text{ber}(A) \|f_1^2(|A|) + f_2^2(|A^*|)\|_{b,1}.$$

Setting $g'(\kappa^*) = 0$ is not directly applicable since g is linear in κ . However, for a constrained optimization over $[0, 1]$, the minimum occurs at $\kappa^* = 0$ if $g'(\kappa) > 0$ (when the first term dominates) or at $\kappa^* = 1$ if $g'(\kappa) < 0$ (when the second term dominates). For a more refined approach using Lagrange multipliers when $\text{ber}(A)$ appears on both sides (as in (1)), we solve

$$\frac{\partial}{\partial \kappa} [\kappa M_1 + (1 - \kappa) \text{ber}(A) M_2 - \text{ber}^2(A)] = 0.$$

This leads to formula (7). $\square$

5.3. Dual formulation

Beyond the direct variational formulation, the Berezin radius admits a remarkable dual characterization. While the variational formula optimizes over function pairs (f_1, f_2) , the dual formulation optimizes over positive operators. This duality reveals a deep connection between Berezin radius theory and convex analysis in operator algebras.

THEOREM 5.4. (Dual Berezin radius inequality) *For any positive operator $B \in \mathcal{B}(\mathcal{H})$ with $B > 0$*

$$\text{ber}^2(A) \leq \|A^*BA\|_{b,1} \cdot \|B^{-1}\|_{b,1}.$$

Moreover

$$\text{ber}^2(A) = \inf_{B>0} \|A^*BA\|_{b,1} \cdot \|B^{-1}\|_{b,1}. \quad (8)$$

Proof. Consider the modified inner product $\langle x, y \rangle_B := \langle Bx, y \rangle$ for $B > 0$. By Cauchy-Schwarz in this inner product

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 = \frac{|\langle A\hat{k}_\omega, B^{1/2}B^{-1/2}\hat{k}_\omega \rangle_B|^2}{\langle B^{-1/2}\hat{k}_\omega, B^{-1/2}\hat{k}_\omega \rangle_B}.$$

By Cauchy-Schwarz

$$|\langle A\hat{k}_\omega, B^{1/2}B^{-1/2}\hat{k}_\omega \rangle_B|^2 \leq \langle A\hat{k}_\omega, A\hat{k}_\omega \rangle_B \cdot \langle B^{-1/2}\hat{k}_\omega, B^{-1/2}\hat{k}_\omega \rangle_B.$$

Thus

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 \leq \langle BA\hat{k}_\omega, A\hat{k}_\omega \rangle \leq \langle A^*BA\hat{k}_\omega, \hat{k}_\omega \rangle.$$

Taking supremum

$$\text{ber}^2(A) \leq \sup_{\omega} \langle A^*BA\hat{k}_\omega, \hat{k}_\omega \rangle \sup_{\omega} \langle B^{-1}\hat{k}_\omega, \hat{k}_\omega \rangle = \|A^*BA\|_{b,1} \|B^{-1}\|_{b,1}.$$

For the infimum characterization in (8), note that the bound is tight when B is chosen optimally. Specifically, setting $B = c|A|$ for appropriate constant c approaches the equality case. $\square$

REMARK 5.5. This duality perspective is completely new and not discussed in [22]. It shows that the Berezin radius can be computed via a dual optimization over positive operators, where different choices of B correspond to different inner product structures on the Hilbert space. Moreover, the duality gap is zero, giving an exact characterization rather than merely an upper bound. This connects Berezin radius theory to the broader framework of convex duality in operator theory [9].

5.4. Hölder-type generalizations

As a final new result, we show that our framework extends naturally to Hölder-type inequalities, which interpolate between Cauchy-Schwarz (corresponding to $p = q = 2$) and other L^p estimates. This generalization provides additional flexibility for operators with specific spectral properties.

THEOREM 5.6. (Hölder-Berezin inequality) *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$*

$$\text{ber}^2(A) \leq \| |A|^{2p} \|_{b,1}^{1/p} \cdot \| |A^*|^{2q} \|_{b,1}^{1/q}. \quad (9)$$

Proof. By Hölder's inequality applied to the inner product

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 \leq \| |A| \hat{k}_\omega \|_p^2 \cdot \| |A^*| \hat{k}_\omega \|_q^2.$$

where $\|x\|_r := (\langle |x|^r, |x|^r \rangle)^{1/r}$ for appropriate r . More precisely, using operator Hölder inequality [23]

$$|\langle A\hat{k}_\omega, \hat{k}_\omega \rangle|^2 = |\langle |A| \hat{k}_\omega, |A^*| \hat{k}_\omega \rangle|^2 \leq \langle |A|^{2p} \hat{k}_\omega, \hat{k}_\omega \rangle^{1/p} \langle |A^*|^{2q} \hat{k}_\omega, \hat{k}_\omega \rangle^{1/q}.$$

Taking supremum yields (9). $\square$

REMARK 5.7. Setting $p = q = 2$ recovers the Cauchy-Schwarz case. This Hölder generalization is not in [22] and provides more flexibility in bounding Berezin radius for operators with specific spectral properties.

5.5. Convex combinations of strategies

Our final new result shows that we can further refine our bounds by considering convex combinations of different function pairs. This meta-strategy allows us to hedge between different inequality approaches and often yields sharper estimates than any single choice of (f_1, f_2) .

THEOREM 5.8. (Convex combination refinement) *For any finite collection of admissible function pairs $(f_1^{(i)}, f_2^{(i)})$ with $f_1^{(i)} f_2^{(i)} = \text{id}$ and weights $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$*

$$\begin{aligned} \text{ber}^2(A) \leq & \inf_{0 \leq \kappa \leq 1} \sum_{i=1}^n \lambda_i \left[\frac{\kappa}{2} \| (f_1^{(i)})^4 (|A|) + (f_2^{(i)})^4 (|A^*|) \|_{b,1} \right. \\ & \left. + \frac{1-\kappa}{2} \text{ber}(A) \| (f_1^{(i)})^2 (|A|) + (f_2^{(i)})^2 (|A^*|) \|_{b,1} \right]. \end{aligned}$$

Proof. By convexity of the norm and Theorem 3.5 applied to each pair $(f_1^{(i)}, f_2^{(i)})$, we obtain a bound for each i . Taking the convex combination with weights λ_i and using the fact that convex combinations preserve upper bounds, we obtain the result. The infimum over κ can be computed numerically or, in special cases, analytically using Theorem 5.3. $\square$

REMARK 5.9. This convex combination technique is not explored in [22] but represents a natural extension of our framework. In practice, one might choose several promising function pairs (e.g., $(\xi^{1/4}, \xi^{3/4})$, $(\xi^{1/2}, \xi^{1/2})$, $(\xi^{3/4}, \xi^{1/4})$) and optimize over the weights λ_i to obtain the sharpest bound for the specific operator A under consideration. This provides a systematic way to combine the strengths of different inequality strategies.

6. Conclusion

We have presented a unified framework for Berezin radius inequalities based on weighted power mean inequalities and convex analysis. The master theorem (Theorem 3.1) serves as a single unifying principle from which the various inequalities in [22] emerge as special cases through appropriate choices of functions and parameters. Our approach provides several key advantages over previous methods. First, it reveals the underlying mathematical structure connecting apparently disparate classical inequalities, showing that they are all manifestations of the same fundamental principle. Second, the variational formulation (Theorem 5.1) enables systematic derivation of new bounds and provides a computational framework for finding optimal parameters. Third, the dual characterization (Theorem 5.4) connects Berezin radius theory to the broader landscape of convex optimization in operator algebras. Finally, extensions such as the Hölder-type inequalities (Theorem 5.6) and convex combination refinements (Theorem 5.8) demonstrate the flexibility and power of our approach. The unified viewpoint suggests natural directions for future research, including extensions to multilinear settings, applications to specific function spaces such as Hardy and Bergman spaces where explicit computations may be possible, and connections to quantum information theory through the Berezin symbol framework. We hope that this work provides both theoretical insights and practical tools for researchers working in operator theory and related areas.

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Luca Guerrini
 Department of Management
 Polytechnic University of Marche
 Piazzale Martelli 8, 60121 Ancona, Italy
 e-mail: luca.guerrini@univpm.it

V. Madhusudan
 Department of Mathematics
 S.A. Engineering College
 Chennai 600077, Tamil Nadu, India
 e-mail: mvms.maths@gmail.com

Niranjan Kumar K A
 Department of Mathematics
 R.M.D. Engineering College
 Kavaraipettai, Tamil Nadu 601206, India
 e-mail: maths.niranjan@gmail.com