

MORE GENERALIZATIONS ON SINGULAR VALUE AND NORM INEQUALITIES OF MATRICES INVOLVING SUBMULTIPLICATIVE FUNCTIONS

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Abstract. In this note, we obtain several singular value and unitarily invariant norm inequalities which generalize Al-Natoor et al.'s results (2025) in [1] and inspired by statements due to Audenh and Kittaneh (2012) in [2], we present the equivalent relations of some known inequalities involving nonnegative and increasing submultiplicative functions.

1. Introduction

Let $\mathbb{C}$ be the set of all complex numbers. The symbol $\mathbb{M}_n(\mathbb{C})$ denotes the space of all complex matrices of size $n \times n$ with its zero matrix 0 and its identity matrix I . For two Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we write $A \geq B$ (or $B \leq A$) to mean that $A - B$ is positive semidefinite. For $A \in \mathbb{M}_n(\mathbb{C})$, A^* stands for the conjugate transpose of A and the absolute value of A is denoted by $|A| = (A^*A)^{\frac{1}{2}}$. For $A \in \mathbb{M}_n(\mathbb{C})$, the singular values of A are denoted by $s_j(A)$, $j = 1, \dots, n$, and they are so arranged such that $s_1(A) \geq \dots \geq s_n(A)$. Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n(\mathbb{C})$ is unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in \mathbb{M}_n(\mathbb{C})$ and any unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. In the following, $\|\cdot\|$ always means a unitarily invariant norm. The operator norms, a special class of unitarily invariant norms, are defined as $\|A\|_\infty = s_1(A)$. A function $f: [0, \infty) \rightarrow \mathbb{R}$ is said to be submultiplicative if $f(xy) \leq f(x)f(y)$ whenever $x, y \in [0, \infty)$.

For $A, B \in \mathbb{M}_n(\mathbb{C})$, the 2×2 block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is denoted by $A \oplus B$ (the direct sum of A and B). It is easy to know that

$$s_j \left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \right) = s_j(A \oplus A^*) = s_j(A \oplus A), \quad j = 1, 2, \dots, n.$$

and $\|A \oplus B\|_\infty = \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_\infty = \max\{\|A\|_\infty, \|B\|_\infty\}.$

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Kittaneh [9] proved a norm inequality: If $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$\|A + B\| \leq \|(A + |B^{\frac{1}{2}}A^{\frac{1}{2}}|) \oplus (B + |A^{\frac{1}{2}}B^{\frac{1}{2}}|)\|. \quad (1)$$

Kittaneh [10] studied the following singular value inequality: If $A, B, X \in \mathbb{M}_n(\mathbb{C})$ with positive semidefinite matrices A and B , then for $j = 1, 2, \dots, n$,

$$s_j(AX - XB) \leq \|X\|_{\infty} s_j(A \oplus B). \quad (2)$$

When $j = 1$ in inequality (2), the result [11] follows that

$$\|AX - XB\|_{\infty} \leq \max\{\|A\|_{\infty}, \|B\|_{\infty}\} \|X\|_{\infty}. \quad (3)$$

As a singular value version of inequality analogue of (3), Kittaneh [12] established the following result: If $A, B, X \in \mathbb{M}_n(\mathbb{C})$ with positive semidefinite matrices A and B , then for $j = 1, 2, \dots, n$,

$$s_j(AX - XB) \leq \max\{\|A\|_{\infty}, \|B\|_{\infty}\} s_j(X \oplus X). \quad (4)$$

Recently, Al-Natoor et al. [1, Theorem 2.2] gave a generalization of inequality (1) by showing that if $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ with positive semidefinite matrices A and B , then for $0 \leq p, q \leq 1$,

$$\|AX - YB\| \leq \frac{1}{2} \left\| \begin{pmatrix} (A^{2p} + A^{1-p}|X^*|^2A^{1-p}) + |B^qY^*A^p - B^{1-q}X^*A^{1-p}| \\ (B^q|Y|^2B^q + B^{2(1-q)} + |A^pYB^q - A^{1-p}XB^{1-q}|) \end{pmatrix} \oplus \right\|. \quad (5)$$

At the same time, they also presented another generalization [1, Corollary 2.3] of inequality (3): if $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ with positive semidefinite matrices A and B , then for $j = 1, \dots, n$

$$\begin{aligned} & s_j(AX - YB) \\ & \leq \max\{\|A\|_{\infty}, \|B\|_{\infty}\} \left(\frac{s_j((X - Y) \oplus (X - Y))}{2} + \max\{\|X\|_{\infty}, \|Y\|_{\infty}\} \right). \end{aligned} \quad (6)$$

By making slight modifications to the arguments used in the proof of inequality (6), Al-Natoor et al. [1, Remark 2.4] got the following two singular value inequalities for $j = 1, \dots, n$:

$$s_j(AX - YB) \leq \max\{\|A\|_{\infty}, \|B\|_{\infty}\} \left(s_j(X \oplus Y) + \frac{\|X - Y\|_{\infty}}{2} \right), \quad (7)$$

$$s_j(AX - YB) \leq \left(\max\{\|X\|_{\infty}, \|Y\|_{\infty}\} + \frac{\|X - Y\|_{\infty}}{2} \right) s_j(A \oplus B). \quad (8)$$

Actually, inequalities (7) and (8) have been obtained in [7].

Audeh and Kittaneh [2, Theorem 2.6] presented equivalent relations of the following several statements:

(a) For $A, B \in \mathbb{M}(\mathbb{C})$, the inequality [3]

$$2s_j(A^*B) \leq s_j(AA^* + BB^*) \quad (9)$$

holds for $j = 1, 2, \dots, n$.

(b) If $A, B \in \mathbb{M}(\mathbb{C})$ are positive semidefinite matrices, then the inequality [14]

$$s_j(A - B) \leq s_j(A \oplus B) \quad (10)$$

holds for $j = 1, 2, \dots, n$.

(c) For $A, B, C \in \mathbb{M}(\mathbb{C})$, if $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geq 0$, then the inequality [13]

$$2s_j(C) \leq s_j\left(\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}\right) \quad (11)$$

holds for $j = 1, 2, \dots, n$.

(d) For Hermitian matrices $A, B \in \mathbb{M}(\mathbb{C})$, where $B \geq 0$ and $\pm A \leq B$, the inequality [2]

$$2s_j(A) \leq s_j((B - A) \oplus (B + A)) \quad (12)$$

holds for $j = 1, 2, \dots, n$.

On the other hand, Al-Natoor et al. [1, Theorem 3.3] obtained a generalization of inequality (9) by showing the following: Let $A, B, P, Q \in \mathbb{M}(\mathbb{C})$ be such that $\max\{\|P\|_\infty, \|Q\|_\infty\} \leq \sqrt{2}$ and let $f : [0, \infty) \rightarrow \mathbb{R}$ be a nonnegative, monotonically increasing convex function satisfying $g(0) = 0$. Then, for $j = 1, 2, \dots, n$, the following inequality holds:

$$2s_j(f(|PA^*BQ^*|)) \leq \max\{\|P\|_\infty^2, \|Q\|_\infty^2\} s_j(f(AA^* + BB^*)). \quad (13)$$

In this paper, we first present several singular value and unitarily invariant norm inequalities for matrices which are generalizations of (1)–(8). And then we give the equivalent relations of inequalities (9)–(12) involving nonnegative and increasing submultiplicative function. Meanwhile, an analogue of inequality (13) is obtained under the condition of submultiplicative functions.

2. Singular value and unitarily invariant norm inequalities for commutators

In this section, we present several generalizations of inequalities (1)–(8). Before proving our main results, we begin with reviewing some known lemmas.

LEMMA 2.1. ([6]) Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be normal. Then,

$$\|A + B\| \leq \| |A| + |B| \|.$$

LEMMA 2.2. ([4, p. 75]) Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then for $j = 1, 2, \dots, n$,

$$s_j(AB) \leq s_j(A)\|B\|_\infty.$$

LEMMA 2.3. ([4, p. 75]) Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then for $j = 1, 2, \dots, n$,

$$s_j(A+B) \leq s_j(A) + \|B\|_\infty.$$

The next result follows from the spectral theorem for Hermitian matrices [8, p. 229].

THEOREM 2.4. Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$, where A and B are positive semidefinite matrices. Then, for $0 \leq p, q \leq 1$, we have

$$\|ZAX - YBW\| \leq \frac{1}{2} \left\| \begin{pmatrix} (A^p|Z|^2A^p + A^{1-p}|X^*|^2A^{1-p}) + |B^qY^*ZA^p - B^{1-q}WX^*A^{1-p}| \\ (B^q|Y|^2B^q + B^{1-q}|W^*|^2B^{1-q} + |A^pZ^*YB^q - A^{1-p}XW^*B^{1-q}|) \end{pmatrix} \oplus \right\|.$$

Proof. Let $E = [ZA^p YB^q]$ and $F = [X^*A^{1-p} - W^*B^{1-q}]$, then by (9) we get

$$\begin{aligned} s_j(ZAX - YBW) &= s_j(EF^*) \\ &\leq \frac{1}{2} s_j(E^*E + F^*F) \\ &= \frac{1}{2} s_j \left(\begin{bmatrix} A^pZ^* \\ B^qY^* \end{bmatrix} [ZA^p YB^q] + \begin{bmatrix} A^{1-p}X \\ -B^{1-q}W \end{bmatrix} [X^*A^{1-p} - W^*B^{1-q}] \right) \\ &= \frac{1}{2} s_j \left(\begin{bmatrix} A^p|Z|^2A^p & A^pZ^*YB^q \\ B^qY^*ZA^p & B^q|Y|^2B^q \end{bmatrix} + \begin{bmatrix} A^{1-p}|X^*|^2A^{1-p} & -A^{1-p}XW^*B^{1-q} \\ -B^{1-q}WX^*A^{1-p} & B^{1-q}|W^*|^2B^{1-q} \end{bmatrix} \right) \\ &= \frac{1}{2} s_j \left(\begin{bmatrix} A^p|Z|^2A^p + A^{1-p}|X^*|^2A^{1-p} & A^pZ^*YB^q - A^{1-p}XW^*B^{1-q} \\ B^qY^*ZA^p - B^{1-q}WX^*A^{1-p} & B^q|Y|^2B^q + B^{1-q}|W^*|^2B^{1-q} \end{bmatrix} \right). \end{aligned} \quad (14)$$

This means that

$$\begin{aligned} \|ZAX - YBW\| &\leq \frac{1}{2} \left\| \begin{bmatrix} A^p|Z|^2A^p + A^{1-p}|X^*|^2A^{1-p} & A^pZ^*YB^q - A^{1-p}XW^*B^{1-q} \\ B^qY^*ZA^p - B^{1-q}WX^*A^{1-p} & B^q|Y|^2B^q + B^{1-q}|W^*|^2B^{1-q} \end{bmatrix} \right\| \\ &= \frac{1}{2} \left\| \begin{bmatrix} A^p|Z|^2A^p + A^{1-p}|X^*|^2A^{1-p} & 0 \\ 0 & B^q|Y|^2B^q + B^{1-q}|W^*|^2B^{1-q} \end{bmatrix} + \begin{bmatrix} 0 & A^pZ^*YB^q - A^{1-p}XW^*B^{1-q} \\ B^qY^*ZA^p - B^{1-q}WX^*A^{1-p} & 0 \end{bmatrix} \right\| \\ &\leq \frac{1}{2} \left\| \begin{bmatrix} A^p|Z|^2A^p + A^{1-p}|X^*|^2A^{1-p} & 0 \\ 0 & B^q|Y|^2B^q + B^{1-q}|W^*|^2B^{1-q} \end{bmatrix} + \begin{bmatrix} 0 & A^pZ^*YB^q - A^{1-p}XW^*B^{1-q} \\ B^qY^*ZA^p - B^{1-q}WX^*A^{1-p} & 0 \end{bmatrix} \right\| \\ &= \frac{1}{2} \left\| \begin{bmatrix} A^p|Z|^2A^p + A^{1-p}|X^*|^2A^{1-p} & 0 \\ 0 & B^q|Y|^2B^q + B^{1-q}|W^*|^2B^{1-q} \end{bmatrix} + \begin{bmatrix} |B^qY^*ZA^p - B^{1-q}WX^*A^{1-p}| & 0 \\ 0 & |A^pZ^*YB^q - A^{1-p}XW^*B^{1-q}| \end{bmatrix} \right\| \\ &= \frac{1}{2} \left\| \begin{pmatrix} (A^p|Z|^2A^p + A^{1-p}|X^*|^2A^{1-p}) + |B^qY^*ZA^p - B^{1-q}WX^*A^{1-p}| \\ (B^q|Y|^2B^q + B^{1-q}|W^*|^2B^{1-q}) + |A^pZ^*YB^q - A^{1-p}XW^*B^{1-q}| \end{pmatrix} \oplus \right\|, \end{aligned}$$

in which the second inequality is by Lemma 2.1. $\square$

REMARK 1. When $Z = W = I$ in Theorem 2.4, our result is inequality (5). Thus, Theorem 2.4 is a generalization of inequality (5).

Replacing W with $-W$ yields the following corollary.

COROLLARY 2.5. Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$, where A and B are positive semidefinite matrices. Then, for $0 \leq p, q \leq 1$, we have

$$\|ZAX + YBW\| \leq \frac{1}{2} \left\| \begin{pmatrix} (A^p|Z|^2A^p + A^{1-p}|X^*|^2A^{1-p}) + |B^qY^*ZA^p + B^{1-q}WX^*A^{1-p}| \\ (B^q|Y|^2B^q + B^{1-q}|W^*|^2B^{1-q}) + |A^pZ^*YB^q + A^{1-p}XW^*B^{1-q}| \end{pmatrix} \oplus \right\|.$$

REMARK 2. If $Z = X = Y = W = I$ and $p = q = \frac{1}{2}$ in Corollary 2.5, then our result is inequality (1).

Next we will present the generalizations of inequalities (3) and (6).

THEOREM 2.6. Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$, where A and B are positive semidefinite matrices, then, for $j = 1, 2, \dots, n$, we have

$$\begin{aligned} s_j(ZAX - YBW) &\leq \max\{\|A\|_\infty, \|B\|_\infty\} \\ &\times \left(\frac{s_j((Z^*Y - XW^*) \oplus (Z^*Y - XW^*))}{2} + \max\{\|X\|_\infty, \|Y\|_\infty\} \max\{\|Z\|_\infty, \|W\|_\infty\} \right). \end{aligned} \quad (15)$$

Proof. Let $p = q = \frac{1}{2}$ in inequality (14). Then

$$\begin{aligned} s_j(ZAX - YBW) &\leq \frac{1}{2} s_j \left(\begin{bmatrix} A^{\frac{1}{2}}|Z|^2A^{\frac{1}{2}} + A^{\frac{1}{2}}|X^*|^2A^{\frac{1}{2}} & A^{\frac{1}{2}}Z^*YB^{\frac{1}{2}} - A^{\frac{1}{2}}XW^*B^{\frac{1}{2}} \\ B^{\frac{1}{2}}Y^*ZA^{\frac{1}{2}} - B^{\frac{1}{2}}WX^*A^{\frac{1}{2}} & B^{\frac{1}{2}}|Y|^2B^{\frac{1}{2}} + B^{\frac{1}{2}}|W^*|^2B^{\frac{1}{2}} \end{bmatrix} \right) \\ &= \frac{1}{2} s_j \left(\begin{bmatrix} A^{\frac{1}{2}}(|Z|^2 + |X^*|^2)A^{\frac{1}{2}} & A^{\frac{1}{2}}(Z^*Y - XW^*)B^{\frac{1}{2}} \\ B^{\frac{1}{2}}(Y^*Z - WX^*)A^{\frac{1}{2}} & B^{\frac{1}{2}}(|Y|^2 + |W^*|^2)B^{\frac{1}{2}} \end{bmatrix} \right) \\ &= \frac{1}{2} s_j \left(\begin{bmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} |Z|^2 + |X^*|^2 & Z^*Y - XW^* \\ Y^*Z - WX^* & |Y|^2 + |W^*|^2 \end{bmatrix} \begin{bmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \right) \\ &\leq \frac{1}{2} \left\| \begin{bmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \right\|_\infty^2 s_j \left(\begin{bmatrix} |Z|^2 + |X^*|^2 & Z^*Y - XW^* \\ Y^*Z - WX^* & |Y|^2 + |W^*|^2 \end{bmatrix} \right) \quad (\text{by Lemma 2.2}) \quad (16) \\ &\leq \frac{1}{2} \max\{\|A\|_\infty, \|B\|_\infty\} \\ &\times \left(\left\| \begin{bmatrix} |Z|^2 + |X^*|^2 & 0 \\ 0 & |Y|^2 + |W^*|^2 \end{bmatrix} \right\|_\infty + s_j \left(\begin{bmatrix} 0 & Z^*Y - XW^* \\ Y^*Z - WX^* & 0 \end{bmatrix} \right) \right) \\ &\quad (\text{by Lemma 2.3}) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \max\{\|A\|_\infty, \|B\|_\infty\} \\
&\quad \times (\max\{\|Z\|_\infty^2, \|W\|_\infty^2\} + \max\{\|X\|_\infty^2, \|Y\|_\infty^2\} + s_j((Z^*Y - XW^*) \oplus (Z^*Y - XW^*))) \\
&\quad \text{(by the triangle inequality).}
\end{aligned}$$

For $t > 0$, we replace X, Y with tX, tY , respectively. Then

$$\begin{aligned}
&s_j(ZAX - YBW) \\
&\leq \inf_{t>0} \max\{\|A\|_\infty, \|B\|_\infty\} \\
&\quad \times \frac{\max\{\|Z\|_\infty^2, \|W\|_\infty^2\} + ts_j((Z^*Y - XW^*) \oplus (Z^*Y - XW^*)) + t^2 \max\{\|X\|_\infty^2, \|Y\|_\infty^2\}}{2t}.
\end{aligned} \tag{17}$$

Suppose the function $f(t) = \frac{\alpha^2 t^2 + \beta t + \gamma}{2t}$, where $\alpha > 0$, $\beta, \gamma \geq 0$. It attains its minimum $\alpha\sqrt{\gamma} + \frac{\beta}{2}$ at $t = \sqrt{\gamma}/\alpha$. By inequality (17), we know that $\alpha = \max\{\|X\|_\infty, \|Y\|_\infty\}$, $\beta = s_j((Z^*Y - XW^*) \oplus (Z^*Y - XW^*))$ and $\gamma = \max\{\|Z\|_\infty^2, \|W\|_\infty^2\}$. So,

$$\begin{aligned}
&s_j(ZAX - YBW) \\
&\leq \max\{\|A\|_\infty, \|B\|_\infty\} \\
&\quad \times \left(\frac{s_j((Z^*Y - XW^*) \oplus (Z^*Y - XW^*))}{2} + \max\{\|Z\|_\infty, \|W\|_\infty\} \max\{\|X\|_\infty, \|Y\|_\infty\} \right).
\end{aligned}$$

□

REMARK 3. Apparently, applying $Z = I$ and $W = I$ on inequality (15) yields (6) and so inequality (3).

The following two corollaries are complements of inequality (15).

COROLLARY 2.7. Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$, where A and B are positive semidefinite matrices, then, for $j = 1, 2, \dots, n$, we have

$$\begin{aligned}
&s_j(ZAX - YBW) \\
&\leq \max\{\|A\|_\infty, \|B\|_\infty\} \left(\max\{\|Z\|_\infty, \|W\|_\infty\} s_j(X \oplus Y) + \frac{\|Z^*Y - XW^*\|_\infty}{2} \right).
\end{aligned}$$

Proof. By inequality (16), we have

$$\begin{aligned}
&s_j(ZAX - YBW) \\
&\leq \frac{1}{2} \left\| \begin{bmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \right\|_\infty^2 s_j \left(\begin{bmatrix} |Z|^2 + |X^*|^2 & Z^*Y - XW^* \\ Y^*Z - WX^* & |Y|^2 + |W^*|^2 \end{bmatrix} \right)
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2} \max\{\|A\|_\infty, \|B\|_\infty\} \\
 &\quad \times \left(\left\| \begin{bmatrix} 0 & Z^*Y - XW^* \\ Y^*Z - WX^* & 0 \end{bmatrix} \right\|_\infty + s_j \left(\begin{bmatrix} |Z|^2 + |X^*|^2 & 0 \\ 0 & |Y|^2 + |W^*|^2 \end{bmatrix} \right) \right) \\
 &\quad \text{(by Lemma 2.3)} \\
 &\leq \frac{1}{2} \max\{\|A\|_\infty, \|B\|_\infty\} \times (\max\{\|Z\|_\infty^2, \|W\|_\infty^2\} + \|Z^*Y - XW^*\|_\infty + s_j^2(X \oplus Y)).
 \end{aligned}$$

Following the steps of the proof in Theorem 2.6 leads to the desired inequality:

$$\begin{aligned}
 &s_j(ZAX - YBW) \\
 &\leq \max\{\|A\|_\infty, \|B\|_\infty\} \left(\max\{\|Z\|_\infty, \|W\|_\infty\} s_j(X \oplus Y) + \frac{\|Z^*Y - XW^*\|_\infty}{2} \right). \quad \square
 \end{aligned}$$

REMARK 4. If $Z = W = I$, then Corollary 2.7 reduces to inequality (7) and so inequality (4). Thus, Corollary 2.7 is a generalization of inequalities (4) and (7).

COROLLARY 2.8. Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$, where A and B are positive semidefinite matrices, then, for $j = 1, 2, \dots, n$, we have

$$\begin{aligned}
 &s_j(ZAX - YBW) \\
 &\leq \left(\max\{\|Z\|_\infty, \|W\|_\infty\} \max\{\|X\|_\infty, \|Y\|_\infty\} + \frac{\|Z^*Y - XW^*\|_\infty}{2} \right) s_j(A \oplus B). \quad (18)
 \end{aligned}$$

Proof. By the triangle inequality, we have

$$\begin{aligned}
 &\left\| \begin{bmatrix} |Z|^2 + |X^*|^2 & Z^*Y - XW^* \\ Y^*Z - WX^* & |Y|^2 + |W^*|^2 \end{bmatrix} \right\|_\infty \\
 &\leq \left\| \begin{bmatrix} |Z|^2 + |X^*|^2 & 0 \\ 0 & |Y|^2 + |W^*|^2 \end{bmatrix} \right\|_\infty + \left\| \begin{bmatrix} 0 & Z^*Y - XW^* \\ Y^*Z - WX^* & 0 \end{bmatrix} \right\|_\infty \quad (19) \\
 &\leq \max\{\|Z\|_\infty^2, \|W\|_\infty^2\} + \max\{\|X\|_\infty^2, \|Y\|_\infty^2\} + \|Z^*Y - XW^*\|_\infty.
 \end{aligned}$$

Combining inequalities (16), (19) with Lemma 2.2, we have

$$\begin{aligned}
 &s_j(ZAX - YBW) \\
 &\leq \frac{1}{2} s_j \left(\begin{bmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} |Z|^2 + |X^*|^2 & Z^*Y - XW^* \\ Y^*Z - WX^* & |Y|^2 + |W^*|^2 \end{bmatrix} \begin{bmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \right) \\
 &\leq \frac{1}{2} \left\| \begin{bmatrix} |Z|^2 + |X^*|^2 & Z^*Y - XW^* \\ Y^*Z - WX^* & |Y|^2 + |W^*|^2 \end{bmatrix} \right\|_\infty s_j \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) \quad (20) \\
 &\leq \frac{1}{2} (\max\{\|Z\|_\infty^2, \|W\|_\infty^2\} + \max\{\|X\|_\infty^2, \|Y\|_\infty^2\} + \|Z^*Y - XW^*\|_\infty) s_j(A \oplus B).
 \end{aligned}$$

The desired result follows by applying the steps of the proof in Theorem 2.6 on the above inequality (20). $\square$

REMARK 5. When $Z = W = I$ in Corollary 2.8, inequality (18) becomes (8) and so (2).

3. Singular value inequalities for matrices involving submultiplicative functions

Inspired by the equivalence of inequalities (9), (10), (11) and (12), we present some generalizations of these four inequalities involving increasing submultiplicative functions in this section. Before giving the main results, we list some lemmas for the convenience of our proofs.

LEMMA 3.1. ([4, p. 5]) *Let $A \in \mathbb{M}_n(\mathbb{C})$ and f be a nonnegative increasing function on $[0, \infty)$. Then for $k = 1, 2, \dots, n$,*

$$f(s_k(A)) = s_k(f(|A|)).$$

LEMMA 3.2. ([15, p. 49]) *Let $A \in \mathbb{M}_n(\mathbb{C})$ have singular values $s_1, \dots, s_n$. Then the matrix*

$$\varphi(A) = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$$

has eigenvalues $s_1, \dots, s_n, -s_n, \dots, -s_1$.

The next theorem is an analogue of inequality (13) under the condition of the increasing submultiplicative functions which is a generalization of inequality (9).

THEOREM 3.3. *Let $A, B, P, Q \in \mathbb{M}_{m \times n}(\mathbb{C})$ and let $f: [0, \infty) \rightarrow \mathbb{R}$ be a nonnegative and increasing submultiplicative function satisfying $f(0) = 0$. Then, for $j = 1, 2, \dots, n$, the following holds:*

$$s_j(f(|PA^*BQ^*|)) \leq f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) s_j(f(AA^* + BB^*)). \quad (21)$$

Proof. For $j = 1, 2, \dots, n$, we have

$$\begin{aligned} s_j(f(|PA^*BQ^*|)) &= f(s_j(PA^*BQ^*)) \quad (\text{by Lemma 3.1}) \\ &\leq f\left(s_j\left(\frac{AP^*PA^* + BQ^*QB^*}{2}\right)\right) \quad (\text{by inequality (9)}) \\ &= f\left(s_j\left(\frac{1}{2} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P^*P & 0 \\ 0 & Q^*Q \end{bmatrix} \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}\right)\right) \\ &\leq f\left(\frac{1}{2} \left\| \begin{bmatrix} P^*P & 0 \\ 0 & Q^*Q \end{bmatrix} \right\|_\infty s_j\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}\right)\right) \\ &\quad (\text{by Lemma 2.2}) \\ &\leq f\left(\frac{1}{2} \left\| \begin{bmatrix} P^*P & 0 \\ 0 & Q^*Q \end{bmatrix} \right\|_\infty\right) f\left(s_j\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}\right)\right) \\ &\quad (\text{since } f \text{ is submultiplicative}) \\ &= f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) f(s_j(AA^* + BB^*)) \\ &= f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) s_j(f(AA^* + BB^*)) \\ &\quad (\text{by Lemma 3.1}). \quad \square \end{aligned}$$

As another generalization of inequality (9), we give a corollary of Theorem 3.3.

COROLLARY 3.4. *Let $A, B, P, Q \in \mathbb{M}_{m \times n}(\mathbb{C})$ and let $f : [0, \infty) \rightarrow \mathbb{R}$ be a non-negative and increasing submultiplicative function satisfying $f(0) = 0$. Then, for $j = 1, 2, \dots, n$, we have*

$$s_j(f(|PA^*BQ^*|)) \leq f(\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}) s_j\left(f\left(\frac{AA^* + BB^*}{2}\right)\right). \quad (22)$$

Proof. (22) can be proved in exactly the same way as in Theorem 3.3. $\square$

In the next theorem we give the equivalence of several inequalities involving increasing submultiplicative functions.

THEOREM 3.5. *Let $P, Q \in \mathbb{M}_n(\mathbb{C})$ and let $f : [0, \infty) \rightarrow \mathbb{R}$ be a nonnegative and increasing submultiplicative function satisfying $f(0) = 0$. Then, for $j = 1, 2, \dots, n$, the following statements are equivalent:*

(a) *For $A, B \in \mathbb{M}_n(\mathbb{C})$,*

$$s_j(f(|PA^*BQ^*|)) \leq f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) s_j(f(AA^* + BB^*)).$$

(b) *For $A, B, C \in \mathbb{M}_n(\mathbb{C})$ with $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geq 0$, the following inequality holds:*

$$s_j(f(|PCQ^*|)) \leq f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) s_j\left(f\left(\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}\right)\right).$$

(c) *For $A, B \in \mathbb{M}_n(\mathbb{C})$ with Hermitian matrix A , positive semidefinite matrix B and $\pm A \leq B$, the following inequality holds:*

$$s_j(f(|PAQ^*|)) \leq f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) s_j(f(B - A) \oplus f(B + A)).$$

(d) *For positive semidefinite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, then*

$$s_j(f(|P(A - B)Q^*|)) \leq f(\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}) s_j(f(A) \oplus f(B)).$$

Proof. (a) $\Rightarrow$ (b) Since

$$\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geq 0,$$

then there exist matrices $X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$ such that

$$\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} = \begin{bmatrix} X & Z \\ Y & W \end{bmatrix}^* \begin{bmatrix} X & Z \\ Y & W \end{bmatrix}.$$

Suppose the $2n \times n$ matrices $N = \begin{bmatrix} X \\ Y \end{bmatrix}$ and $M = \begin{bmatrix} Z \\ W \end{bmatrix}$. So,

$$\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} = [N \ M]^* [N \ M] = \begin{bmatrix} N^*N & N^*M \\ M^*N & M^*M \end{bmatrix},$$

which means that $C = N^*M$. By $s_j(E^*E) = s_j(EE^*)$, we have

$$\begin{aligned} s_j \left(\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \right) &= s_j ([N \ M]^* [N \ M]) \\ &= s_j ([N \ M] [N \ M]^*) \\ &= s_j (NN^* + MM^*). \end{aligned} \tag{23}$$

Combining (23) and (a) for matrices N and M yields

$$\begin{aligned} s_j(f(|PCQ^*|)) &= s_j(f(|PN^*MQ^*|)) \\ &\leq f \left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2} \right) s_j(f(NN^* + MM^*)) \\ &= f \left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2} \right) s_j \left(f \left(\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \right) \right). \end{aligned}$$

(b) $\Rightarrow$ (a) Note that

$$\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} = [A \ B]^* [A \ B],$$

and

$$[A \ B] [A \ B]^* = AA^* + BB^*.$$

Therefore,

$$T = \begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} \geq 0.$$

By the property $s_j(X^*X) = s_j(XX^*)$, we have

$$s_j \left(\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} \right) = s_j(AA^* + BB^*). \tag{24}$$

Thus, (a) follows from (b) for the matrix T and (24).

(b) $\Rightarrow$ (c) Let

$$X = \begin{bmatrix} B & A \\ A & B \end{bmatrix}$$

and

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}.$$

Then

$$U^*XU = (B - A) \oplus (B + A) \geq 0. \quad (25)$$

By (25), $X = \begin{bmatrix} B & A \\ A & B \end{bmatrix}$ is positive semidefinite. So using (b), we have

$$\begin{aligned} s_j(f(|PAQ^*|)) &\leq f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) s_j(f(X)) \\ &= f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) f(s_j(X)) \quad (\text{by Lemma 3.1}) \\ &= f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) f(s_j((B - A) \oplus (B + A))) \quad (\text{by (25)}) \\ &= f\left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2}\right) s_j(f((B - A) \oplus (B + A))) \quad (\text{by Lemma 3.1}). \end{aligned}$$

(c) $\Rightarrow$ (b) Let

$$X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geq 0$$

and

$$U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Then

$$X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} = U^* \begin{bmatrix} A & -C \\ -C^* & B \end{bmatrix} U \geq 0.$$

Meanwhile,

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \frac{X + UXU^*}{2}$$

and

$$\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} = \frac{X - UXU^*}{2}.$$

Since $X \geq 0$, then $X + UXU^* \geq 0$. Moreover, $X - UXU^*$ is Hermitian and so

$$\pm \frac{X - UXU^*}{2} \leq \frac{X + UXU^*}{2}.$$

Consider

$$S = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}.$$

By (c), we obtain

$$\begin{aligned} &s_j\left(f\left(\left|S\left(\frac{X - UXU^*}{2}\right)S^*\right|\right)\right) \\ &\leq f\left(\frac{\|S\|_\infty^2}{2}\right) s_j\left(f\left(\frac{X + UXU^*}{2} - \frac{X - UXU^*}{2}\right) \oplus f\left(\frac{X + UXU^*}{2} + \frac{X - UXU^*}{2}\right)\right). \end{aligned}$$

Hence,

$$s_j \left(f \left(\left\| \begin{bmatrix} 0 & PCQ^* \\ QC^*P^* & 0 \end{bmatrix} \right\| \right) \right) \leq f \left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2} \right) s_j(f(X \oplus UXU^*)).$$

Using Lemma 3.1, we have

$$f \left(s_j \left(\left\| \begin{bmatrix} 0 & PCQ^* \\ QC^*P^* & 0 \end{bmatrix} \right\| \right) \right) \leq f \left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2} \right) f(s_j(X \oplus UXU^*)).$$

So applying Lemma 3.2 on the above inequality yields

$$\begin{aligned} f(s_j(PCQ^*)) &\leq f \left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2} \right) f(s_j(X)) \\ &= f \left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2} \right) f \left(s_j \left(\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \right) \right) \\ &= f \left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2} \right) s_j \left(f \left(\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \right) \right). \end{aligned}$$

(c) $\Rightarrow$ (d) Since A and B are positive semidefinite matrices, then

$$\pm \frac{A-B}{2} \leq \frac{A+B}{2}.$$

Using (c), we have

$$\begin{aligned} s_j(f(|P(A-B)Q^*|)) &= s_j(f(|\sqrt{2}P(\frac{A-B}{2})\sqrt{2}Q^*|)) \\ &\leq f \left(\frac{\max\{2\|P\|_\infty^2, 2\|Q\|_\infty^2\}}{2} \right) s_j(f(A) \oplus f(B)) \\ &= f(\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}) s_j(f(A) \oplus f(B)). \end{aligned}$$

(d) $\Rightarrow$ (c) Since A is a Hermitian matrix and $\pm A \leq B$, then $A+B$ and $B-A$ are positive semidefinite.

Using (d), we get

$$\begin{aligned} s_j(f(|P(A)Q^*|)) &= s_j(f(|P(\frac{(A+B)-(B-A)}{2})Q^*|)) \\ &\leq f \left(\frac{\max\{\|P\|_\infty^2, \|Q\|_\infty^2\}}{2} \right) s_j(f(A+B) \oplus f(B-A)). \quad \square \end{aligned}$$

At last, we present a corollary which is also an equivalence of variants of the inequalities in Theorem 3.5.

COROLLARY 3.6. *Let $P, Q \in \mathbb{M}_n(\mathbb{C})$ and let $f : [0, \infty) \rightarrow \mathbb{R}$ be a nonnegative and increasing submultiplicative function satisfying $f(0) = 0$. Then, for $j = 1, 2, \dots, n$, the following statements are equivalent:*

(a) For $A, B \in \mathbb{M}_n(\mathbb{C})$,

$$s_j(f(|PA^*BQ^*|)) \leq f(\{\|P\|_\infty^2, \|Q\|_\infty^2\})s_j\left(f\left(\frac{AA^* + BB^*}{2}\right)\right).$$

(b) For $A, B, C \in \mathbb{M}_n(\mathbb{C})$ with $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geq 0$, the following inequality holds:

$$s_j(f(|PCQ^*|)) \leq f(\max\{\|P\|_\infty^2, \|Q\|_\infty^2\})s_j\left(f\left(\frac{1}{2}\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}\right)\right).$$

(c) For $A, B \in \mathbb{M}_n(\mathbb{C})$ with Hermitian matrix A , positive semidefinite matrix B and $\pm A \leq B$, the following inequality holds:

$$s_j(f(|PAQ^*|)) \leq f(\max\{\|P\|_\infty^2, \|Q\|_\infty^2\})s_j\left(f\left(\frac{B-A}{\sqrt{2}}\right) \oplus f\left(\frac{B+A}{\sqrt{2}}\right)\right).$$

(d) For positive semidefinite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(f(|P(A-B)Q^*|)) \leq f(\max\{\|P\|_\infty^2, \|Q\|_\infty^2\})s_j(f(A) \oplus f(B)).$$

Proof. The proof is similar to that of Theorem 3.5. $\square$

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