

ON SOME GENERALIZED INTEGRATION OPERATORS FROM BESOV SPACES TO BLOCH TENT SPACES

ZHI-JIE JIANG AND STEVO STEVIĆ*

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Abstract. Let $H(\mathbb{D})$ be the class of all analytic functions on the open unit disk $\mathbb{D}$, $\vec{u} = (u_0, u_1, \dots, u_{n-1})$ with $u_l \in H(\mathbb{D})$ for $l = \overline{0, n-1}$. We study the following operators

$$T_{\vec{u}}^{(n)}(f)(z) = I^n(u_0^{(n)}f + u_1^{(n-1)}f' + \dots + u_{n-1}'f^{(n-1)})(z), \quad f \in H(\mathbb{D}),$$

where I is the integration operator

$$I(f)(z) = \int_0^z f(\zeta) d\zeta,$$

and I^n is the n th iteration of I , characterizing the boundedness and compactness of these operators acting from the Besov spaces B_p to the Bloch tent spaces for $p \geq 1$, and estimate the essential norm of the operators for $p > 1$.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, that is, the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, that is, the unit circle, $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$, $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$, $S(\mathbb{D})$ be the class of analytic self-maps of $\mathbb{D}$, and $dA(z) = \frac{1}{\pi} dx dy$ be the normalized Lebesgue area measure on $\mathbb{D}$. If $s, t \in \mathbb{N}_0$ are such that $s \leq t$, then the notation $j = \overline{s, t}$ is an abbreviation for the expression: $s \leq j \leq t$, $j \in \mathbb{N}_0$.

For any $w \in \mathbb{D}$, let $\phi_w(z)$ be the Möbius transformation of $\mathbb{D}$ which interchanges the origin and w , that is,

$$\phi_w(z) = \frac{w - z}{1 - \overline{w}z}, \quad z \in \mathbb{D}.$$

The pseudo-hyperbolic distance between z and w is defined by

$$\rho(z, w) = |\phi_w(z)| = \left| \frac{w - z}{1 - \overline{w}z} \right|.$$

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* Corresponding author.

The Bergman metric between z and w is defined by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

For a fixed $w \in \mathbb{D}$ and $r > 0$, let

$$D(w, r) = \{z \in \mathbb{D} : \beta(z, w) < r\}$$

denote the Bergman metric disk centered at w with radius r . It is known that

$$(1 - |w|^2)^2 \asymp |1 - \bar{z}w|^2 \asymp (1 - |z|^2)^2 \asymp A(D(w, r)) \quad (1)$$

for all $z \in D(w, r)$.

For $1 < p < +\infty$, the Besov space $B_p(\mathbb{D}) = B_p$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{sB_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p d\lambda(z) < +\infty,$$

where

$$d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}$$

is the Möbius invariant measure on $\mathbb{D}$. The functional $\|\cdot\|_{sB_p}$ is a semi-norm on B_p , and it is Möbius invariant in the sense that $\|f \circ \varphi\|_{sB_p} = \|f\|_{sB_p}$ for all $f \in B_p$ and $\varphi \in \text{Aut}(\mathbb{D})$. B_p becomes a Banach space with the norm

$$\|f\|_{B_p} = |f(0)| + \|f\|_{sB_p}.$$

B_2 is the classical Dirichlet space D^2 . But, the Besov space B_1 is defined in a different fashion because $(1 - |z|^2)f'(z) \in L^1(\mathbb{D}, d\lambda)$ if and only if f is constant. B_1 is the space of analytic functions on $\mathbb{D}$ which can be written as

$$f(z) = \sum_{j=1}^{+\infty} a_j \varphi_{\lambda_j}(z)$$

for some sequence $(a_j)_{j \in \mathbb{N}}$ in l^1 and $(\lambda_j)_{j \in \mathbb{N}} \subset \mathbb{D}$. The norm on B_1 is defined by

$$\|f\|_{B_1} = \inf \left\{ \sum_{j=1}^{+\infty} |a_j| : f(z) = \sum_{j=1}^{+\infty} a_j \varphi_{\lambda_j}(z) \right\}.$$

B_1 is a Banach space with the norm ([51]).

From the definition, we see that the functions $\varphi_w(z)$ and $f(z) = z$ belong to B_1 . But, whether the functions $f_k(z) = z^k$ ($k \geq 2$) belong to B_1 is difficult to determine by the definition alone. If $1 < p < +\infty$, then $(B_p)^* \cong B_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $(B_1)^* \cong \mathcal{B}$. Hence, B_p is reflexive for $1 < p < +\infty$, but B_1 is not. For these and some other information on the Besov spaces see, for example, [51].

For $0 < q < +\infty$, the Bloch tent space BT^q consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{sBT^q}^q = \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} |f'(z)|(1 - |z|^2) \right)^q |d\zeta| < +\infty,$$

where $|d\zeta|$ is the normalized arc-length measure on $\mathbb{T}$ and $\Gamma_M(\zeta)$ is the Stolz angle of aperture M centered at ζ , that is,

$$\Gamma_M(\zeta) = \left\{ z \in \mathbb{D} : |\zeta - z| < \frac{M}{2}(1 - |z|^2) \right\}, \quad M > 1, \quad \zeta \in \mathbb{T}.$$

BT^q is a Banach space with the norm

$$\|f\|_{BT^q} = |f(0)| + \|f\|_{sBT^q}.$$

It should be said that any two apertures generate the same function space with equivalent (quasi-)norms. Some basic properties of the space can be found in [26]. In addition to other results, there was proved that this space can be identified by using higher order derivatives of functions when $q > 1$. Additionally, properties such as growth properties of functions in BT^q have been recently studied in [6]. Most recently, in [5] was extended the result related to higher order derivatives of functions in [26] to the case $0 < q \leq 1$.

The Bloch space $\mathcal{B}(\mathbb{D}) = \mathcal{B}$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < +\infty.$$

It is a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$. The little Bloch space $\mathcal{B}_0(\mathbb{D}) = \mathcal{B}_0$, consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |f'(z)| = 0.$$

Some information on these and related spaces can be found, for example, in [7, 51].

For $u \in H(\mathbb{D})$, the integration operator I_u and its companion operator J_u on $H(\mathbb{D})$ are defined by

$$(I_u f)(z) = \int_0^z f(\zeta) u'(\zeta) d\zeta \quad \text{and} \quad (J_u f)(z) = \int_0^z f'(\zeta) u(\zeta) d\zeta.$$

These operators and their n -dimensional extensions have been studied extensively on various analytic function spaces (see, for example, [1, 15, 18–21, 27, 33] and the related references therein) and contain as special cases a number of operators such as the integral operator (I_u with $u(z) = z$) and the Cesàro operator (I_u with $u(z) = \log(1/(1 - z))$).

The multiplication operator M_u on $H(\mathbb{D})$ is defined by

$$(M_u f)(z) = u(z) f(z).$$

It should be mentioned the following relation containing the operators I_u , J_u and M_u

$$(I_u f)(z) + (J_u f)(z) = (M_u f)(z) - u(0)f(0).$$

To generalize several concrete operators containing the multiplication operator, the composition operator $C_\varphi f(z) = f(\varphi(z))$ and the differentiation operator $Df(z) = f'(z)$, Stević and Sharma proposed studying the following operator

$$T_{u_1, u_2, \varphi} f(z) = u_1(z)f(\varphi(z)) + u_2(z)f'(\varphi(z)),$$

where $u_1, u_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, and conducted the first investigations in [41]. Then, they introduced and studied the following generalization of the operator $T_{u_1, u_2, \varphi}$

$$T_{u_1, u_2, \varphi}^n f(z) = u_1(z)f^{(n)}(\varphi(z)) + u_2(z)f^{(n+1)}(\varphi(z)),$$

where $n \in \mathbb{N}_0$ ([42]), which is a sum of two weighted differentiation composition operators (or generalized weighted composition operators), that is, of the operators in the form

$$D_{\varphi, u}^k(f)(z) = u(z)f^{(k)}(\varphi(z))$$

(see, for example, [12, 22, 45, 47, 48]). The operators $T_{u_1, u_2, \varphi}$ and $T_{u_1, u_2, \varphi}^n$ have attracted a lot of attention (see, for example, [2, 10, 11, 13, 17, 24, 43, 46] and the related references therein). Having published [42], Stević noticed that the results therein and in some other papers can be easily generalized and came up with an idea of studying finite sums of the weighted differentiation composition operators, that is, of the operator

$$P_{\varphi, \vec{u}}^m(f)(z) := \sum_{j=0}^m u_j(z)f^{(j)}(\varphi(z)), \quad (2)$$

where $m \in \mathbb{N}_0$, $\varphi \in S(\Omega)$, $u_j \in H(\Omega)$, $j = \overline{0, m}$, Ω is a domain in $\mathbb{C}$, and proposed the investigation to some of his collaborators. He has also proposed studying some n -dimensional analogs (see, for example, [39, 40]). For a further generalization of the operator in [40] see, for example, [16]. For some results in the direction see also [8, 14, 25, 49].

Besides, motivated by his previous investigations on product-type operators which include the integration operator (see, for example, [30, 35–37]), Stević proposed studying iterations of integral operators. Namely, in [38] he studied the operator

$$C_{\varphi, k}^g(f)(z) = I^k D_{\varphi, g}^k(f)(z),$$

where I denotes the integration operator

$$I(f)(z) = \int_0^z f(w)dw.$$

One of the general ideas by Stević is to investigate the products of above-mentioned operators.

In [4] was considered the following special case of the product of an operator in (2) and the iterated integration operator on $H(\mathbb{D})$

$$I_{\vec{a},u}(f)(z) = I^n(a_0 u^{(n)} f + a_1 u^{(n-1)} f' + \cdots + a_{n-1} u' f^{(n-1)})(z),$$

where $\vec{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{C}^n$ and $u \in H(\mathbb{D})$, and were characterized the bounded and compact operators $I_{\vec{a},u} : H^p \rightarrow H^q$, when $0 < p \leq q < +\infty$. Recently, research has been conducted for the case $0 < q < p < +\infty$ in [5].

In [3] was considered the following special case of the product of an operator in (2) and the iterated integration operator

$$I_{\vec{u}}^{(n)}(f)(z) = I^n(u_0 f + u_1 f' + \cdots + u_{n-1} f^{(n-1)})(z), \quad (3)$$

where $\vec{u} = (u_0, u_1, \dots, u_{n-1})$ with $u_k \in H(\mathbb{D})$ for $k = \overline{0, n-1}$.

Here, we consider the following operator, which is essentially operator (3),

$$T_{\vec{u}}^{(n)}(f)(z) = I^n(u_0^{(n)} f + u_1^{(n-1)} f' + \cdots + u_{n-1}' f^{(n-1)})(z),$$

where $n \in \mathbb{N}$ and $u_k \in H(\mathbb{D})$ for $k = \overline{0, n-1}$, but the form is more suitable for formulating our main results.

Recently, the authors of [28, 50] studied the following product of the iterated integration operator and a weighted differentiation composition operator

$$T_u^{n,k}(f)(z) = I^n(u^{(n-k)} f^{(k)})(z), \quad f \in H(\mathbb{D}).$$

Note that the operator $T_{\vec{u}}^{(n)}$ is a sum operator of the operators $T_{u_k}^{n,k}$, $k = \overline{0, n-1}$, that is,

$$T_{\vec{u}}^{(n)} = \sum_{k=0}^{n-1} T_{u_k}^{n,k}.$$

Here, we study the boundedness and compactness of the operator $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$. We prove that $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ is bounded if and only if $T_{u_k}^{n,k} : B_p \rightarrow BT^q$ are bounded for $k = \overline{0, n-1}$, if and only if $u_k \in \mathcal{B}$ for $k = \overline{0, n-1}$. We also prove that $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ is compact if and only if $T_{u_k}^{n,k} : B_p \rightarrow BT^q$ are compact for $k = \overline{0, n-1}$, if and only if $u_k \in \mathcal{B}_0$ for $k = \overline{0, n-1}$. Additionally, we would also like to mention that compared to other literature (for example, [28, 50]), in this paper we consider not only the case $p > 1$, but also the case $p = 1$.

As usual, in this paper the positive numbers are denoted by C , and they may vary in different situations. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a positive number C such that $a \leq Cb$ (resp. $a \geq Cb$). If $a \lesssim b$ and $b \gtrsim a$, we write $a \asymp b$.

2. Auxiliary lemmas

First, we need the following result (see, for example, [51]).

LEMMA 1. *Let $z \in \mathbb{D}$, $t > -1$, $c \geq 0$, and*

$$J_{t,c}(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - \bar{z}w|^{t+2+c}} dA(w).$$

Then, the following asymptotic relations hold.

(a) *If $c > 0$, then*

$$J_{t,c}(z) \asymp \frac{1}{(1 - |z|^2)^c}.$$

(b) *If $c = 0$, then*

$$J_{t,c}(z) \asymp \log \frac{1}{1 - |z|^2}.$$

For $q > 1$, the following result was obtained in [26], while, when $0 < q \leq 1$, it was obtained in [5].

LEMMA 2. *Let $f \in H(\mathbb{D})$, $0 < q < +\infty$ and $n \in \mathbb{N}$. Then, $f \in BT^q$ if and only if*

$$\int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} |f^{(n)}(z)| (1 - |z|^2)^n \right)^q |d\zeta| < +\infty.$$

Moreover,

$$\|f\|_{BT^q} \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)| + \left(\int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} |f^{(n)}(z)| (1 - |z|^2)^n \right)^q |d\zeta| \right)^{1/q}. \quad (4)$$

The following lemma connects the hyperbolic distance and the region $\Gamma_M(\zeta)$ (see [44]).

LEMMA 3. *Let $M > 0$, $r > 0$ and $M^* = (M + 1)e^{2r}$. Then*

$$\bigcup_{z \in \Gamma_M(\zeta)} D(z, r) \subset \Gamma_{M^*}(\zeta). \quad (5)$$

By Lemma 3, we get the following result in a standard way. We give a proof of it for completeness.

LEMMA 4. *Let $0 < q < +\infty$ and $n \in \mathbb{N}$. If $f \in BT^q$, then*

$$|f^{(n)}(z)| \lesssim \frac{\|f\|_{sBT^q}}{(1 - |z|^2)^n}.$$

Proof. Let $r > 0$. Then, by the Cauchy estimate, the subharmonicity of the function $|f'(z)|$, and (1), we have

$$|f^{(n)}(z)| \lesssim \frac{1}{(1 - |z|^2)^{n+1}} \int_{D(z,r)} |f'(w)| dA(w).$$

Hence, by (1) we have

$$\begin{aligned}
 (1 - |z|^2)^n |f^{(n)}(z)| &\lesssim \frac{1}{1 - |z|^2} \int_{D(z,r)} |f'(w)| dA(w) \\
 &\lesssim \frac{1}{(1 - |z|^2)^2} \int_{D(z,r)} |f'(w)|(1 - |w|^2) dA(w) \\
 &\lesssim \frac{A(D(z,r))}{(1 - |z|^2)^2} \sup_{w \in D(z,r)} |f'(w)|(1 - |w|^2) \\
 &\asymp \sup_{w \in D(z,r)} |f'(w)|(1 - |w|^2).
 \end{aligned}$$

Lemma 3 shows that there exist $M^* > M > 1$ such that $D(z, r) \subset \Gamma_{M^*}(\zeta)$ for all $z \in \Gamma_M(\zeta)$. Hence, for each $z \in \Gamma_M(\zeta)$, we have

$$\begin{aligned}
 (1 - |z|^2)^n |f^{(n)}(z)| &\lesssim \sup_{w \in D(z,r)} |f'(w)|(1 - |w|^2) \\
 &\leq \sup_{w \in \Gamma_{M^*}(\zeta)} |f'(w)|(1 - |w|^2),
 \end{aligned}$$

which shows

$$(1 - |z|^2)^{nq} |f^{(n)}(z)|^q \lesssim \int_{\mathbb{T}} \left(\sup_{w \in \Gamma_{M^*}(\zeta)} |f'(w)|(1 - |w|^2) \right)^q |d\zeta| = \|f\|_{sBT^q}^q.$$

From this, the desired result follows. $\square$

LEMMA 5. Let $1 \leq p < +\infty$ and $0 < q < +\infty$. If $f \in B_p$, then

$$\|f\|_{sBT^q} \lesssim \|f\|_{sB_p},$$

where we regard that $\|f\|_{sB_1} = \|f\|_{B_1}$.

Proof. By the definition of the Besov space when $p = 1$, and Proposition 4.3.8 in [51], we have

$$\sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) \lesssim \|f\|_{sB_p},$$

(see [21]).

From this, it follows that

$$\|f\|_{sBT^q}^q = \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} |f'(z)|(1 - |z|^2) \right)^q |d\zeta| \lesssim \|f\|_{sB_p}^q,$$

from which the lemma follows. $\square$

The following result, which allows us to determine easily whether a function is in B_1 , can be found, for example, in [51].

LEMMA 6. *Let $f \in H(\mathbb{D})$. Then $f \in B_1$ if and only if $\int_{\mathbb{D}} |f''(z)| dA(z) < +\infty$.*

For a fixed $w \in \mathbb{D}$ and $j \in \mathbb{N}$, set

$$k_w(z) = \frac{1 - |w|^2}{1 - \bar{w}z}, \quad z \in \mathbb{D}, \quad (6)$$

and

$$k_{w,j}(z) = k_w^j(z), \quad z \in \mathbb{D}.$$

For $1 \leq p < +\infty$, from Lemma 1 and Lemma 6, it easily follows that $k_{w,j} \in B_p$ for every $w \in \mathbb{D}$ and $j \in \mathbb{N}$. Moreover, we have

$$\sup_{w \in \mathbb{D}} \|k_{w,j}\|_{B_p} \lesssim 1. \quad (7)$$

Beside this, we also see that the family of functions $k_{w,j}$ uniformly converges to zero on compact subsets of $\mathbb{D}$ as $|w| \rightarrow 1$.

In the following lemma we use the functions $k_{w,j}$ to construct some test functions in B_p . To do this we use a method developed by Stević in [34]. Since the proof can be regarded as standard, it is omitted.

LEMMA 7. *Let $1 \leq p < +\infty$ and $w \in \mathbb{D}$. Then, for each fixed $k \in \{0, 1, \dots, n-1\}$, there exist constants $c_{k,1}, c_{k,2}, \dots, c_{k,n}$ such that the function*

$$f_{w,k}(z) = \sum_{j=1}^n c_{k,j} k_{w,j}(z)$$

satisfies

$$f_{w,k}^{(k)}(w) = \frac{\bar{w}^k}{(1 - |w|^2)^k} \quad \text{and} \quad f_{w,k}^{(j)}(w) = 0$$

for $j \in \{0, 1, \dots, n-1\} \setminus \{k\}$.

Moreover,

$$\sup_{w \in \mathbb{D}} \|f_{w,k}\|_{B_p} \lesssim 1.$$

The following result follows from Theorem 5.1.5 and Theorem 5.2.6 in [51].

LEMMA 8. *Let $g \in H(\mathbb{D})$. Then, the following statements hold.*

(a) $g \in \mathcal{B}$ if and only if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-k} |g^{(n-k)}(z)| < +\infty,$$

for some $k \in \{0, 1, \dots, n-2\}$, and the asymptotic relation holds

$$\|g\|_{\mathcal{B}} \asymp \sum_{j=0}^{n-k-1} |g^{(j)}(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-k} |g^{(n-k)}(z)|. \quad (8)$$

Specially, if $g(0) = g'(0) = \dots = g^{(n-k-1)}(0) = 0$, we have

$$\|g\|_{\mathcal{B}} \asymp \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-k} |g^{(n-k)}(z)|. \quad (9)$$

(b) $g \in \mathcal{B}_0$ if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{n-k} |g^{(n-k)}(z)| = 0,$$

for some $k \in \{0, 1, \dots, n-2\}$.

LEMMA 9. Let $1 \leq p < +\infty$, $0 < q < +\infty$, $u \in H(\mathbb{D})$, $n \in \mathbb{N}$, and $0 \leq k \leq n-1$. Then, the operator $T_u^{n,k} : B_p \rightarrow BT^q$ is bounded if and only if $u \in \mathcal{B}$.

Moreover, if $u(0) = u'(0) = \dots = u^{(n-k-1)}(0) = 0$, then

$$\|T_u^{n,k}\|_{B_p \rightarrow BT^q} \asymp \|u\|_{\mathcal{B}}. \quad (10)$$

Proof. Assume that $T_u^{n,k} : B_p \rightarrow BT^q$ is bounded. For a fixed $w \in \mathbb{D}$, consider the function defined in (6). By the boundedness of the operator, Lemma 4 and some calculations, we have

$$\frac{\|T_u^{n,k} k_w\|_{BT^q}}{(1 - |w|^2)^n} \gtrsim |(T_u^{n,k} k_w)^{(n)}(w)| \gtrsim \frac{|u^{(n-k)}(w)| |w|^k}{(1 - |w|^2)^k}, \quad (11)$$

from which, along with (7), we obtain

$$\sup_{|w| > 1/2} (1 - |w|^2)^{n-k} |u^{(n-k)}(w)| \lesssim \|T_u^{n,k} k_w\|_{BT^q} \lesssim \|T_u^{n,k}\|_{B_p \rightarrow BT^q} < +\infty. \quad (12)$$

Set

$$f_k(z) = \frac{z^k}{k!}, \quad z \in \mathbb{D}.$$

From Lemma 6, we have $f_k \in B_p$. Using this fact and the boundedness of the operator $T_u^{n,k} : B_p \rightarrow BT^q$, we obtain $T_u^{n,k} f_k \in BT^q$.

By Lemma 4, we have

$$|u^{(n-k)}(z) f_k^{(k)}(z)| = |(T_u^{n,k} f_k)^{(n)}(z)| \lesssim \frac{\|T_u^{n,k} f_k\|_{BT^q}}{(1 - |z|^2)^n} \lesssim \frac{\|T_u^{n,k}\|_{B_p \rightarrow BT^q}}{(1 - |z|^2)^n}$$

for all $z \in \mathbb{D}$, from which, it follows that

$$\sup_{|w| \leq 1/2} (1 - |w|^2)^{n-k} |u^{(n-k)}(w)| \lesssim \|T_u^{n,k}\|_{B_p \rightarrow BT^q} < +\infty. \quad (13)$$

From (12) and (13), we see that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-k} |u^{(n-k)}(z)| \lesssim \|T_u^{n,k}\|_{B_p \rightarrow BT^q} < +\infty, \quad (14)$$

which, by Lemma 8, is equivalent with $u \in \mathcal{B}$.

Next, we assume that $u \in \mathcal{B}$. Let $f \in B_p$. Because $(T_u^{n,k}f)^{(j)}(0) = 0$ for $j = \overline{0, n-1}$, from (4), it follows that

$$\begin{aligned} \|T_u^{n,k}f\|_{BT^q}^q &\asymp \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} |(T_u^{n,k}f)^{(n)}(z)|(1-|z|^2)^n \right)^q |d\zeta| \\ &= \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} |u^{(n-k)}(z)f^{(k)}(z)|(1-|z|^2)^n \right)^q |d\zeta|. \end{aligned} \quad (15)$$

By using (8), Lemma 4 and Lemma 5, in (15), we have

$$\begin{aligned} \|T_u^{n,k}f\|_{BT^q}^q &\lesssim \|u\|_{\mathcal{B}}^q \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} |f^{(k)}(z)|(1-|z|^2)^k \right)^q |d\zeta| \\ &\lesssim \|u\|_{\mathcal{B}}^q \|f\|_{sBT^q}^q \\ &\lesssim \|u\|_{\mathcal{B}}^q \|f\|_{B_p}^q, \end{aligned}$$

which shows that $T_u^{n,k} : B_p \rightarrow BT^q$ is bounded and

$$\|T_u^{n,k}\|_{B_p \rightarrow BT^q} \lesssim \|u\|_{\mathcal{B}}. \quad (16)$$

If $u(0) = u'(0) = \dots = u^{(n-k-1)}(0) = 0$, then from (9), (14) and (16), we get (10). $\square$

Before we present our results on the compactness of the operator we provide some explanations. Let $X = X(\mathbb{D})$ be a Banach space of holomorphic functions on $\mathbb{D}$, $\|\cdot\|_X$ be its norm, and B_X be the closed unit ball of X . Let τ_0 denote the compact open topology on X and τ be any locally convex topology coarser than the norm topology of Y . By w we denote the weak topology on the Banach space X , and by w^* we denote the weak-star topology on X^* . To study compactness on functional spaces, we need some applicable characterizations. For subspaces of $H(\mathbb{D})$, one of the first such characterizations was given in [29], where the author studied compactness of the composition operator on the Hardy spaces H^p , $1 \leq p \leq +\infty$ ([9]). Later, many authors used some similar arguments to characterize the compactness of various concrete operators (see, for example, [18, 20, 32, 33]). However, it is also a frequent situation that authors only formulate such results without proving it, and only say that the results can be similarly proved. In these studies, the spaces involved, such as H^∞ , have some characteristics not typical for the Hardy spaces, which was considered in [29]. In 2003 Stević found a gap in the proof of a related result in the literature, which seemed applicable, and still is used by some authors.

In response to this phenomenon, Lindström, Norrbo, and Stević in [23] proved the following result.

LEMMA 10. *Let (B_X, τ_0) be compact and $T : X \rightarrow Y$ be a bounded linear operator. Let τ be any locally convex topology coarser than the norm topology of Y . Then, $T : X \rightarrow Y$ is compact and $T|_{B_X} : (B_X, w^*) \rightarrow (Y, \tau)$ is continuous if and only if T has the (SC)-property, that is, that every bounded sequence $(f_j)_{j \in \mathbb{N}}$ in X such that $f_j \rightarrow 0$*

uniformly on any compact subset of $\mathbb{D}$ as $j \rightarrow +\infty$, implies that $Tf_j \rightarrow 0$ with respect to the norm of Y as $j \rightarrow +\infty$.

The next result is a corollary of Lemma 10 (see [23]).

LEMMA 11. *Let (B_X, τ_0) be compact, X be reflexive and $T : X \rightarrow Y$ be a bounded linear operator. Then, $T : X \rightarrow Y$ is compact if and only if T has the (SC)-property.*

Let $X = B_p$. By using the point evaluation estimates, Montel's theorem, and Fatou's lemma, it is easy to see that (B_X, τ_0) is compact. Since B_p is reflexive for $1 < p < +\infty$, by Lemma 11 we get the following lemma.

LEMMA 12. *Let $1 < p < +\infty$, $0 < q < +\infty$, and the linear operator $T : B_p \rightarrow BT^q$ be bounded. Then, $T : B_p \rightarrow BT^q$ is compact if and only if $\|Tf_j\|_{BT^q} \rightarrow 0$ as $j \rightarrow +\infty$ whenever $(f_j)_{j \in \mathbb{N}}$ is bounded in B_p and $f_j \rightarrow 0$ uniformly on any compact subset of $\mathbb{D}$ as $j \rightarrow +\infty$.*

Lindström, Norrbo, and Stević in [23] also pointed out that compactness of the operator $T : X \rightarrow Y$ when X is not reflexive, is not a sufficient condition for the (SC)-property.

Hence, Lemma 12 does not hold for $X = B_1$, since B_1 is not reflexive. But, for the operator $T_{\vec{u}}^{(n)} : B_1 \rightarrow BT^q$, Lemma 12 holds, which is a frequent situation for concrete operators. The proof is standard and follows the arguments, for example, in [29, 33]. We give a proof for the completeness.

LEMMA 13. *Let $0 < q < +\infty$, $n \in \mathbb{N}$, and $\vec{u} = (u_0, u_1, \dots, u_{n-1})$ with $u_k \in H(\mathbb{D})$ for $k = \overline{0, n-1}$, such that $T_{\vec{u}}^{(n)} : B_1 \rightarrow BT^q$ is bounded. Then, $T_{\vec{u}}^{(n)} : B_1 \rightarrow BT^q$ is compact if and only if $\|T_{\vec{u}}^{(n)} f_j\|_{BT^q} \rightarrow 0$ as $j \rightarrow +\infty$ whenever $(f_j)_{j \in \mathbb{N}}$ is a bounded sequence in B_1 and uniformly converges to zero on any compact subset of $\mathbb{D}$ as $j \rightarrow +\infty$.*

Proof. Assume that for any bounded sequence $(f_j)_{j \in \mathbb{N}} \in B_1$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, it follows that $\|T_{\vec{u}}^{(n)} f_j\|_{BT^q} \rightarrow 0$ as $j \rightarrow +\infty$. Let $\sup_{j \in \mathbb{N}} \|f_j\|_{B_1} = M$. Since there is a positive constant C such that

$$|f(z)| \leq C \|f\|_{B_1}$$

for every $f \in B_1$, we have $|f_j(z)| \leq CM$ on $\{z \in \mathbb{D} : |z| \leq r\}$ for each $j \in \mathbb{N}$. Since every compact subset K of $\mathbb{D}$ is contained in $\{z : |z| \leq r\}$ for some $r > 0$, it follows that $(f_j)_{j \in \mathbb{N}}$ is uniformly bounded on every compact subset of $\mathbb{D}$. Hence, Montel's theorem allows us to choose a subsequence $(f_{j_k})_{k \in \mathbb{N}}$ of $(f_j)_{j \in \mathbb{N}}$ which converges uniformly on every compact subset of $\mathbb{D}$ to an analytic function f . Since, then $f_{j_k}''(z) \rightarrow f''(z)$ uniformly on compacts of $\mathbb{D}$, as $k \rightarrow +\infty$, we have that $|f_{j_k}''(z)| \rightarrow |f''(z)|$ for every $z \in \mathbb{D}$, as $k \rightarrow +\infty$, then by Fatou's lemma we have

$$\int_{\mathbb{D}} |f''(z)| dA(z) \leq \liminf_{k \rightarrow +\infty} \int_{\mathbb{D}} |f_{j_k}''(z)| dA(z) \leq M < +\infty,$$

from which, together with Lemma 6, we obtain $f \in B_1$. Then $(f_{j_k} - f)_{k \in \mathbb{N}}$ is a bounded sequence in B_1 and $f_{j_k} - f \rightarrow 0$ uniformly on any compact subset of $\mathbb{D}$, as $k \rightarrow +\infty$. The assumption guarantees that $\|T_u^{(n)}(f_{j_k} - f)\|_{BT^q} \rightarrow 0$ as $k \rightarrow +\infty$. This shows that $T_u^{(n)} : B_1 \rightarrow BT^q$ is compact.

Conversely, assume that $T_u^{(n)} : B_1 \rightarrow BT^q$ is compact. Let B be the closed unit ball in B_1 . The compactness of $T_u^{(n)} : B_1 \rightarrow BT^q$ means that the set $T_u^{(n)}(B)$ is a relatively compact subset of BT^q . Let $(f_j)_{j \in \mathbb{N}} \subset B$ be a bounded sequence converging to zero uniformly on compacts of $\mathbb{D}$, as $j \rightarrow +\infty$. From the Cauchy estimate, it follows that $f_j^{(k)} \rightarrow 0$ uniformly on compacts of $\mathbb{D}$, as $j \rightarrow +\infty$, for each $k \in \mathbb{N}_0$. We need to prove that $\|T_u^{(n)} f_j\|_{BT^q} \rightarrow 0$ as $j \rightarrow +\infty$. To do this, it is enough to prove that the zero function is a unique limit point of the sequence $(T_u^{(n)} f_j)_{j \in \mathbb{N}}$. The compactness implies that there is $g \in BT^q$ such that

$$\lim_{j \rightarrow +\infty} \|T_u^{(n)} f_j - g\|_{BT^q} = 0. \quad (17)$$

Using (17), Lemma 2 and Lemma 4, we have

$$|u_0^{(n)}(z)f_j(z) + u_1^{(n-1)}(z)f_j'(z) + \cdots + u_{n-1}'(z)f_j^{(n-1)}(z) - g^{(n)}(z)| \rightarrow 0 \quad (18)$$

as $j \rightarrow +\infty$, for each $z \in \mathbb{D}$. Using the uniform convergence on compacts of the sequences $(f_j^{(l)})_{j \in \mathbb{N}}$, $l = \overline{0, n-1}$, in (18), we conclude that $g^{(n)}(z) = 0$ for all $z \in \mathbb{D}$.

On the other hand, by the definition of $T_u^{(n)}$, we have

$$(T_u^{(n)} f_j)^{(k)}(0) = 0$$

for $j \in \mathbb{N}$ and $k = \overline{0, n-1}$. From this, and Lemma 2, we obtain $g^{(k)}(0) = 0$ for $k = \overline{0, n-1}$, from which, together with the Taylor formula for the function g , it follows that $g \equiv 0$. This shows that the zero function is a unique limit of the sequence $(T_u^{(n)} f_j)_{j \in \mathbb{N}}$. $\square$

LEMMA 14. Let $1 \leq p < +\infty$, $0 < q < +\infty$, $u \in H(\mathbb{D})$, $n \in \mathbb{N}$, $0 \leq k \leq n-1$, and the operator $T_u^{n,k} : B_p \rightarrow BT^q$ is bounded. Then, the operator $T_u^{n,k} : B_p \rightarrow BT^q$ is compact if and only if $u \in \mathcal{B}_0$.

Proof. Assume that $T_u^{n,k} : B_p \rightarrow BT^q$ is compact. Then, the operator is bounded. Let the sequence $(k_{w_j})_{j \in \mathbb{N}}$, where $w_j \in \mathbb{D}$ is such that $|w_j| \rightarrow 1$ as $j \rightarrow +\infty$, be defined by

$$k_{w_j}(z) = \frac{1 - |w_j|^2}{1 - \overline{w_j}z}, \quad z \in \mathbb{D}.$$

Then, the sequence is bounded in B_p and uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \rightarrow +\infty$. Hence, by Lemma 12 and Lemma 13, we have

$$\lim_{j \rightarrow +\infty} \|T_u^{n,k} k_{w_j}\|_{BT^q} = 0. \quad (19)$$

By the boundedness of $T_u^{n,k}$ and (11), we have

$$|u^{(n-k)}(w_j)|(1-|w_j|^2)^{n-k} \lesssim \|T_u^{n,k}k_{w_j}\|_{BT^q}$$

for $|w_j| \geq 1/2$, from which together with (19) and the arbitrariness of the sequence $(w_j)_{j \in \mathbb{N}}$ satisfying the condition $|w_j| \rightarrow 1$ as $j \rightarrow +\infty$, we get

$$\lim_{|z| \rightarrow 1-0} |u^{(n-k)}(z)|(1-|z|^2)^{n-k} = 0, \quad (20)$$

which, by Lemma 8, is equivalent with $u \in \mathcal{B}_0$.

Now, assume that $u \in \mathcal{B}_0$. Then, we have that (20) holds, that is, for every $\varepsilon > 0$ there is $r \in (0, 1)$ such that

$$|u^{(n-k)}(z)|(1-|z|^2)^{n-k} < \varepsilon, \quad (21)$$

for $r < |z| < 1$.

Let $(f_j)_{j \in \mathbb{N}} \subset B_p$ be a bounded sequence converging to zero uniformly on compacts of $\mathbb{D}$ as $j \rightarrow +\infty$. Then, the sequence $(f_j^{(k)})_{j \in \mathbb{N}}$ also converges to zero uniformly on compacts of $\mathbb{D}$ as $j \rightarrow +\infty$. So, for every $\varepsilon > 0$ there is $j_0 \in \mathbb{N}$ such that

$$|f_j^{(k)}(z)| < \varepsilon \quad (22)$$

for $j \geq j_0$ and every $|z| \leq r$, where r is chosen as above in (21).

Let $f \in B_p$. As we have already noticed, since $T_u^{n,k}(f_j)^{(j)}(0) = 0$ for $j = \overline{0, n-1}$, from Lemma 2 and Lemma 5, we have

$$\|T_u^{n,k}f_j\|_{BT^q}^q \asymp \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} |u^{(n-k)}(z)f_j^{(k)}(z)|(1-|z|^2)^n \right)^q |d\zeta|. \quad (23)$$

From (21), (22), (23), Lemma 2 and Lemma 5, we have

$$\begin{aligned} \|T_u^{n,k}f_j\|_{BT^q}^q &\lesssim \|u\|_{\mathcal{B}}^q \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta) \cap r\overline{\mathbb{D}}} |f_j^{(k)}(z)|(1-|z|^2)^k \right)^q |d\zeta| \\ &\quad + \varepsilon^q \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta) \cap |z| > r} |f_j^{(k)}(z)|(1-|z|^2)^k \right)^q |d\zeta| \\ &\lesssim \varepsilon^q \|u\|_{\mathcal{B}}^q + \varepsilon^q \sup_{j \in \mathbb{N}} \|f_j\|_{B_p}^q. \end{aligned} \quad (24)$$

From (24), and since ε is an arbitrary positive number, we get

$$\lim_{j \rightarrow +\infty} \|T_u^{n,k}f_j\|_{BT^q} = 0,$$

from which, together with Lemma 12 and Lemma 13, the compactness of the operator $T_u^{n,k} : B_p \rightarrow BT^q$ follows. $\square$

By the definition of the Bloch space and the little Bloch space, it follows that if $f \in \mathcal{B}$, then $f_r(z) = f(rz) \in \mathcal{B}_0$. So, we have the following result.

LEMMA 15. Let $1 \leq p < +\infty$, $0 < q < +\infty$, $n \in \mathbb{N}$, and $0 \leq k \leq n-1$. If $0 < r < 1$ and $u \in \mathcal{B}$, then the operator $T_{u_r}^{n,k} : B_p \rightarrow BT^q$ is compact.

Proof. Since $u \in \mathcal{B}$ implies that $u_r \in \mathcal{B}_0 \subset \mathcal{B}$, it follows from Lemma 9 that the operator $T_{u_r}^{n,k} : B_p \rightarrow BT^q$ is bounded. Then, by Lemma 14, the operator is also compact. $\square$

Let $u_k \in H(\mathbb{D})$, $u_{k,r}(z) = u_k(rz)$, and $\vec{u}_r = (u_{0,r}, u_{1,r}, \dots, u_{n-1,r})$. Then, from Lemma 15 we obtain the following result.

LEMMA 16. Let $1 \leq p < +\infty$, $0 < q < +\infty$, $n \in \mathbb{N}$, $\vec{u} = (u_0, u_1, \dots, u_{n-1})$ with $u_k \in H(\mathbb{D})$ for $k = \overline{0, n-1}$. If $0 < r < 1$ and $u_k \in \mathcal{B}$ for $k = \overline{0, n-1}$, then the operator $T_{\vec{u}_r}^{(n)} : B_p \rightarrow BT^q$ is compact.

3. Boundedness and essential norm

In this section we formulate and prove the results on the boundedness and essential norm of the operator $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$, and as a consequence give some characterizations for the compactness of the operator.

THEOREM 1. Let $1 \leq p < +\infty$, $0 < q < +\infty$, $n \in \mathbb{N}$, $\vec{u} = (u_0, u_1, \dots, u_{n-1})$ with $u_k \in H(\mathbb{D})$ for $k = \overline{0, n-1}$. Then, the operator $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ is bounded if and only if the operators $T_{u_k}^{n,k} : B_p \rightarrow BT^q$, $k = \overline{0, n-1}$, are bounded.

If the operator $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ is bounded, then

$$\|T_{\vec{u}}^{(n)}\|_{B_p \rightarrow BT^q} \asymp \sum_{k=0}^{n-1} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-k} |u_k^{(n-k)}(z)|. \quad (25)$$

Moreover, if $u_k(0) = u'_k(0) = \dots = u_k^{(n-k-1)}(0) = 0$ for each $k = \overline{0, n-1}$, then

$$\|T_{\vec{u}}^{(n)}\|_{B_p \rightarrow BT^q} \asymp \sum_{k=0}^{n-1} \|u_k\|_{\mathcal{B}}. \quad (26)$$

Proof. Assume that the operator $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ is bounded. For $w \in \mathbb{D}$ and $k \in \{0, 1, \dots, n-1\}$, let $f_{w,k}$ be the function defined in Lemma 7. Then, from Lemma 4, it follows that

$$\begin{aligned} \frac{\|T_{\vec{u}}^{(n)} f_{w,k}\|_{BT^q}}{(1 - |w|^2)^n} &\gtrsim |(I^n(u_0^{(n)} f_{w,k} + u_1^{(n-1)} f'_{w,k} + \dots + u'_{n-1} f_{w,k}^{(n-1)}))^{(n)}(w)| \\ &= |u_0^{(n)}(w) f_{w,k}(w) + u_1^{(n-1)}(w) f'_{w,k}(w) \\ &\quad + \dots + u_k^{(n-k)}(w) f_{w,k}^{(k)}(w) + \dots + u'_{n-1}(w) f_{w,k}^{(n-1)}(w)| \\ &= \frac{|w|^k}{(1 - |w|^2)^k} |u_k^{(n-k)}(w)|. \end{aligned}$$

From this, it follows that

$$\sup_{|w|>1/2} (1-|w|^2)^{n-k} |u_k^{(n-k)}(w)| \lesssim \|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q} < +\infty. \quad (27)$$

Employing the function

$$f_0(z) = 1, \quad z \in \mathbb{D},$$

and Lemma 4, we have

$$|u_0^{(n)}(z)| = |(T_{\bar{u}}^{(n)} f_0)^{(n)}(z)| \lesssim \frac{\|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q}}{(1-|z|^2)^n},$$

which implies that

$$(1-|z|^2)^n |u_0^{(n)}(z)| \lesssim \|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q}$$

for all $z \in \mathbb{D}$. So, we have

$$\sup_{|z| \leq 1/2} (1-|z|^2)^n |u_0^{(n)}(z)| \lesssim \|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q}. \quad (28)$$

By using the function

$$f_1(z) = z, \quad z \in \mathbb{D},$$

and Lemma 4, we have

$$|u_0^{(n)}(z)z + u_1^{(n-1)}(z)| = |(T_{\bar{u}}^{(n)} f_1)^{(n)}(z)| \lesssim \frac{\|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q}}{(1-|z|^2)^n},$$

from which and (28), we get

$$\sup_{|z| \leq 1/2} (1-|z|^2)^{n-1} |u_1^{(n-1)}(z)| \lesssim \sup_{|z| \leq 1/2} \frac{\|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q}}{1-|z|^2} \lesssim \|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q}.$$

Assume that we have proved

$$\sup_{|z| \leq 1/2} (1-|z|^2)^{n-j} |u_j^{(n-j)}(z)| \lesssim \|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q}, \quad (29)$$

for $j = \overline{0, k-1}$, for some $2 \leq k < n-1$.

Let

$$f_k(z) = \frac{z^k}{k!}, \quad z \in \mathbb{D}.$$

Then, we have

$$\begin{aligned} \left| u_0^{(n)}(z) \frac{z^k}{k!} + u_0^{(n-1)}(z) \frac{z^{k-1}}{(k-1)!} + \cdots + u_k^{(n-k)}(z) \right| &= |(T_{\bar{u}}^{(n)} f_k)^{(n)}(z)| \\ &\lesssim \frac{\|T_{\bar{u}}^{(n)}\|_{B_p \rightarrow BT^q}}{(1-|z|^2)^n}, \end{aligned} \quad (30)$$

for $z \in \mathbb{D}$.

From (30) and the induction hypothesis (29) we obtain

$$\sup_{|z| \leq 1/2} (1 - |z|^2)^{n-k} |u_k^{(n-k)}(z)| \lesssim \|T_{\vec{u}}^{(n)}\|_{B_p \rightarrow BT^q}. \quad (31)$$

So, (29) holds for $k = \overline{0, n-1}$.

Combining (27) and (31), we obtain

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{(n-k)} |u_k^{(n-k)}(z)| \lesssim \|T_{\vec{u}}^{(n)}\|_{B_p \rightarrow BT^q} < +\infty. \quad (32)$$

Hence, by Lemma 8, we have $u_k \in \mathcal{B}$, so by Lemma 9, the operator $T_{u_k}^{n,k} : B_p \rightarrow BT^q$ is bounded.

Conversely, if the operators $T_{u_k}^{n,k} : B_p \rightarrow BT^q$, $k = \overline{0, n-1}$, are bounded, then it is clear that the operator $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ is bounded.

From (15), it easily follows that

$$\|T_{u_k}^{n,k}\|_{B_p \rightarrow BT^q} \lesssim \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-k} |u_k^{(n-k)}(z)|.$$

Since

$$T_{\vec{u}}^{(n)} = \sum_{k=0}^{n-1} T_{u_k}^{n,k},$$

we get

$$\|T_{\vec{u}}^{(n)}\|_{B_p \rightarrow BT^q} \lesssim \sum_{k=0}^{n-1} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n-k} |u_k^{(n-k)}(z)|. \quad (33)$$

From (32) and (33), the relation (25) follows. From Lemma 9 and (25), the relation (26) follows, completing the proof. $\square$

Next, we study the essential norm of the operators $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$. Let X and Y be two Banach spaces, and $T : X \rightarrow Y$ be a bounded linear operator. The essential norm of the operator $T : X \rightarrow Y$, denoted by $\|T\|_{e, X \rightarrow Y}$, is defined by

$$\|T\|_{e, X \rightarrow Y} = \inf \{ \|T - K\|_{X \rightarrow Y} : K \text{ is compact from } X \text{ to } Y \}.$$

By this, $T : X \rightarrow Y$ is compact if and only if $\|\cdot\|_{e, X \rightarrow Y} = 0$. For some results on the essential norm of concrete linear operators, see, for example, [31, 38] and the related references therein.

THEOREM 2. *Let $1 < p < +\infty$, $0 < q < +\infty$, $n \in \mathbb{N}$, and $\vec{u} = (u_0, u_1, \dots, u_{n-1})$ with $u_k \in H(\mathbb{D})$ for $k = \overline{0, n-1}$. Let $u_k(0) = u'_k(0) = \dots = u_k^{(n-k-1)}(0) = 0$ for each $k = \overline{0, n-1}$, and the operator $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ be bounded. Then*

$$\|T_{\vec{u}}^{(n)}\|_{e, B_p \rightarrow BT^q} \asymp \sum_{l=0}^{n-1} \limsup_{|z| \rightarrow 1} (1 - |z|^2)^{n-l} |u_l^{(n-l)}(z)|.$$

Proof. Let $(z_j)_{j \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|z_j| \rightarrow 1$ as $j \rightarrow +\infty$. For each j , set

$$f_{j,k}(z) = \left(\frac{1 - |z_j|^2}{1 - \bar{z}_j z} \right)^k, \quad z \in \mathbb{D}.$$

For each fixed $l \in \{0, 1, \dots, n-1\}$, by Lemma 7 there exist constants $c_{l,1}, c_{l,2}, \dots, c_{l,n}$ such that the function

$$f_j(z) = \sum_{k=1}^n c_{l,k} f_{j,k}(z)$$

satisfies

$$f_j^{(l)}(z_j) = \frac{\bar{z}_j^l}{(1 - |z_j|^2)^l} \quad \text{and} \quad f_j^{(l')}(z_j) = 0$$

for all $l' \in \{0, 1, \dots, n-1\} \setminus \{l\}$,

$$\sup_{j \in \mathbb{N}} \|f_j\|_{B_p} \lesssim 1,$$

and f_j uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \rightarrow +\infty$.

Assume that the operator $T : B_p \rightarrow BT^q$ is compact. Then, by using Lemma 12, it follows that

$$\lim_{j \rightarrow +\infty} \|T f_j\|_{BT^q} = 0. \quad (34)$$

From (34), Lemma 2, and Lemma 4, it follows that

$$\begin{aligned} \|T_{\bar{u}}^{(n)} - T\|_{B_p \rightarrow BT^q} &\gtrsim \limsup_{j \rightarrow +\infty} \|(T_{\bar{u}}^{(n)} - T)f_j\|_{BT^q} \\ &\gtrsim \limsup_{j \rightarrow +\infty} \|T_{\bar{u}}^{(n)} f_j\|_{BT^q} - \limsup_{j \rightarrow +\infty} \|T f_j\|_{BT^q} \\ &= \limsup_{j \rightarrow +\infty} \|T_{\bar{u}}^{(n)} f_j\|_{BT^q} \\ &\gtrsim \limsup_{j \rightarrow +\infty} (1 - |z_j|^2)^n |(T_{\bar{u}}^{(n)} f_j)^{(n)}(z_j)| \\ &\asymp \limsup_{j \rightarrow +\infty} (1 - |z_j|^2)^{n-l} |u_l^{(n-l)}(z_j)|, \end{aligned}$$

which implies that

$$\|T_{\bar{u}}^{(n)}\|_{e, B_p \rightarrow BT^q} \gtrsim \limsup_{j \rightarrow +\infty} (1 - |z_j|^2)^{n-l} |u_l^{(n-l)}(z_j)|.$$

From the arbitrariness of $(z_j)_{j \in \mathbb{N}}$, it follows that

$$\|T_{\bar{u}}^{(n)}\|_{e, B_p \rightarrow BT^q} \gtrsim \limsup_{|z| \rightarrow 1} (1 - |z|^2)^{n-l} |u_l^{(n-l)}(z)|.$$

Hence, we obtain

$$\|T_{\vec{u}}^{(n)}\|_{e, B_p \rightarrow BT^q} \gtrsim \sum_{l=0}^{n-1} \limsup_{|z| \rightarrow 1} (1 - |z|^2)^{n-l} |u_l^{(n-l)}(z)|. \quad (35)$$

On the other hand, by Lemma 16, we have that for any $r \in (0, 1)$ the operator $T_{\vec{u}_r}^{(n)} : B_p \rightarrow BT^q$ is compact. Therefore, we have

$$\begin{aligned} \|T_{\vec{u}}^{(n)}\|_{e, B_p \rightarrow BT^q}^q &\leq \|T_{\vec{u}}^{(n)} - T_{\vec{u}_r}^{(n)}\|_{B_p \rightarrow BT^q}^q \\ &= \|T_{\vec{u} - \vec{u}_r}^{(n)}\|_{B_p \rightarrow BT^q}^q \\ &= \sup_{\|f\|_{B_p} \leq 1} \|T_{\vec{u} - \vec{u}_r}^{(n)}(f)\|_{BT^q}^q \\ &\lesssim \sup_{\|f\|_{B_p} \leq 1} \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} \left| \sum_{l=0}^{n-1} (u_l - (u_l)_r)^{(n-l)}(z) f^{(l)}(z) \right| (1 - |z|^2)^n \right)^q |d\zeta|. \end{aligned} \quad (36)$$

Since the operator $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ is bounded, by Theorem 1, we have, $u_l \in \mathcal{B}$ for $l = \overline{0, n-1}$. By the uniform continuity of the functions and their derivatives on compact subsets in $\mathbb{D}$, we have that for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there is $r \in (0, 1)$ such that

$$(1 - |z|^2)^{n-l} |(u_l - (u_l)_r)^{(n-l)}(z)| < \varepsilon, \quad (37)$$

for every $z \in \overline{\rho\mathbb{D}}$ and $l = \overline{0, n-1}$.

Employing (37), Lemma 4, Lemma 5 and (8), in (36), we obtain

$$\begin{aligned} &\|T_{\vec{u}}^{(n)}\|_{e, B_p \rightarrow BT^q}^q \\ &\lesssim \sup_{\|f\|_{B_p} \leq 1} \int_{\mathbb{T}} \left(\sup_{z \in \Gamma_M(\zeta)} \left| \sum_{l=0}^{n-1} (u_l - (u_l)_r)^{(n-l)}(z) f^{(l)}(z) \right| (1 - |z|^2)^n \right)^q |d\zeta| \\ &\lesssim \sup_{\|f\|_{B_p} \leq 1} \int_{\mathbb{T}} \left(\varepsilon + \sup_{\rho < |z| < 1} \left| \sum_{l=0}^{n-1} (u_l - (u_l)_r)^{(n-l)}(z) f^{(l)}(z) \right| (1 - |z|^2)^n \right)^q |d\zeta| \\ &\lesssim \varepsilon^q + \left(\sum_{l=0}^{n-1} \sup_{\rho < |z| < 1} (1 - |z|^2)^{n-l} |(u_l - (u_l)_r)^{(n-l)}(z)| \right)^q. \end{aligned} \quad (38)$$

Letting $\rho \rightarrow 1 - 0$ in (38), and using the fact that

$$\limsup_{|z| \rightarrow 1} \sum_{l=0}^{n-1} |h_l(z)| \leq \sum_{l=0}^{n-1} \limsup_{|z| \rightarrow 1} |h_l(z)|,$$

for any complex-valued functions h_l , $l = \overline{0, n-1}$, on $\mathbb{D}$, it follows that

$$\|T_{\vec{u}}^{(n)}\|_{e, B_p \rightarrow BT^q}^q \lesssim \varepsilon^q + \left(\sum_{l=0}^{n-1} \limsup_{|z| \rightarrow 1} (1 - |z|^2)^{n-l} |u_l^{(n-l)}(z)| \right)^q,$$

from which, together with the arbitrariness of $\varepsilon > 0$, we obtain

$$\|T_{\vec{u}}^{(n)}\|_{e, B_p \rightarrow BT^q} \lesssim \sum_{l=0}^{n-1} \limsup_{|z| \rightarrow 1} (1 - |z|^2)^{n-l} |u_l^{(n-l)}(z)|. \quad (39)$$

From (35) and (39), the desired result follows. $\square$

As an application of Theorem 2 and Lemma 15, we get the following corollary.

COROLLARY 1. *Let $1 < p < +\infty$, $0 < q < +\infty$, $n \in \mathbb{N}$, and $\vec{u} = (u_0, u_1, \dots, u_{n-1})$ with $u_k \in H(\mathbb{D})$ for $k = \overline{0, n-1}$. Let $u_k(0) = u'_k(0) = \dots = u_k^{(n-k-1)}(0) = 0$ for $k = \overline{0, n-1}$, and $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ be bounded. Then, the following statements are equivalent.*

- (a) $T_{\vec{u}}^{(n)} : B_p \rightarrow BT^q$ is compact;
- (b) $T_{u_k}^{n,k} : B_p \rightarrow BT^q$ are compact for $k = \overline{0, n-1}$;
- (c) $u_k \in \mathcal{B}_0$ for $k = \overline{0, n-1}$.

The next result deals with the case of B_1 space and provides the following characterization.

THEOREM 3. *Let $0 < q < +\infty$, $n \in \mathbb{N}$, and $\vec{u} = (u_0, u_1, \dots, u_{n-1})$ with $u_k \in H(\mathbb{D})$ for $k = \overline{0, n-1}$. Let $u_k(0) = u'_k(0) = \dots = u_k^{(n-k-1)}(0) = 0$ for each $k = \overline{0, n-1}$, and $T_{\vec{u}}^{(n)} : B_1 \rightarrow BT^q$ be bounded. Then, the following statements are equivalent.*

- (a) $T_{\vec{u}}^{(n)} : B_1 \rightarrow BT^q$ is compact;
- (b) $T_{u_k}^{n,k} : B_1 \rightarrow BT^q$ are compact for $k = \overline{0, n-1}$;
- (c) $u_k \in \mathcal{B}_0$ for $k = \overline{0, n-1}$.

Proof. (b) $\Leftrightarrow$ (c) The equivalence follows from Lemma 14.

(b) $\Rightarrow$ (a) The implication follows directly from the assumption, the definition of the operator $T_{\vec{u}}^{(n)}$ and the fact that the compact operators form a linear space.

(a) $\Rightarrow$ (c) Assume that the operator $T_{\vec{u}}^{(n)} : B_1 \rightarrow BT^q$ is compact. Then, the operator is bounded. For $w \in \mathbb{D}$ and $k \in \{0, 1, \dots, n-1\}$, let $f_{w,k}$ be the function defined in Lemma 7. Then it follows from Lemma 4 that the relation holds

$$(1 - |w|^2)^{n-k} |u_k^{(n-k)}(w)| \lesssim \|T_{\vec{u}}^{(n)} f_{w,k}\|_{BT^q}, \quad (40)$$

for $|w| > 1/2$.

Let $(w_j)_{j \in \mathbb{N}} \subset \mathbb{D}$ be such that $|w_j| \rightarrow 1$ as $j \rightarrow +\infty$. Then, the sequence $(f_{w_j,k})_{j \in \mathbb{N}}$ is bounded in B_1 and converges uniformly to zero on compacts of $\mathbb{D}$. Since the operator $T_{\vec{u}}^{(n)} : B_1 \rightarrow BT^q$ is compact, by Lemma 13 we have

$$\lim_{j \rightarrow +\infty} \|T_{\vec{u}}^{(n)} f_{w_j,k}\|_{BT^q} = 0. \quad (41)$$

Since $(w_j)_{j \in \mathbb{N}}$ is an arbitrary sequence such that $|w_j| \rightarrow 1$ as $j \rightarrow +\infty$, from (40) and (41), it follows that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{n-k} |u_k^{(n-k)}(z)| = 0,$$

for each $k \in \{0, 1, \dots, n-1\}$, from which, together with Lemma 8, it follows that $u_k \in \mathcal{B}_0$, for each $k \in \{0, 1, \dots, n-1\}$. $\square$

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Zhi-jie Jiang
School of Mathematics and Statistics
Sichuan University of Science and Engineering
Zigong, Sichuan, 643000, P.R. China
e-mail: matjzj@126.com

Stevo Stević
Mathematical Institute of the
Serbian Academy of Sciences and Arts
Knez Mihailova 36/III, 11000 Beograd, Serbia
and
Serbian Academy of Sciences and Arts
Knez Mihailova 35, 11000 Beograd, Serbia
and
Department of Medical Research
China Medical University Hospital
China Medical University
Taichung 40402, Taiwan, Republic of China
e-mail: sscitel@gmail.com