

A NEW DETERMINANTAL FORMULA AND A NEW RECURSIVE RELATION FOR COEFFICIENTS IN THE MACLAURIN EXPANSION OF THE RECIPROCAL OF THE GAMMA FUNCTION

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Abstract. In the work, the authors establish a new determinantal formula and a new recursive relation for coefficients in the Maclaurin power series expansion of the reciprocal of the classical Euler gamma function.

1. Introduction

The classical Euler gamma function $\Gamma(z)$ may be defined [21, Chapter 3] by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

It is known from [1, p. 255, Entry 6.1.3] that the reciprocal $\frac{1}{\Gamma(z)}$ is an entire function possessing simple zeros at the points $1-j$ for $j \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

On [12, p. 139] and in [23, Section 3], a recursive relation of the coefficients c_j in the Maclaurin power series expansion

$$\frac{1}{\Gamma(z)} = \sum_{j=1}^{\infty} c_j z^j, \quad |z| < \infty \quad (1)$$

was given by $c_1 = 1$, $c_2 = \gamma$, and

$$c_j = \frac{1}{j-1} [\gamma c_{j-1} - \zeta(2)c_{j-2} + \zeta(3)c_{j-3} - \dots + (-1)^j \zeta(j-1)c_1] \quad (2)$$

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for $j \geq 3$, where $\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$ for $\Re(z) > 1$ denotes the classical Riemann zeta function and

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n \right) = 0.57721566490153286060 \dots$$

is the Euler–Mascheroni constant whose irrationality or rationality is still unknown.

The recursive relation (2) can be rewritten as

$$c_n = nc_1c_n - c_2c_{n-1} + \sum_{k=2}^{\infty} (-1)^k \zeta(k) c_{n-k}, \quad n \in \mathbb{N}.$$

Entry 6.1.34 on [1, p. 256] reads that the coefficients c_j for $1 \leq j \leq 6$ in the Maclaurin power series expansion (1) can be numerically calculated by

$$\begin{aligned} c_1 &= 1.0000000000000000 \dots, & c_2 &= 0.5772156649015329 \dots, \\ c_3 &= -0.6558780715202538 \dots, & c_4 &= -0.0420026350340952 \dots, \\ c_5 &= 0.1665386113822915 \dots, & c_6 &= -0.0421977345555443 \dots \end{aligned} \quad (3)$$

In [23, Section 3], Wrench derived the formula

$$\frac{1}{\Gamma(z)} = z \exp \left[\gamma z - \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} z^k \right],$$

from which the recursive relation (2) was deduced by Wrench therein. In [23, 24], there existed more accurate numerical computation of c_j for $2 \leq j \leq 41$.

On [10, p. 2] and in [23], it is available that

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + \frac{1}{12} (6\gamma^2 - \pi^2) z^3 + \frac{1}{12} [2\gamma^3 - \gamma\pi^2 + 4\zeta(3)] z^4 + \dots, \quad (4)$$

$$\frac{1}{\Gamma(z+1)} = 1 + \gamma z + \left(\gamma^2 - \frac{\pi^2}{6} \right) \frac{z^2}{2} + \dots, \quad (5)$$

$$\frac{1}{\Gamma(z+2)} = 1 + (\gamma-1)z + \left[1 + \frac{1}{2}(\gamma-2)\gamma - \frac{1}{12}\pi^2 \right] z^2 + \dots \quad (6)$$

What is the general and explicit formula for the coefficient c_j in (1)? In this paper, we aim to answer this question by establishing a new determinantal formula and a new recursive relation for the coefficients c_j in the Maclaurin power series expansion (1) of the reciprocal $\frac{1}{\Gamma(z)}$ of the classical Euler gamma function $\Gamma(z)$.

2. A determinantal formula of coefficients in series expansion for $\frac{1}{\Gamma(z)}$

We now start out to establish a sound, general, and explicit formula of c_j in the Maclaurin power series expansion (1).

THEOREM 1. *For any $j \in \mathbb{N}$, the coefficients c_j in the Maclaurin power series expansion (1) can be determinantly computed by*

$$c_j = \frac{(-1)^j}{j!} \left(\frac{2}{\pi} \right)^{j+1} \begin{vmatrix} 0 & \binom{0}{0} I_0 & 0 & \dots & 0 & 0 \\ \frac{\pi}{2} & \binom{1}{0} I_1 & \binom{1}{1} I_0 & \dots & 0 & 0 \\ 0 & \binom{2}{0} I_2 & \binom{2}{1} I_1 & \dots & 0 & 0 \\ -\left(\frac{\pi}{2}\right)^3 & \binom{3}{0} I_3 & \binom{3}{1} I_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \left(\frac{\pi}{2}\right)^{j-1} \sin \frac{(j-1)\pi}{2} & \binom{j-1}{0} I_{j-1} & \binom{j-1}{1} I_{j-2} & \dots & \binom{j-1}{j-2} I_1 & \binom{j-1}{j-1} I_0 \\ \left(\frac{\pi}{2}\right)^j \sin \frac{j\pi}{2} & \binom{j}{0} I_j & \binom{j}{1} I_{j-1} & \dots & \binom{j}{j-2} I_2 & \binom{j}{j-1} I_1 \end{vmatrix} \quad (7)$$

$$= \frac{(-1)^j}{j!} \left(\frac{2}{\pi} \right)^{j+1} |e_{i,j}|_{(j+1) \times (j+1)},$$

where

$$I_j = \int_0^\infty \frac{\sin t}{t} (\ln t)^j dt, \quad j \in \mathbb{N}_0 = \{0\} \cup \mathbb{N} \quad (8)$$

and

$$e_{i,j} = \begin{cases} \left(\frac{\pi}{2}\right)^{i-1} \sin \frac{(i-1)\pi}{2}, & 1 \leq i \leq j+1, j=1 \\ \binom{i-1}{j-2} I_{i-j+1}, & 1 \leq i \leq j+1, 2 \leq j \leq j+1 \end{cases}$$

with the convention that the binomial coefficients $\binom{\ell}{m} = 0$ for $\ell, m \in \mathbb{N}_0$ such that $\ell < m$.

Proof. Entry 5.9.7 on p. 140 in [12] reads that

$$\Gamma(z) \sin \frac{\pi z}{2} = \int_0^\infty t^{z-1} \sin t dt, \quad -1 < \Re(z) < 1.$$

We rewrite this integral representation as

$$\frac{1}{\Gamma(z)} = \frac{\sin \frac{\pi z}{2}}{\int_0^\infty t^{z-1} \sin t dt}, \quad -1 < \Re(z) < 1.$$

This means that

$$c_j = \frac{1}{j!} \lim_{z \rightarrow 0} \left[\frac{\sin \frac{\pi z}{2}}{\int_0^\infty t^{z-1} \sin t dt} \right]^{(j)}, \quad j \in \mathbb{N}. \quad (9)$$

It is easy to see that, for $j \in \mathbb{N}_0$,

$$\lim_{z \rightarrow 0} \left(\sin \frac{\pi z}{2} \right)^{(j)} = \left(\frac{\pi}{2} \right)^j \lim_{z \rightarrow 0} \sin \left(\frac{j\pi}{2} + \frac{\pi z}{2} \right) = \left(\frac{\pi}{2} \right)^j \sin \frac{j\pi}{2}$$

and

$$\lim_{z \rightarrow 0} \left(\int_0^\infty t^{z-1} \sin t \, dt \right)^{(j)} = \lim_{z \rightarrow 0} \int_0^\infty t^{z-1} (\ln t)^j \sin t \, dt = \int_0^\infty \frac{\sin t}{t} (\ln t)^j \, dt = I_j,$$

where

$$I_0 = \int_0^\infty \frac{\sin t}{t} \, dt = \frac{\pi}{2} \quad (10)$$

and its generalizations can be found in [7, 8, 9, 15, 17, 20]. Accordingly, we obtain

$$c_j = \frac{(-1)^j}{j!} \left(\frac{2}{\pi} \right)^{j+1} \begin{vmatrix} 0 & \binom{0}{0} \frac{\pi}{2} & 0 & \dots & 0 \\ \frac{\pi}{2} & \binom{1}{0} I_1 & \binom{1}{1} \frac{\pi}{2} & \dots & 0 \\ 0 & \binom{2}{0} I_2 & \binom{2}{1} I_1 & \dots & 0 \\ -\left(\frac{\pi}{2}\right)^3 & \binom{3}{0} I_3 & \binom{3}{1} I_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \left(\frac{\pi}{2}\right)^{j-1} \sin \frac{(j-1)\pi}{2} & \binom{j-1}{0} I_{j-1} & \binom{j-1}{1} I_{j-2} & \dots & \binom{j-1}{j-1} \frac{\pi}{2} \\ \left(\frac{\pi}{2}\right)^j \sin \frac{j\pi}{2} & \binom{j}{0} I_j & \binom{j}{1} I_{j-1} & \dots & \binom{j}{j-1} I_1 \end{vmatrix}$$

for $j \in \mathbb{N}$, where we used the formula (9) and the determinantal formula

$$\left[\frac{p(x)}{q(x)} \right]^{(k)} = \frac{(-1)^k}{q^{k+1}} \begin{vmatrix} p & q & 0 & \dots & 0 & 0 \\ p' & q' & q & \dots & 0 & 0 \\ p'' & q'' & \binom{2}{1} q' & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p^{(k-2)} & q^{(k-2)} & \binom{k-2}{1} q^{(k-3)} & \dots & q & 0 \\ p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1} q^{(k-2)} & \dots & \binom{k-1}{k-2} q' & q \\ p^{(k)} & q^{(k)} & \binom{k}{1} q^{(k-1)} & \dots & \binom{k}{k-2} q'' & \binom{k}{k-1} q' \end{vmatrix}, \quad k \geq 0 \quad (11)$$

for higher derivatives of the ratio $\frac{p(x)}{q(x)}$ of two differentiable functions $p(x)$ and $q(x)$. The formula (11) can be found on [3, p. 40, Entry 5] and in [14, p. 51, Section 5]. The proof of Theorem 1 is thus proved. $\square$

3. A recursive relation of coefficients in series expansion for $\frac{1}{\Gamma(z)}$

In this section, we aim to derive a new recursive relation of coefficients c_j in the series expansion (1) for the reciprocal $\frac{1}{\Gamma(z)}$ of the Euler gamma function $\Gamma(z)$.

THEOREM 2. For $j \in \mathbb{N}$, the coefficients c_j satisfy the recursive relation

$$c_j = \frac{1}{j!} \left(\frac{\pi}{2} \right)^{j-1} \sin \frac{j\pi}{2} + \gamma c_{j-1} - \frac{2}{\pi} \sum_{r=1}^{j-2} \frac{I_{j-r}}{(j-r)!} c_r \quad (12)$$

with the initial value $c_0 = 0$, where I_j is defined by (8) with (10) and an empty sum is understood to be 0.

Proof. Let $M_0 = 1$ and

$$M_j = \begin{vmatrix} m_{1,1} & m_{1,2} & 0 & \dots & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & \dots & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{j-2,1} & m_{j-2,2} & m_{j-2,3} & \dots & m_{j-2,j-1} & 0 \\ m_{j-1,1} & m_{j-1,2} & m_{j-1,3} & \dots & m_{j-1,j-1} & m_{j-1,j} \\ m_{j,1} & m_{j,2} & m_{j,3} & \dots & m_{j,j-1} & m_{j,j} \end{vmatrix}$$

for $j \in \mathbb{N}$. Theorem in [4, p. 222] states that the sequence M_j for $j \geq 0$ satisfies $M_1 = m_{1,1}$ and

$$M_j = m_{j,j} M_{j-1} + \sum_{r=1}^{j-1} (-1)^{j-r} m_{j,r} \left(\prod_{s=r}^{j-1} m_{s,s+1} \right) M_{r-1}, \quad j \geq 2. \quad (13)$$

Applying $M_0 = 1$ and

$$M_{j+1} = \begin{vmatrix} 0 & \binom{0}{0} \frac{\pi}{2} & 0 & \dots & 0 & 0 \\ \frac{\pi}{2} & \binom{1}{0} I_1 & \binom{1}{1} \frac{\pi}{2} & \dots & 0 & 0 \\ 0 & \binom{2}{0} I_2 & \binom{2}{1} I_1 & \dots & 0 & 0 \\ -\left(\frac{\pi}{2}\right)^3 & \binom{3}{0} I_3 & \binom{3}{1} I_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \left(\frac{\pi}{2}\right)^{j-1} \sin \frac{(j-1)\pi}{2} & \binom{j-1}{0} I_{j-1} & \binom{j-1}{1} I_{j-2} & \dots & \binom{j-1}{j-2} I_1 & \binom{j-1}{j-1} \frac{\pi}{2} \\ \left(\frac{\pi}{2}\right)^j \sin \frac{j\pi}{2} & \binom{j}{0} I_j & \binom{j}{1} I_{j-1} & \dots & \binom{j}{j-2} I_2 & \binom{j}{j-1} I_1 \end{vmatrix}$$

for $j \in \mathbb{N}_0$ to the recursive relation (13) yields

$$\begin{aligned} M_j &= \binom{j-1}{j-2} I_1 M_{j-1} + (-1)^{j-1} m_{j,1} \left(\frac{\pi}{2}\right)^{j-1} M_0 \\ &\quad + \sum_{r=2}^{j-1} (-1)^{j-r} m_{j,r} \left(\frac{\pi}{2}\right)^{j-r} M_{r-1} \\ &= \binom{j-1}{j-2} I_1 M_{j-1} + \left(-\frac{\pi^2}{4}\right)^{j-1} \sin \frac{(j-1)\pi}{2} \\ &\quad + \sum_{r=2}^{j-1} (-1)^{j-r} \binom{j-1}{r-2} I_{j-r+1} \left(\frac{\pi}{2}\right)^{j-r} M_{r-1} \end{aligned}$$

for $j \geq 2$. Further replacing M_{j+1} by $(-1)^j j! \left(\frac{\pi}{2}\right)^{j+1} c_j$ for $j \in \mathbb{N}_0$ in the above recursive relation and simplifying lead to

$$\begin{aligned} (j-1)! \left(\frac{\pi}{2}\right)^j c_{j-1} &= -(j-1)! \left(\frac{\pi}{2}\right)^{j-1} I_1 c_{j-2} + \left(\frac{\pi^2}{4}\right)^{j-1} \sin \frac{(j-1)\pi}{2} \\ &\quad - \sum_{r=2}^{j-1} \frac{(j-1)!}{(j-r+1)!} \left(\frac{\pi}{2}\right)^{r-1} I_{j-r+1} \left(\frac{\pi}{2}\right)^{j-r} c_{r-2} \end{aligned}$$

for $j \geq 2$. By virtue of the value

$$I_1 = -\frac{\pi}{2} \gamma, \quad (14)$$

which is a special case $a = 1$ of the formula

$$\int_0^\infty \ln x \sin(ax) \frac{dx}{x} = -\frac{\pi}{2} (\gamma + \ln a), \quad a > 0 \quad (15)$$

on [5, p. 597, Entry 4.421], the above equation can be rearranged as the recursive relation (12). The proof of Theorem 2 is complete. $\square$

4. Remarks

In this section, we list several remarks as follows.

REMARK 1. As the first three special cases of I_j for $0 \leq j \leq 2$, the formulas (10), (14), and

$$I_2 = \frac{\pi}{24} (12\gamma^2 + \pi^2) \quad (16)$$

are known or be deduced from a little general formulas. The formulas (10) is famous in textbooks of complex functions and integral transforms for undergraduates (see also [1, p. 22, Example 10; p. 78, Entry 4.3.142]). It has been generalized in [7, 8, 9, 15, 17, 20].

The formula (14) is a special case of the formula (15).

The formula (16) can be deduced from the formula

$$\int_0^\infty (\ln x)^2 \sin(ax) \frac{dx}{x} = \frac{\pi}{2} \gamma^2 + \frac{\pi^3}{24} + \pi \gamma \ln a + \frac{\pi}{2} (\ln a)^2, \quad a > 0$$

on [5, p. 598, Entry 4.424].

REMARK 2. Because the sequence I_j defined by (8) is not a closed-form expression, the determinantal formula (7) for c_j in Theorem 1 is not a closed-form expression, but it is an explicit expression in mathematics.

Making use of the formulas in (16), we numerically figure out

$$\begin{aligned} c_3 &= -\frac{1}{3!} \left(\frac{2}{\pi} \right)^4 \begin{vmatrix} 0 & \binom{0}{0} \frac{\pi}{2} & 0 & 0 \\ \frac{\pi}{2} & \binom{1}{0} I_1 & \binom{1}{1} \frac{\pi}{2} & 0 \\ 0 & \binom{2}{0} I_2 & \binom{2}{1} I_1 & \binom{2}{2} \frac{\pi}{2} \\ -(\frac{\pi}{2})^3 & \binom{3}{0} I_3 & \binom{3}{1} I_2 & \binom{3}{2} I_1 \end{vmatrix} \\ &= \frac{1}{3!} \left(\frac{2}{\pi} \right)^4 \left(\begin{vmatrix} \binom{0}{0} \frac{\pi}{2} & 0 & 0 \\ \binom{2}{0} I_2 & \binom{2}{1} I_1 & \binom{2}{2} \frac{\pi}{2} \\ \binom{3}{0} I_3 & \binom{3}{1} I_2 & \binom{3}{2} I_1 \end{vmatrix} - \left(\frac{\pi}{2} \right)^3 \begin{vmatrix} \binom{0}{0} \frac{\pi}{2} & 0 & 0 \\ \binom{1}{0} I_1 & \binom{1}{1} \frac{\pi}{2} & 0 \\ \binom{2}{0} I_2 & \binom{2}{1} I_1 & \binom{2}{2} \frac{\pi}{2} \end{vmatrix} \right) \\ &= \frac{1}{3!} \left(\frac{2}{\pi} \right)^4 \left\{ \frac{\pi}{2} \frac{\pi}{2} \left[\binom{2}{1} \binom{3}{2} I_1^2 - \binom{3}{1} \binom{2}{2} I_2 \frac{\pi}{2} \right] - \left(\frac{\pi}{2} \right)^3 \left(\frac{\pi}{2} \right)^3 \right\} \\ &= \frac{1}{12} (6\gamma^2 - \pi^2), \\ c_2 &= \frac{1}{2} \left(\frac{2}{\pi} \right)^3 \begin{vmatrix} 0 & \binom{0}{0} \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & \binom{1}{0} I_1 & \binom{1}{1} \frac{\pi}{2} \\ 0 & \binom{2}{0} I_2 & \binom{2}{1} I_1 \end{vmatrix} = -\frac{1}{2} \left(\frac{2}{\pi} \right)^2 \begin{vmatrix} \binom{0}{0} \frac{\pi}{2} & 0 \\ \binom{2}{0} I_2 & \binom{2}{1} I_1 \end{vmatrix} = -\left(\frac{2}{\pi} \right)^2 \frac{\pi}{2} I_1 = \gamma, \\ c_1 &= -\left(\frac{2}{\pi} \right)^2 \begin{vmatrix} 0 & \binom{0}{0} \frac{\pi}{2} \\ \frac{\pi}{2} & \binom{1}{0} I_1 \end{vmatrix} = \frac{2}{\pi} \frac{\pi}{2} = 1, \end{aligned}$$

which coincide with the corresponding coefficients in (3) and (4).

REMARK 3. The integrals in (8) for $1 \leq j \leq 8$ can be numerically computed using the software MATHEMATICA 14.0.0 FOR MICROSOFT WINDOWS as follows:

$$\begin{aligned} I_1 &= -0.9066882461958017498065\dots, & I_2 &= 1.8152828538988069366097\dots, \\ I_3 &= -6.3156252198389770505131\dots, & I_4 &= 23.589428053244358339696\dots, \\ I_5 &= -120.27782124285607922585\dots, & I_6 &= 720.00907144925147679440\dots, \\ I_7 &= -5038.9887610683195440419\dots, & I_8 &= 40321.955508718785145561\dots \end{aligned}$$

These values motivate us to conjecture that

$$(-1)^j I_j > 0, \quad j \in \mathbb{N}_0.$$

Furthermore, it may be interesting to discuss the improper integral

$$I(z) = \int_0^\infty \frac{\sin t}{t} (\ln t)^z dt$$

as a function of $z \in \mathbb{C}$.

REMARK 4. Comparing two recursive relations (2) and (12), we arrive at

$$\frac{\pi}{2} \gamma_{j-1}^{j-2} c_{j-1} = \sum_{\ell=1}^{j-2} \left[\frac{I_{j-\ell}}{(j-\ell)!} + (-1)^{j-\ell} \frac{\pi}{2} \frac{1}{j-1} \zeta(j-\ell) \right] c_\ell - \frac{1}{j!} \left(\frac{\pi}{2} \right)^j \sin \frac{j\pi}{2}$$

for $j \geq 3$.

REMARK 5. As done in Theorems 1 and 2, we can establish determinantal expressions and recursive relations for the Maclaurin power series expansions (5) and (6) of the functions $\frac{1}{\Gamma(z+1)}$ and $\frac{1}{\Gamma(z+2)}$.

Generally, as done in Theorems 1 and 2, we can establish determinantal expressions and recursive relations for the Maclaurin power series expansions of the functions $\frac{1}{\Gamma(z+k)}$ at the point $z = 0$ for $k \in \mathbb{N}_0$.

REMARK 6. In [21, p. 71, Eq. (3.38)], we find the integral representation

$$\frac{1}{\Gamma(z)} = \frac{e^z z^{1-z}}{\pi} \int_0^\pi e^{-z\Phi(\theta)} d\theta, \quad \Re(z) \geq 0,$$

where

$$\begin{aligned} \Phi(\theta) &= 1 - \frac{\theta}{\tan \theta} + \ln \frac{\theta}{\sin \theta} \\ &= \sum_{j=1}^{\infty} \frac{2j+1}{2j} |B_{2j}| \frac{(2\theta)^{2j}}{(2j)!} \\ &= \frac{1}{2} \theta^2 + \frac{1}{36} \theta^4 + \frac{1}{405} \theta^6 + \frac{1}{4200} \theta^8 + \frac{1}{42525} \theta^{10} + \dots, \quad |\theta| < \pi. \end{aligned}$$

Recall from the monographs [11, Chapter XIII], [18], and [22, Chapter IV] that,

1. an infinitely differentiable real function $f(x)$ defined on an interval I is said to be absolutely monotonic in $x \in I$ if and only if all of its derivatives satisfy $f^{(n)}(x) \geq 0$ for $n \in \mathbb{N}_0$ and $x \in I$;
2. an infinitely differentiable real function $f(x)$ defined on I is said to be completely monotonic in $x \in I$ if and only if all of its derivatives satisfy the inequalities $(-1)^n f^{(n)}(x) \geq 0$ for $n \in \mathbb{N}_0$ and $x \in I$.

A function $f(x)$ is completely monotonic on (a, b) if and only if it is absolutely monotonic on $(-b, -a)$; see [22, p. 145, Definition 2c].

In [6, Definition 1] and [16, Definition 1], the concepts of logarithmically absolutely (completely) monotonic functions were defined as follows:

1. A positive function $f(x)$ is said to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(x)]^{(n)} \geq 0$ for $x \in I$ and $n \in \mathbb{N}$.
2. A positive function $f(x)$ is said to be logarithmically completely monotonic on an interval I if it has derivatives of all orders and $(-1)^n [\ln f(x)]^{(n)} \geq 0$ for $x \in I$ and $n \in \mathbb{N}$.

In [6, Theorem 1], the authors proved that a logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I , but not conversely. In the paper [2], [6, Theorem 4], and [16, Theorem 1], the authors proved that a logarithmically completely monotonic function on an interval I is also completely monotonic on I , but not conversely.

In the paper [25], we considered the normalized remainder

$$T_{2n+1}[\Phi(\theta)] = \begin{cases} \frac{2n+2}{2n+3} \frac{1}{|B_{2n+2}|} \frac{(2n+2)!}{(2\theta)^{2n+2}} \left[\Phi(\theta) - \sum_{j=1}^n \frac{2j+1}{2j} |B_{2j}| \frac{(2\theta)^{2j}}{(2j)!} \right], & \theta \neq 0 \\ 1, & \theta = 0 \end{cases}$$

for $n \in \mathbb{N}_0$ and $\theta \in (-\pi, \pi)$. Our main results in the paper [25] are as follows:

1. For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ is positive in $\theta \in (-\pi, \pi)$, is decreasing in $\theta \in (-\pi, 0)$, and is increasing in $\theta \in (0, \pi)$.
2. For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ is convex in $\theta \in (-\pi, \pi)$.
3. For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ is an absolutely monotonic function in $\theta \in (0, \pi)$ and is a completely monotonic function in $\theta \in (-\pi, 0)$.
4. For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ is logarithmically convex in $\theta \in (-\pi, \pi)$.
5. For $n \in \mathbb{N}_0$, the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ is decreasing in $\theta \in (-\pi, 0)$ and is increasing in $\theta \in (0, \pi)$.

In the paper [19], we generalized the main results in [25] and obtained the following conclusions:

1. For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ is a logarithmically absolutely monotonic function in $\theta \in (0, \pi)$ and a logarithmically completely monotonic function in $\theta \in (-\pi, 0)$.

2. For $n \in \mathbb{N}_0$, the function $\frac{1}{\theta} \frac{d \ln T_{2n+1}[\Phi(\theta)]}{d\theta}$ is an absolutely monotonic function in $(0, \pi)$ and a completely monotonic function in $(-\pi, 0)$.
3. For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ and $\frac{1}{\theta} \frac{d \ln T_{2n+1}[\Phi(\theta)]}{d\theta}$ can be extended analytically into the complex z -plane and are analytic in the disc $|z| < \pi$.
4. For $n \in \mathbb{N}_0$, the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$
 - (a) is an absolutely monotonic function in $\theta \in (0, \pi)$,
 - (b) is a completely monotonic function in $\theta \in (-\pi, 0)$,
 - (c) can be extended analytically into the complex z -plane and is analytic in the disc $|z| < \pi$.
5. The function $\frac{1}{\Phi(\theta)}$ for $0 < |\theta| < \pi$ was expanded into a Laurent series.

For more information and recent developments on normalized remainders, please refer to the review and survey article [13].

5. Conclusions

In this work, we mainly established three conclusions: For $j \in \mathbb{N}$, the coefficients c_j in the Maclaurin power series expansion (1) of the reciprocal $\frac{1}{\Gamma(z)}$ can be explicitly computed by the determinantal formula (7) in Theorem 1, while they can be derived from the recursive relation (12) in Theorem 2.

To consider and solve the conjecture and the problems in Remarks 3 and 5 are much more interesting.

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