

VARIETY OF NUMERICAL RADIUS INEQUALITIES

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Abstract. This paper presents various bounds for the numerical radius and operator norm of bounded linear operators on a complex Hilbert space. The new results employ convexity and operator-matrix techniques in a way that allows obtaining refinements of existing results and new forms for such bounds.

1. Introduction

Let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the Banach space of all bounded linear operators from a complex Hilbert space $\mathcal{H}$ to another $\mathcal{K}$. When $\mathcal{H} = \mathcal{K}$, we write $\mathcal{B}(\mathcal{H})$ instead of $\mathcal{B}(\mathcal{H}, \mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$, T^* denotes the adjoint of T . An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $T = T^*$. An operator T is called positive if, for every $x \in \mathcal{H}$, one has $\langle Tx, x \rangle \geq 0$. In this case, we write $T \geq O$, where O is the zero operator. Positive operators play an important role in understanding the geometry of a Hilbert space and form a distinguished subclass of self-adjoint operators.

Every operator $T \in \mathcal{B}(\mathcal{H})$ admits a Cartesian decomposition of the form

$$T = \Re T + i\Im T,$$

where $\Re T = \frac{T+T^*}{2}$ and $\Im T = \frac{T-T^*}{2i}$ are the real and imaginary parts of T , respectively. Another canonical representation of T is given by its polar decomposition: for operator $T \in \mathcal{B}(\mathcal{H})$, there exists a partial isometry $U \in \mathcal{B}(\mathcal{H})$ such that

$$T = U|T|, \quad \text{where} \quad |T| = (T^*T)^{\frac{1}{2}}.$$

Every $T \in \mathcal{B}(\mathcal{H})$ is naturally associated with three fundamental quantities: the spectral radius $\rho(T)$, the numerical radius $\omega(T)$, and the operator norm $\|T\|$. These quantities satisfy the well-known inequalities

$$\rho(T) \leq \omega(T) \leq \|T\|, \quad T \in \mathcal{B}(\mathcal{H}),$$

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with equality throughout when T is self-adjoint, that is,

$$\|T\| = \omega(T) = \rho(T).$$

While the spectral radius does not define a norm on $\mathcal{B}(\mathcal{H})$, the numerical radius does. Moreover, the numerical radius and the operator norm are equivalent in the sense that [4, Theorem 1.3-1]

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|, \quad T \in \mathcal{B}(\mathcal{H}). \quad (1)$$

Furthermore, the Cartesian components of T satisfy the following reverse inequalities [8, Theorem 2.1]

$$\|\Re T\| \leq \omega(T), \quad \|\Im T\| \leq \omega(T). \quad (2)$$

Considerable research has been devoted to sharpening the above inequalities and obtaining new possible relations between $\|\cdot\|$ and $\omega(\cdot)$. This is partially because $\|\cdot\|$ is much easier to compute than $\omega(\cdot)$.

Although the numerical radius is originally defined by

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle|,$$

an alternative and useful characterization is given by the identity [13]

$$\omega(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re(e^{i\theta} T) \right\|. \quad (3)$$

Moreover, when $\dim(\mathcal{H}) < \infty$, the supremum in (3) is attained, which yields the specific expression

$$\omega(T) = \left\| \Re(e^{-i\gamma} T) \right\|,$$

where $\gamma = \arg\langle Tx_0, x_0 \rangle$ and $x_0 \in \mathcal{H}$ is a unit vector such that $\omega(T) = |\langle Tx_0, x_0 \rangle|$, see [1, Theorem 3.1] for example.

One of the targets of this paper is to establish new inequalities for the numerical radius by means of convexity. Recall that a function $f: J \rightarrow \mathbb{R}$ is said to be convex on the interval J if it satisfies $f((1-\lambda)a + \lambda b) \leq (1-\lambda)f(a) + \lambda f(b)$ for all $a, b \in J$ and $0 \leq \lambda \leq 1$. In convex analysis, the Hermite-Hadamard inequality states that for a convex function f on $[0, 1]$, one has

$$f\left(\frac{1}{2}\right) \leq \int_0^1 f(t) dt \leq \frac{f(0) + f(1)}{2}. \quad (4)$$

This inequality plays a central role in convexity theory and may be viewed as a refinement of the mid-convexity condition of f .

To achieve our goal, we explore the connection between convex functions and operator inequalities. For this, we begin by noting that since $\omega(\cdot)$ and $\|\cdot\|$ are norms, one can easily verify that the functions

$$f(t) = \omega((1-t)T + tT^*), \quad \text{and} \quad g(t) = \|(1-t)T + tT^*\|$$

are convex on $[0, 1]$.

With a considerable amount of research devoted to inequalities of convex functions, the following inequalities, which have been shown in [3] for a convex function $f : [0, 1] \rightarrow \mathbb{R}$, have played a useful role in the literature

$$f(t) \leq (1-t)f(0) + tf(1) - 2r \left(\frac{f(0) + f(1)}{2} - f\left(\frac{1}{2}\right) \right),$$

and

$$(1-t)f(0) + tf(1) \leq f(t) + 2R \left(\frac{f(0) + f(1)}{2} - f\left(\frac{1}{2}\right) \right), \quad (5)$$

where $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$, and $0 \leq t \leq 1$. We refer the reader to [9, 10] for some applications and further discussion of these inequalities.

Applying these inequalities to the convex functions f and g above implies the following refinements and reverses of (2), for $0 < t < 1$,

$$\frac{\omega(T) - \omega((1-t)T + tT^*)}{2R} \leq \omega(T) - \|\Re T\| \leq \frac{\omega(T) - \omega((1-t)T + tT^*)}{2r}. \quad (6)$$

Furthermore, for $0 < t < 1$,

$$\frac{\|T\| - \|(1-t)T + tT^*\|}{2R} \leq \|T\| - \|\Re T\| \leq \frac{\|T\| - \|(1-t)T + tT^*\|}{2r}.$$

Using this approach, we will be able to present refined versions and generalizations of most of the above inequalities. Such refinements emphasize the role of convexity in developing sharper bounds for the numerical radius and the operator norm.

2. Auxiliary results

In this section, we present the lemmas needed for the proofs of our main results.

Let $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ be defined by

$$T = \begin{bmatrix} A & C^* \\ C & B \end{bmatrix}, \quad (7)$$

where $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

LEMMA 1. [6] *Let $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ be defined by (7). Then the following two statements are equivalent:*

(i) $T \geqslant O$;

(ii) $A \geqslant O$, $B \geqslant O$ and

$$|\langle Cu, v \rangle|^2 \leq |\langle Au, u \rangle| |\langle Bv, v \rangle|, \quad \forall u \in \mathcal{H}, v \in \mathcal{K}.$$

As an interesting refinement of the triangle inequality of norms, using the numerical radius of an operator matrix, we have the following

LEMMA 2. [8] *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A + B\| \leq 2\omega \left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix} \right) \leq \|A\| + \|B\|$$

The Cauchy-Schwarz inequality states that for any vectors u, v in an inner product space

$$|\langle u, v \rangle| \leq \|u\| \|v\|,$$

where $\langle \cdot, \cdot \rangle$ is the inner product and $\|u\| = \sqrt{\langle u, u \rangle}$. Several generalizations of the Cauchy-Schwarz inequality have been established in the literature. The Mixed Schwarz inequality is a remarkably general concept.

LEMMA 3. (Mixed Schwarz Inequality) [5, pp. 75–76] *For every $u, v \in \mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$,*

$$|\langle Tu, v \rangle|^2 \leq \langle T|u, u \rangle \langle T^*|v, v \rangle.$$

While a matrix-valued analogue of the following result was established by Tao in [12], a corresponding statement for general operators has not yet been addressed. To the best of our knowledge, such a result does not appear in the literature, and we therefore provide a proof here.

LEMMA 4. *Let $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ be given by (7) such that $T \geq O$. Then*

$$\|C\| \leq \frac{1}{2}\|T\|.$$

Proof. Let $x \in \mathcal{H}$ and $y \in \mathcal{H}$ be such that $\|x\| = \|y\| = 1$, and set

$$u = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since $T \geq O$, then $\langle Tu, u \rangle \geq 0$. Now by Lemma 1, we have $A, B \geq O$, and

$$|\langle Cx, y \rangle|^2 \leq \langle Ax, x \rangle \langle By, y \rangle.$$

Thus,

$$\begin{aligned} 2|\langle Cx, y \rangle| + 2\Re \langle Cx, y \rangle &\leq 2\sqrt{\langle Ax, x \rangle \langle By, y \rangle} + 2\Re \langle Cx, y \rangle \\ &\leq \langle Ax, x \rangle + \langle By, y \rangle + \langle Cx, y \rangle + \langle C^*y, x \rangle \\ &\quad \text{(by the AM-GM inequality).} \\ &= \left\langle \begin{pmatrix} Ax + C^*y \\ Cx + By \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \langle Tu, u \rangle. \end{aligned}$$

But $\langle Cx, y \rangle \in \mathbb{C}$, so there exists $\theta \in \mathbb{R}$ such that

$$\langle Cx, y \rangle = e^{i\theta} |\langle Cx, y \rangle|.$$

Now replace x by $e^{-i\theta}x$ and let $\hat{u} = \begin{pmatrix} e^{-i\theta}x \\ y \end{pmatrix}$. Then

$$\begin{aligned} 4|\langle Cx, y \rangle| &= 2|\langle Cx, y \rangle| + 2|\langle Cx, y \rangle| = 2 \left| \left\langle C(e^{-i\theta}x), y \right\rangle \right| + 2\Re \left\langle C(e^{-i\theta}x), y \right\rangle \\ &\leq \langle T\hat{u}, \hat{u} \rangle \\ &\leq \|T\| \|\hat{u}\|^2 \leq 2\|T\|. \end{aligned}$$

That is,

$$|\langle Cx, y \rangle| \leq \frac{1}{2} \|T\|.$$

Taking the supremum over all $x \in \mathcal{H}$ and $y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$, we obtain

$$\|C\| = \sup_{\|x\|=\|y\|=1} |\langle Cx, y \rangle| \leq \frac{1}{2} \|T\|. \quad \square$$

A useful supremum representation for the numerical radius of off-diagonal blocks is provided below.

LEMMA 5. [11, Proposition 1.1] *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$2\omega \left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right) = \sup_{\alpha^2 + \beta^2 = 1} \left\| \alpha(A + B^*) + \beta \frac{A - B^*}{i} \right\|.$$

3. Main results

3.1. Integral refinements via convexity and the Hermite-Hadamard inequality

In this subsection, we explore integral refinements of numerical radius inequalities by leveraging the convexity properties of the operator norm and the Hermite-Hadamard inequality. The obtained results provide sharper bounds and alternative formulations, offering deep insight into the relationship between the numerical radius and the operator norm.

We start by applying (4) to the functions

$$f(t) = \omega((1-t)T + tT^*) \quad \text{and} \quad g(t) = \|(1-t)T + tT^*\|,$$

leading to refinements of (2).

PROPOSITION 1. Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$\|\Re T\| \leq \int_0^1 \omega((1-t)T + tT^*) dt \leq \omega(T), \quad (8)$$

and

$$\|\Im T\| \leq \int_0^1 \omega((1-t)T^* - tT) dt \leq \omega(T). \quad (9)$$

Moreover,

$$\|\Re T\| \leq \int_0^1 \|(1-t)T + tT^*\| dt \leq \|T\|, \quad (10)$$

and

$$\|\Im T\| \leq \int_0^1 \|(1-t)T^* - tT\| dt \leq \|T\|. \quad (11)$$

The identity (3) provides an alternative formula to evaluate the numerical radius without appealing to the inner product. Interestingly, the inequalities (8) and (9) yield the following alternative identities, which provide a better understanding of how the numerical radius behaves.

COROLLARY 1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \omega(T) &= \sup_{\theta \in \mathbb{R}} \left(\int_0^1 \omega((1-t)e^{i\theta}T + te^{-i\theta}T^*) dt \right) \\ &= \sup_{\theta \in \mathbb{R}} \left(\int_0^1 \omega((1-t)e^{-i\theta}T^* - te^{i\theta}T) dt \right). \end{aligned}$$

Proof. Replacing T by $e^{i\theta}T$ in (8), we get

$$\left\| \Re(e^{i\theta}T) \right\| \leq \int_0^1 \omega((1-t)e^{i\theta}T + te^{-i\theta}T^*) dt \leq \omega(T).$$

By taking the supremum over $\theta \in \mathbb{R}$, (3) yields the first identity. The second identity follows from (9) and noting that

$$\sup_{\theta \in \mathbb{R}} \left\| \Im(e^{i\theta}T) \right\| = \omega(T). \quad \square$$

The following result involves an integral refinement of the second inequality in (1).

THEOREM 1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega(T) \leq \min\{\lambda_1, \lambda_2\} \leq \|T\|,$$

where

$$\lambda_1 = \sup_{\theta \in \mathbb{R}} \left(\int_0^1 \left\| (1-t)e^{i\theta}T + te^{-i\theta}T^* \right\| dt \right)$$

and

$$\lambda_2 = \sup_{\theta \in \mathbb{R}} \left(\int_0^1 \left\| (1-t)e^{-i\theta}T^* - te^{i\theta}T \right\| dt \right).$$

Proof. By the inequality (10), we have

$$\sup_{\theta \in \mathbb{R}} \|\Re e^{i\theta}T\| \leq \sup_{\theta \in \mathbb{R}} \left(\int_0^1 \left\| (1-t)e^{i\theta}T + te^{-i\theta}T^* \right\| dt \right) \leq \|T\|.$$

Now, by (3) we get

$$\omega(T) \leq \sup_{\theta \in \mathbb{R}} \left(\int_0^1 \left\| (1-t)e^{i\theta}T + te^{-i\theta}T^* \right\| dt \right) \leq \|T\|.$$

By a similar argument and using (11), we obtain

$$\omega(T) \leq \sup_{\theta \in \mathbb{R}} \left(\int_0^1 \left\| (1-t)e^{-i\theta}T^* - te^{i\theta}T \right\| dt \right) \leq \|T\|.$$

This completes the proof. $\square$

Continuing with the convexity of the norms, the inequality (6) may be used to get the following refinement of the first inequality in (1).

THEOREM 2. Let $T \in \mathcal{B}(\mathcal{H})$. Then for any $0 \leq t \leq 1$,

$$\frac{1}{2}\|T\| + \frac{1}{4R} \left(2\omega(T) - \left(\omega((1-t)T^* - tT) + \omega((1-t)T + tT^*) \right) \right) \leq \omega(T),$$

where $R = \max\{t, 1-t\}$.

Proof. The first inequality in (6) can be written as

$$\|\Re T\| + \frac{\omega(T) - \omega((1-t)T + tT^*)}{2R} \leq \omega(T). \quad (12)$$

Replacing T by iT^* , we infer that

$$\|\Im T\| + \frac{\omega(T) - \omega((1-t)T^* - tT)}{2R} \leq \omega(T). \quad (13)$$

Adding the inequalities (12) and (13) and dividing by two, we obtain

$$\begin{aligned}
 & \frac{1}{2} \|T\| + \frac{1}{4R} (2\omega(T) - (\omega((1-t)T^* - tT) + \omega((1-t)T + tT^*))) \\
 &= \frac{1}{2} \|\Re T + i\Im T\| + \frac{1}{4R} (2\omega(T) - (\omega((1-t)T^* - tT) + \omega((1-t)T + tT^*))) \\
 &\leq \frac{1}{2} (\|\Re T\| + \|\Im T\|) + \frac{1}{4R} (2\omega(T) - (\omega((1-t)T^* - tT) + \omega((1-t)T + tT^*))) \\
 &\quad (\text{by the triangle inequality for the usual operator norm}) \\
 &\leq \omega(T).
 \end{aligned}$$

This completes the proof. $\square$

Implementing convexity, we can derive the following refinement of the triangle inequality by virtue of the numerical radius of operator matrices.

THEOREM 3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then for any $0 \leq t \leq 1$,*

$$\|A + B\| \leq \|A\| + \|B\| - \frac{\omega\left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix}\right) - \omega\left(\begin{bmatrix} O & (1-t)A + tB \\ (1-t)B^* + tA^* & O \end{bmatrix}\right)}{R},$$

where $R = \max\{t, 1-t\}$.

Proof. Let $T = \begin{bmatrix} O & A \\ B^* & O \end{bmatrix}$, and define $f(t) = \omega((1-t)T + tT^*)$. By applying (5), and noting that $\omega\left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix}\right) \leq \frac{\|A\| + \|B\|}{2}$, we obtain the desired conclusion. $\square$

The following result provides an integral version of Lemma 2, where the numerical radius of convex combinations of operator matrices is used to refine the triangle inequality.

THEOREM 4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A + B\| \leq 2 \int_0^1 \omega\left(\begin{bmatrix} O & (1-t)A + tB \\ (1-t)B^* + tA^* & O \end{bmatrix}\right) dt \leq \|A\| + \|B\|.$$

Proof. Let $T = \begin{bmatrix} O & A \\ B^* & O \end{bmatrix}$, and define $f(t) = \omega((1-t)T + tT^*)$. By (4), we have

$$\omega(\Re T) \leq \int_0^1 \omega((1-t)T + tT^*) dt \leq \omega(T).$$

Multiplying the entire inequality by 2 gives

$$2\omega(\Re T) \leq 2 \int_0^1 \omega((1-t)T + tT^*) dt \leq 2\omega(T).$$

Now observe that $2\omega(\Re T) = \omega\left(\begin{bmatrix} O & A+B \\ A^*+B^* & O \end{bmatrix}\right) = \|A+B\|$, and that

$$2\omega(T) \leq \|A\| + \|B\|,$$

by Lemma 2. Combining these observations, the desired inequality follows. $\square$

4. Mixed Schwarz inequalities and applications

4.1. Numerical radius bounds utilizing mixed Schwarz and matrix positivity

In this section, we employ the mixed Schwarz inequality together with positivity arguments for block operator matrices to establish new upper bounds for the numerical radius.

Suppose that $A, B \in \mathcal{B}(\mathcal{H})$ are self-adjoint. Let

$$T = A + iB, \quad S = A - iB.$$

Then for each $x \in \mathcal{H}$,

$$|\langle Tx, x \rangle|^2 = |\langle Sx, x \rangle|^2 = \langle Ax, x \rangle^2 + \langle Bx, x \rangle^2,$$

which implies that

$$\max\{\|A\|, \|B\|\} = \max\{\omega(A), \omega(B)\} \leq \omega(T) = \omega(S). \quad (14)$$

The quantity $\frac{1}{2} \| |T| + |T^*| \|$ has appeared in numerous studies treating numerical radius bounds. To the best of our knowledge, it appeared for the first time in [7]. In the following theorem, we present a new bound involving this quantity.

THEOREM 5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\max \left\{ \omega \left(\frac{|T| - |T^*|}{2} \pm i\Re T \right), \omega \left(\frac{|T| - |T^*|}{2} \pm i\Im T \right) \right\} \leq \frac{1}{2} \| |T| + |T^*| \|.$$

Proof. Let $x \in \mathcal{H}$ be arbitrary. Define

$$\mathcal{A}(x) = \begin{bmatrix} \langle |T|x, x \rangle & \langle Tx, x \rangle \\ \langle T^*x, x \rangle & \langle |T^*|x, x \rangle \end{bmatrix}. \quad (15)$$

Clearly $\mathcal{A}(x)$ is a 2×2 Hermitian matrix with $\det(\mathcal{A}(x)) \geq 0$ and $\text{Tr}(\mathcal{A}(x)) \geq 0$. Hence, $\mathcal{A}(x) \geq O$ for all $x \in \mathcal{H}$. Now consider the matrix

$$\mathcal{L}(x) = \begin{bmatrix} \langle \left(\frac{|T| + |T^*|}{2} - \Im T \right) x, x \rangle & \langle \left(\frac{|T^*| - |T|}{2} + i\Re T \right) x, x \rangle \\ \langle \left(\frac{|T^*| - |T|}{2} - i\Re T \right) x, x \rangle & \langle \left(\frac{|T| + |T^*|}{2} + \Im T \right) x, x \rangle \end{bmatrix}.$$

A direct computation shows that

$$\mathcal{L}(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \mathcal{A}(x) \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}.$$

Since $\mathcal{A}(x) \geq O$, it follows that $\mathcal{L}(x) \geq O$, and hence $\det(\mathcal{L}(x)) \geq 0$. Computing the determinant yields

$$\left\langle \left(\frac{|T|+|T^*|}{2} - \Im T \right) x, x \right\rangle \left\langle \left(\frac{|T|+|T^*|}{2} + \Im T \right) x, x \right\rangle - \left| \left\langle \left(\frac{|T^*|-|T|}{2} + i\Re T \right) x, x \right\rangle \right|^2 \geq 0$$

Consequently,

$$\left| \left\langle \left(\frac{|T|-|T^*|}{2} - i\Re T \right) x, x \right\rangle \right|^2 \leq \left\langle \left(\frac{|T|+|T^*|}{2} - \Im T \right) x, x \right\rangle \left\langle \left(\frac{|T|+|T^*|}{2} + \Im T \right) x, x \right\rangle.$$

Using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} \left| \left\langle \left(\frac{|T|-|T^*|}{2} - i\Re T \right) x, x \right\rangle \right|^2 &\leq \frac{1}{4} \left(\langle (|T|+|T^*|)x, x \rangle \right)^2 \\ &\leq \frac{1}{4} \omega^2(|T|+|T^*|) = \frac{1}{4} \| |T|+|T^*| \|^2. \end{aligned}$$

Since x was arbitrary, we conclude that

$$\omega \left(\frac{|T|-|T^*|}{2} \pm i\Re T \right) \leq \frac{1}{2} \| |T|+|T^*| \|.$$

Noting that $\Re(-iT) = \Im(T)$, the desired conclusion follows immediately by replacing T with $-iT$. $\square$

REMARK 1. A combination of (14) and Theorem 5 gives

$$\| |T| - |T^*| \| = \omega(|T| - |T^*|) \leq \| |T| + |T^*| \|$$

for every $T \in \mathcal{B}(\mathcal{H})$.

REMARK 2. If we replace T with iT in Theorem 5 for each $T \in \mathcal{B}(\mathcal{H})$, we get

$$\omega(|T| - |T^*| \pm (T - T^*)) \leq \| |T| + |T^*| \|.$$

Specifically, if $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ and $T = \begin{bmatrix} O & A \\ B^* & O \end{bmatrix}$ for arbitrary $A, B \in \mathcal{B}(\mathcal{K})$, then the above inequality ensures that

$$\max\{\alpha_1, \alpha_2\} \leq \max\{\| |A| + |B| \|, \| |A^*| + |B^*| \| \},$$

where

$$\begin{aligned} \alpha_1 &= \omega \left(\begin{bmatrix} |B^*| - |A^*| & A - B \\ -(A - B)^* & |A| - |B| \end{bmatrix} \right), \\ \alpha_2 &= \omega \left(\begin{bmatrix} |B^*| - |A^*| & B - A \\ -(B - A)^* & |A| - |B| \end{bmatrix} \right). \end{aligned}$$

THEOREM 6. For every $T \in \mathcal{B}(\mathcal{H})$, we have

$$\omega(T) \leq \frac{1}{4} \| |T| + |T^*| \| + \frac{1}{2} \sqrt{\omega^2(T) + \frac{1}{4} \| |T| - |T^*| \|^2}.$$

Proof. Given any $x \in \mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$, let

$$\mathcal{C}(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \times \mathcal{A}(x) \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad (16)$$

where $\mathcal{A}(x)$ is defined by (15). Since $\mathcal{A}(x) \geq 0$, we have $\mathcal{C}(x) \geq 0$, and a direct computation yields

$$\mathcal{C}(x) = \begin{bmatrix} \left\langle \left(\frac{|T|+|T^*|}{2} - \Re T \right) x, x \right\rangle & \left\langle \left(\frac{|T|-|T^*|}{2} + i\Im T \right) x, x \right\rangle \\ \left\langle \left(\frac{|T|-|T^*|}{2} - i\Im T \right) x, x \right\rangle & \left\langle \left(\frac{|T|+|T^*|}{2} + \Re T \right) x, x \right\rangle \end{bmatrix}.$$

Observe that for each pair of real numbers a and b with $a+b \geq 0$, it is easy to verify that

$$\left\| \begin{pmatrix} a & \bar{c} \\ c & b \end{pmatrix} \right\| = \frac{a+b + \sqrt{(a-b)^2 + 4|c|^2}}{2}$$

for any complex number c . So, we may conclude that

$$\|\mathcal{C}(x)\| = \frac{1}{2} \langle (|T| + |T^*|)x, x \rangle + \sqrt{\Re T x, x)^2 + \mathcal{D}(x)},$$

where

$$\begin{aligned} \mathcal{D}(x) &= \left| \left\langle \left(\frac{|T|-|T^*|}{2} - i\Im T \right) x, x \right\rangle \right|^2 \\ &= \langle \Im T x, x \rangle^2 + \frac{1}{4} \langle (|T^*| - |T|)x, x \rangle^2. \end{aligned}$$

Hence,

$$\langle \Re T x, x \rangle^2 + \mathcal{D}(x) = |\langle T x, x \rangle|^2 + \frac{1}{4} \langle (|T^*| - |T|)x, x \rangle^2.$$

It follows from (15), Lemma 4 and (16) that

$$\begin{aligned} |\langle T x, x \rangle| &\leq \frac{1}{2} \|\mathcal{A}(x)\| = \frac{1}{2} \|\mathcal{C}(x)\| \\ &= \frac{1}{4} \langle (|T| + |T^*|)x, x \rangle + \frac{1}{2} \sqrt{|\langle T x, x \rangle|^2 + \frac{1}{4} \langle (|T^*| - |T|)x, x \rangle^2}. \end{aligned}$$

By taking supremum over all unit vectors $x \in \mathcal{H}$, we obtain

$$\omega(T) \leq \frac{1}{4} \| |T| + |T^*| \| + \frac{1}{2} \sqrt{\omega^2(T) + \frac{1}{4} \| |T| - |T^*| \|^2},$$

as desired. $\square$

REMARK 3. For each $x \in \mathcal{H}$, let

$$\mathcal{E}(x) = \left\langle \left(\frac{|T| - |T^*|}{2} - i\Im T \right) x, x \right\rangle.$$

Arguing as in the proof of Theorem 6, we have

$$\begin{aligned} 2|\mathcal{E}(x)| &\leq \|\mathcal{E}(x)\| = \|\mathcal{A}(x)\| \\ &= \frac{1}{2} \langle (|T| + |T^*|)x, x \rangle + \frac{1}{2} \sqrt{\langle (|T| - |T^*|)x, x \rangle^2 + 4|\langle Tx, x \rangle|^2} \\ &\leq \frac{1}{2} \langle (|T| + |T^*|)x, x \rangle + \frac{1}{2} |\langle (|T| - |T^*|)x, x \rangle| + |\langle Tx, x \rangle| \\ &= \max \{ \langle |T|x, x \rangle, \langle |T^*|x, x \rangle \} + |\langle Tx, x \rangle| \\ &\leq \|T\| + \omega(T), \end{aligned}$$

which implies that

$$\omega \left(\frac{|T| - |T^*|}{2} \pm i\Im T \right) \leq \min\{\beta_1, \beta_2\},$$

in which

$$\beta_1 = \frac{1}{4} \left(\| |T| + |T^*| \| + \sqrt{\| |T| - |T^*| \|^2 + 4\omega^2(T)} \right), \quad \beta_2 = \frac{\|T\| + \omega(T)}{2}.$$

Moreover, a replacement of T with iT will lead to

$$\omega \left(\frac{|T| - |T^*|}{2} \pm i\Re T \right) \leq \min\{\beta_1, \beta_2\}.$$

Numerical verification. To demonstrate that the bound obtained in (17) and the classical bound (18) are mutually non-comparable, we present the following numerical test using random matrices generated in Mathematica.

EXAMPLE 1. In Theorem 6, we have shown that

$$\omega(T) \leq \frac{1}{4} \| |T| + |T^*| \| + \frac{1}{2} \sqrt{\omega^2(T) + \frac{1}{4} \| |T| - |T^*| \|^2}. \quad (17)$$

On the other hand, as one of the most celebrated upper bounds for $\omega(T)$, we have [7]

$$\omega(T) \leq \frac{1}{2} \| |T| + |T^*| \|. \quad (18)$$

In the following two tables, we give four matrices, where either the bound found in (17) or (18) is better than the other. This shows that neither (17) nor (18) is uniformly

better than the other.

No.	T	$\omega(T)$	Bound in (17)	Bound in (18)
1	$\begin{pmatrix} 2 & 1 \\ 2 & 9 \end{pmatrix}$	9.30789	9.3146	9.31493
2	$\begin{pmatrix} 5 & 7 \\ 0 & 6 \end{pmatrix}$	9.03553	9.36738	9.502

No.	T	$\omega(T)$	Bound in (17)	Bound in (18)
1	$\begin{pmatrix} 0 & 0 \\ 9 & 10 \end{pmatrix}$	11.7268	12.1437	11.7268
2	$\begin{pmatrix} 0 & 2 \\ 6 & 0 \end{pmatrix}$	4.	4.23607	4

4.2. The rhombic numerical radius and applications to block matrices

Beyond the classical numerical radius, various generalizations have been introduced to capture more refined operator behavior. One such notion is the rhombic numerical radius, which symmetrically combines contributions from two operators. Building upon mixed Schwarz-type arguments, we establish inequalities for this quantity and illustrate its usefulness in analyzing block operator matrices.

LEMMA 6. *Let $a, b \in \mathbb{C}$. Then*

$$\sup_{\substack{\alpha, \beta \in \mathbb{R} \\ \alpha^2 + \beta^2 = 1}} \left| \alpha(a + \bar{b}) + \beta \left(\frac{a - \bar{b}}{i} \right) \right| = |a| + |b|.$$

Proof. Notice that

$$\begin{aligned} e^{i\theta}a + e^{-i\theta}\bar{b} &= (\cos \theta + i \sin \theta)a + (\cos \theta - i \sin \theta)\bar{b} \\ &= \cos \theta (a + \bar{b}) - \sin \theta \left(\frac{a - \bar{b}}{i} \right). \end{aligned} \tag{19}$$

Letting $\alpha = \cos \theta$ and $\beta = -\sin \theta$, in (19), we obtain

$$\left| e^{i\theta}a + e^{-i\theta}\bar{b} \right| = \left| \alpha(a + \bar{b}) + \beta \left(\frac{a - \bar{b}}{i} \right) \right|.$$

If we take the supremum over $\alpha^2 + \beta^2 = 1$, we infer the desired result, because

$$\sup_{\theta \in \mathbb{R}} \left| e^{i\theta}a + e^{-i\theta}\bar{b} \right| = |a| + |b|. \quad \square$$

The rhombic numerical radius has been defined in [2] in the following manner:

$$\omega_R(A, B) = \sup_{\substack{x \in \mathbb{H} \\ \|x\|=1}} (|\langle Ax, x \rangle| + |\langle Bx, x \rangle|).$$

THEOREM 7. Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

$$\frac{1}{\sqrt{2}} \omega \left(A + B^* + \frac{A - B^*}{i} \right) \leq \omega_R(A, B) \leq 2\omega \left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right).$$

Proof. Using Lemma 6 and Lemma 5, we obtain

$$\begin{aligned} |\langle Ax, x \rangle| + |\langle Bx, x \rangle| &= \sup_{\alpha^2 + \beta^2 = 1} \left| \alpha (\langle Ax, x \rangle + \langle x, Bx \rangle) + \beta \left(\frac{\langle Ax, x \rangle - \langle x, Bx \rangle}{i} \right) \right| \\ &= \sup_{\alpha^2 + \beta^2 = 1} \left| \left\langle \left(\alpha (A + B^*) + \beta \left(\frac{A - B^*}{i} \right) \right) x, x \right\rangle \right| \\ &\leq \sup_{\alpha^2 + \beta^2 = 1} \left\| \left(\alpha (A + B^*) + \beta \left(\frac{A - B^*}{i} \right) \right) x \right\| \\ &\leq \sup_{\alpha^2 + \beta^2 = 1} \left\| \alpha (A + B^*) + \beta \left(\frac{A - B^*}{i} \right) \right\| \\ &= 2\omega \left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right) \end{aligned}$$

for any unit vector $x \in \mathcal{H}$. After taking the supremum over all unit vectors $x \in \mathcal{H}$, we obtain

$$\omega_R(A, B) \leq 2\omega \left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right).$$

On the other hand,

$$\begin{aligned} |\langle Ax, x \rangle| + |\langle Bx, x \rangle| &= \sup_{\alpha^2 + \beta^2 = 1} \left| \left\langle \left(\alpha (A + B^*) + \beta \left(\frac{A - B^*}{i} \right) \right) x, x \right\rangle \right| \\ &\geq \frac{1}{\sqrt{2}} \left| \left\langle \left(A + B^* + \frac{A - B^*}{i} \right) x, x \right\rangle \right| \end{aligned}$$

which in turn implies that

$$\frac{1}{\sqrt{2}} \omega \left(A + B^* + \frac{A - B^*}{i} \right) \leq \omega_R(A, B). \quad \square$$

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