

FURTHER INEQUALITIES FOR $(\mathcal{M}_\phi, G)$ h -CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we introduce several new properties and inequalities for $(\mathcal{M}_\phi, G)$ h -convex functions, complementing the existing results in this field. Additionally, we explore multi-term multiplicative refinements of the classical Jensen-type inequality for $(\mathcal{M}_\phi, G)$ h -convex functions. As an application, by selecting specific $(\mathcal{M}_\phi, G)$ h -convex functions, we present novel inequalities for unitarily invariant norms. Our results extend and generalize some recent publications in this field, including [5, 11, 12].

1. Introduction and preliminaries

Convex functions are essential in numerous fields of mathematics, including analysis, optimization, mathematical physics, functional analysis, and operator theory. A convex function $f : I \rightarrow \mathbb{R}$ is distinguished by the property that it adheres to

$$f((1 - \eta)u + \eta v) \leq (1 - \eta)f(u) + \eta f(v), \quad (1.1)$$

for every $u, v \in I$ and $\eta \in [0, 1]$. A function f is called log-convex if it is positive and $\log f$ is convex. A recent research direction in mathematical inequalities focuses on minimizing the gap between the two sides of (1.1) by introducing specific terms. This inequality has been improved in the literature and has many applications in both scalar and matrix settings. Inspired by the numerous classes of convexity, we propose a new type that unifies several of them.

In this paper, we use I to denote a ϕ -convex subset of $\mathbb{R}$, which ϕ is a continuous function that is strictly monotone on an interval I and

$$\mathcal{M}_\phi(\eta, u, v) = \phi^{-1}[(1 - \eta)\phi(u) + \eta\phi(v)].$$

A subset $I \subset \mathbb{R}$ is called ϕ -convex if for all $u, v \in I$ and $\eta \in [0, 1]$, the element $\mathcal{M}_\phi(\eta, u, v) \in I$. Let $h : J \rightarrow \mathbb{R}$ be a non-negative, non-zero function with J as a subset of $\mathbb{R}$, and let $f : I \rightarrow \mathbb{R}$ be a function. Recall that f is called $(\mathcal{M}_\phi, A)$ h -convex [15],

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if I is a ϕ -convex set and for every $u, v \in I$ and $\eta \in [0, 1]$, the following inequality holds

$$f(\mathcal{M}_\phi(\eta, u, v)) \leq h(1 - \eta)f(u) + h(\eta)f(v), \quad (1.2)$$

with the notations previously defined. We recall that f is called a $(\mathcal{M}_\phi, G)$ h -convex function if it meets the following inequality

$$f(\mathcal{M}_\phi(\eta, u, v)) \leq f^{h(1-\eta)}(u)f^{h(\eta)}(v). \quad (1.3)$$

The well-known Jensen-type inequality for h -convex functions in the context of $(\mathcal{M}_\phi, G)$, which extends inequality (1.3) to n parameters, is given as follows

$$f\left(\phi^{-1}\left(\sum_{k=1}^n \eta_k \phi(u_k)\right)\right) \leq \prod_{k=1}^n f^{h(\eta_k)}(u_k), \quad \sum_{k=1}^n \eta_k = 1.$$

It is evident that if $h = \text{id}$ and $\phi = \text{id}$ (where id represents the identity function) in (1.2), we recover the classical definition of convexity. Moreover, if $\phi(x) = x^p$ for $p \in \mathbb{R}$, we obtain the familiar definition of (p, h) -convexity [4]. Additionally, when $\phi(x) = \ln(x)$ and $h = \text{id}$ in inequality (1.3), we recover the classical definition of geometrically convex functions (or (G-G) convex functions) [12], which can be expressed in the following inequality

$$f(u^{1-\eta}v^\eta) \leq f^{1-\eta}(u)f^\eta(v). \quad (1.4)$$

The concept of (G-G)-convex functions is becoming increasingly significant in the advancement of operator and norm inequalities. In various studies [6, 9, 12], the authors leveraged the (G-G)-convexity of certain functions to introduce several improvements and refinements to norm and numerical radius inequalities for bounded linear operators on a complex Hilbert space. A reversed form of inequality (1.4), for the case $\eta > 0$ or $\eta < -1$ can be stated as follows.

PROPOSITION 1.1. ([12]) *Let f be geometrically convex function on the interval $(0, \infty)$ and $u, v > 0$. Then for any $\eta > 0$ or $\eta < -1$,*

$$f(u^{1+\eta}v^{-\eta}) \geq f^{1+\eta}(u)f^{-\eta}(v).$$

Equivalently,

$$f(u^{1-\eta}v^\eta) \geq f^{1-\eta}(u)f^\eta(v), \quad \eta \notin [0, 1].$$

A significant refinement and reversal of inequality (1.4), established by Sababheh et al. in [12], is as follows

THEOREM 1.2. *Let f be a geometrically convex on the interval I and $u, v \in I$. Then for any $\eta \in [0, 1]$,*

$$\left(\frac{\sqrt{f(u)f(v)}}{f(\sqrt{uv})}\right)^{2r} f(u^{1-\eta}v^\eta) \leq f^{1-\eta}(u)f^\eta(v) \leq \left(\frac{\sqrt{f(u)f(v)}}{f(\sqrt{uv})}\right)^{2R} f(u^{1-\eta}v^\eta), \quad (1.5)$$

where $r = \min\{\eta, 1 - \eta\}$ and $R = \max\{\eta, 1 - \eta\}$.

The classical Jensen-type inequality for (G, G) -convex functions can be formulated as follows

$$f\left(\prod_{k=1}^n u_k^{\eta_k}\right) \leq \prod_{k=1}^n f^{\eta_k}(u_k), \quad \sum_{k=1}^n \eta_k = 1.$$

Moreover, the first inequality in Proposition 1.1 can be generalized as follows.

PROPOSITION 1.3. *Let $u, v_k, \eta_k \geq 0$ and let $\xi = \sum_{k=1}^n \eta_k$. If the function $f : (0, \infty) \rightarrow (0, \infty)$ is geometrically convex, then*

$$f\left(u^{1+\xi} \prod_{k=1}^n v_k^{-\eta_k}\right) \geq f^{1+\xi}(u) \prod_{k=1}^n f^{-\eta_k}(v_k).$$

This generalization mirrors similar extensions for convex and log-convex functions found in [14]. For additional results on Jensen's type inequality related to specific types of convexity, readers are encouraged to consult the following references [1, 3, 7, 10, 13].

In this paper, we focused on $(\mathcal{M}_\phi, G)$ h -convex functions. More specifically, we extended the inequalities discussed in the introduction to a broader class of convex functions.

This paper is organized as follows: In Section 2, we present new properties of $(\mathcal{M}_\phi, G)$ h -convex functions, including the $\mathcal{M}_\phi$ -super-additivity of the function f under certain conditions. We also extend the inequalities in Theorem 1.2 to the class of $(\mathcal{M}_\phi, G)$ h -convex functions and introduce additional multiplicative refinement terms. Furthermore, we discuss the case when $\eta \notin [0, 1]$. In Section 3, we explore a reverse version of Jensen's type inequality for $(\mathcal{M}_\phi, G)$ convex functions. In Section 4, by selecting several rich examples of such $(\mathcal{M}_\phi, G)$ convex functions, we derive new inequalities for unitarily invariant norms.

2. Preliminaries and auxiliary results

This section is divided into three parts: in the first part, we examine several important properties of the class of $(\mathcal{M}_\phi, G)$ h -convex functions. In the second part, we extend and present new improvements to the inequalities (1.3), (1.4) and (1.5). In the final part, we explore the case when $\eta \notin (0, 1)$.

We begin this section by revisiting several categories of functions. A function $h : J \rightarrow \mathbb{R}$ is termed super-multiplicative if, for all $u, v \in J$, $uv \in J$ and the following condition holds:

$$h(u)h(v) \leq h(uv). \quad (2.1)$$

If the inequality (2.1) is reversed, h is referred to as a sub-multiplicative function. If equality holds in (2.1), h is called a multiplicative function. Besides, if $u + v \in J$ and

$$h(u) + h(v) \leq h(u + v), \quad (2.2)$$

the function h is considered super-additive if the inequality (2.2) holds. If the inequality is reversed, h is called sub-additive. If the equality in (2.2) is satisfied, h is described

as an additive function. Specific examples of super-additive and super-multiplicative functions can be found in [8].

2.1. Additional properties for $(\mathcal{M}_\phi, G)$ and $(\mathcal{M}_\phi, A)$ h -convex functions

We commence our findings with the following proposition, offering an equivalent representation of the definition of $(\mathcal{M}_\phi, G)$ h -convexity. For the remainder of this subsection, we assume that $0 \in \phi(I)$. Given that ϕ is continuous and strictly monotonic, the Intermediate Value Theorem guarantees the existence of a number c such that $\phi(c) = 0$.

PROPOSITION 2.1. *Let $f : (0, \infty) \rightarrow (0, \infty)$ and $h : J \rightarrow \mathbb{R}^+$ be a super-additive and super-multiplicative function. The conditions listed below are equivalent.*

i) f is $(\mathcal{M}_\phi, G)$ h -convex function with $f(c) \leq 1$.

ii) For any $u, v > 0$, and $\eta, \theta \geq 0$ with $\eta + \theta \leq 1$,

$$f(\phi^{-1}(\eta\phi(u) + \theta\phi(v))) \leq f^{h(\eta)}(u)f^{h(\theta)}(v).$$

Proof. Assume first that f is $(\mathcal{M}_\phi, G)$ h -convex function with $f(c) \leq 1$. Then, according to the definition of $(\mathcal{M}_\phi, G)$ h -convexity, and $\phi(c) = 0$, it follows that

$$\begin{aligned} & f(\phi^{-1}(\eta\phi(u) + \theta\phi(v))) \\ &= f\left[\phi^{-1}\left((\eta + \theta)\phi\left(\phi^{-1}\left(\frac{\eta}{\eta + \theta}\phi(u) + \frac{\theta}{\eta + \theta}\phi(v)\right)\right) + (1 - \eta - \theta)\phi(c)\right)\right] \\ &\leq f^{h(\eta + \theta)}\left(\phi^{-1}\left(\frac{\eta}{\eta + \theta}\phi(u) + \frac{\theta}{\eta + \theta}\phi(v)\right)\right)f^{h(1 - (\eta + \theta))}(c) \\ &\leq f^{h(\eta + \theta)}\left(\phi^{-1}\left(\frac{\eta}{\eta + \theta}\phi(u) + \frac{\theta}{\eta + \theta}\phi(v)\right)\right) \\ &\leq \left(f^{h\left(\frac{\eta}{\eta + \theta}\right)}(u)f^{h\left(\frac{\theta}{\eta + \theta}\right)}(v)\right)^{h(\eta + \theta)} \\ &\leq f^{h(\eta)}(u)f^{h(\theta)}(v). \end{aligned}$$

This establishes the implication (i) $\implies$ (ii). To prove the implication (ii) $\implies$ (i), we take $u = v = c$ in the inequality given in (ii)

$$f(c) = f(\phi^{-1}(0)) \leq f^{h(\eta) + h(\theta)}(c) \leq f^{h(\eta + \theta)}(c).$$

Since that h is a super-additive function, it follows that h is increasing, and thus we have $h(\eta + \theta) \leq h(1) \leq 1$. This shows that $f^{1 - h(1)}(c) \leq 1$. So $f(c) \leq 1$. $\square$

REMARK 2.2. If we choose $\phi(x) = \ln(x)$ and $h(x) = x$, we recover Proposition 2.1 from [12].

In the next proposition, we demonstrate that the class of $(\mathcal{M}_\phi, G)$ h -convex functions is stable under multiplication and addition.

PROPOSITION 2.3. *Let f_1, f_2 be two non-negative $(\mathcal{M}_\phi, G)$ h -convex functions and $\lambda > 0$, and suppose that $h : J \rightarrow \mathbb{R}^+$ satisfies $h(1 - \eta) + h(\eta) = 1$. Then, the functions λf_1 , $f_1 + f_2$ and $f_1 f_2$ are also $(\mathcal{M}_\phi, G)$ h -convex.*

Proof. We have

$$\begin{aligned} (\lambda f_1)(\mathcal{M}_\phi(\eta, u, v)) &\leq \lambda^{h(1-\eta)+h(\eta)} f_1^{h(1-\eta)}(u) f_1^{h(\eta)}(v) \\ &\leq (\lambda f_1)^{h(1-\eta)}(u) (\lambda f_1)^{h(\eta)}(v). \end{aligned}$$

This proves that λf_1 is $(\mathcal{M}_\phi, G)$ h -convex. For $f_1 + f_2$ we have

$$\begin{aligned} (f_1 + f_2)(\mathcal{M}_\phi(\eta, u, v)) &\leq f_1^{h(1-\eta)}(u) f_1^{h(\eta)}(v) + f_2^{h(1-\eta)}(u) f_2^{h(\eta)}(v) \\ &\leq (f_1(u) + f_2(u))^{h(1-\eta)} (f_1(v) + f_2(v))^{h(\eta)} \\ &= (f_1 + f_2)^{h(1-\eta)}(u) (f_1 + f_2)^{h(\eta)}(v), \end{aligned}$$

where the first inequality is derived using the following classical inequality:

$$a^{h(1-\eta)} b^{h(\eta)} + c^{h(1-\eta)} d^{h(\eta)} \leq (a + c)^{h(1-\eta)} (b + d)^{h(\eta)}.$$

This can be inferred from the concavity of the function $f(t) = \left(\frac{a+b}{c+d}\right)^{h(\alpha)}$ with the weights $\eta = \frac{c}{c+d}$ and $\theta = \frac{d}{c+d}$. Similarly, for $f_1 f_2$, we obtain

$$\begin{aligned} (f_1 f_2)(\mathcal{M}_\phi(\eta, u, v)) &\leq f_1^{h(1-\eta)}(u) f_1^{h(\eta)}(v) f_2^{h(1-\eta)}(u) f_2^{h(\eta)}(v) \\ &\leq (f_1 f_2)^{h(1-\eta)}(u) (f_1 f_2)^{h(\eta)}(v). \quad \square \end{aligned}$$

It's well established that for any convex function f with $f(0) = 0$, is super-additive in the sense that $f(u + v) \geq f(u) + f(v)$. The following theorems extend this property to the class of $(\mathcal{M}_\phi, G)$ and $(\mathcal{M}_\phi, A)$ h -convex functions.

THEOREM 2.4. *Let f be a non-negative $(\mathcal{M}_\phi, G)$ h -convex function on I , and $c \in I$ be such that $\phi(c) = 0$. Then for $f(c) \leq 1$ we have*

$$f^{h\left(\frac{\phi(u)}{\phi(u)+\phi(v)}\right)+h\left(\frac{\phi(v)}{\phi(u)+\phi(v)}\right)}(\phi^{-1}(\phi(u) + \phi(v))) \geq f(u)f(v).$$

Specifically, if h is a super-additive function, then we have

$$f^{h(1)}(\phi^{-1}(\phi(u) + \phi(v))) \geq f(u)f(v).$$

Proof. We have

$$\begin{aligned} f(\phi^{-1}(\eta\phi(u))) &= f(\mathcal{M}_\phi(\eta, c, u)) \\ &\leq f^{h(\eta)}(u)f^{h(1-\eta)}(c) \\ &\leq f^{h(\eta)}(u). \end{aligned} \quad (2.3)$$

By substituting η with $\frac{\phi(u)}{\phi(u)+\phi(v)}$ in the inequality above, we obtain the following

$$\begin{aligned} f^{h\left(\frac{\phi(u)}{\phi(u)+\phi(v)}\right)}(\phi^{-1}(\phi(u)+\phi(v))) &\geq f\left(\phi^{-1}\left(\frac{\phi(u)}{\phi(u)+\phi(v)}(\phi(u)+\phi(v))\right)\right) \\ &= f(u). \end{aligned}$$

In a similar manner, we have

$$f^{h\left(\frac{\phi(v)}{\phi(u)+\phi(v)}\right)}(\phi^{-1}(\phi(u)+\phi(v))) \geq f(v).$$

By combining the last two inequalities, we arrive at the desired result. $\square$

REMARK 2.5. If we choose $\phi(x) = \ln(x)$ and $h(x) = x$, we recover Theorem 2.3 from [12].

THEOREM 2.6. Let f be a non-negative $(\mathcal{M}_\phi, A)$ h -convex function on I , and $c \in I$ be such that $\phi(c) = 0$. Then for $f(c) = 0$ we have

$$\left(h\left(\frac{\phi(u)}{\phi(u)+\phi(v)}\right) + h\left(\frac{\phi(v)}{\phi(u)+\phi(v)}\right)\right) f(\phi^{-1}(\phi(u)+\phi(v))) \geq f(u) + f(v).$$

Specifically, if h is a super-additive function, then we have

$$h(1)f(\phi^{-1}(\phi(u)+\phi(v))) \geq f(u) + f(v).$$

Proof. We have

$$\begin{aligned} f(\phi^{-1}(\eta\phi(u))) &= f(\mathcal{M}_\phi(\eta, c, u)) \\ &\leq h(\eta)f(u) + h(1-\eta)f(c) \\ &\leq h(\eta)f(u). \end{aligned}$$

By substituting η with $\frac{\phi(u)}{\phi(u)+\phi(v)}$ and u by $\phi^{-1}(\phi(u)+\phi(v))$ in the inequality above, we obtain the following

$$\begin{aligned} h\left(\frac{\phi(u)}{\phi(u)+\phi(v)}\right) f(\phi^{-1}(\phi(u)+\phi(v))) &\geq f\left(\phi^{-1}\left(\frac{\phi(u)}{\phi(u)+\phi(v)}(\phi(u)+\phi(v))\right)\right) \\ &= f(u). \end{aligned}$$

In a similar manner, we have

$$h\left(\frac{\phi(v)}{\phi(u)+\phi(v)}\right) f(\phi^{-1}(\phi(u)+\phi(v))) \geq f(v).$$

By consolidating the last two inequalities, we reach the desired conclusion. $\square$

REMARK 2.7. By selecting $\phi(x) = x$ and $h(x) = x$ in the proposition above, we retrieve the classical super-additivity property of a convex function f , assuming the condition $f(0) = 0$.

In the following section, we introduce a refined approach to the concept of $(\mathcal{M}_\phi, G)$ h -convex functions. These refinements build upon the inequality stated in (1.2), enhancing their mathematical framework.

2.2. The case $\eta \in [0, 1]$

In the next theorem, we introduce a substantial refinement and a graceful reversal of inequality (1.3), bringing forth a profound improvement to its formulation. This presents a refined version of inequality (1.3) by introducing two weights.

THEOREM 2.8. Let f be a non-negative $(\mathcal{M}_\phi, G)$ h -convex function on I , and $0 < \eta \leq \theta < 1$. we assume that h is non-negative and both super-multiplicative and super-additive. Then for any $u, v \in I$, we have

$$\left(\frac{f^{h(1-\theta)}(u)f^{h(\theta)}(v)}{f(\mathcal{M}_\phi(\theta, u, v))} \right)^{h(\frac{\eta}{\theta})} \leq \frac{f^{h(1-\eta)}(u)f^{h(\eta)}(v)}{f(\mathcal{M}_\phi(\eta, u, v))} \leq \left(\frac{f^{h(1-\theta)}(u)f^{h(\theta)}(v)}{f(\mathcal{M}_\phi(\theta, u, v))} \right)^{h(\frac{1-\eta}{1-\theta})}.$$

Proof. Since f be a non-negative $(\mathcal{M}_\phi, G)$ h -convex function on I , and h is non-negative and both super-multiplicative and super-additive. Then

$$\begin{aligned} & f^{h(1-\eta)}(u)f^{h(\eta)}(v) \times \left(\frac{f^{h(1-\theta)}(u)f^{h(\theta)}(v)}{f(\mathcal{M}_\phi(\theta, u, v))} \right)^{-h(\frac{\eta}{\theta})} \\ &= f^{h(1-\eta)-h(1-\theta)h(\frac{\eta}{\theta})}(u)f^{h(\eta)-h(\theta)h(\frac{\eta}{\theta})}(v) \times f(\mathcal{M}_\phi(\theta, u, v))^{h(\frac{\eta}{\theta})} \\ &\geq f^{h(1-\frac{\eta}{\theta})}(u) \times f(\mathcal{M}_\phi(\theta, u, v))^{h(\frac{\eta}{\theta})} \\ &\geq f\left(\left(1-\frac{\eta}{\theta}\right)u + \frac{\eta}{\theta}\mathcal{M}_\phi(\theta, u, v)\right) \\ &= f(\mathcal{M}_\phi(\eta, u, v)). \end{aligned}$$

The second inequality is satisfied directly by using the fact that if $0 \leq \eta \leq \theta \leq 1$, then $0 \leq 1-\theta \leq 1-\eta \leq 1$, and by substituting η, θ, u , and v with $1-\theta, 1-\eta, v$, and u , respectively. $\square$

A special case of Theorem 2.8 provides the following extension of Theorem 1.2 to the concept of $(\mathcal{M}_\phi, G)$ h -convexity.

COROLLARY 2.9. Let f be a non-negative $(\mathcal{M}_\phi, G)$ h -convex function on I , and $0 \leq \eta \leq 1$. We assume that h is non-negative and both super-multiplicative and super-additive. Then for any $u, v \in I$, we have the following

$$\left(\frac{f^{h(\frac{1}{2})}(u)f^{h(\frac{1}{2})}(v)}{f(\mathcal{M}_\phi(\frac{1}{2}, u, v))} \right)^{h(2r_0)} \leq \frac{f^{h(1-\eta)}(u)f^{h(\eta)}(v)}{f(\mathcal{M}_\phi(\eta, u, v))} \leq \left(\frac{f^{h(\frac{1}{2})}(u)f^{h(\frac{1}{2})}(v)}{f(\mathcal{M}_\phi(\frac{1}{2}, u, v))} \right)^{h(2R_0)},$$

where $r_0 = \min\{\eta, 1 - \eta\}$ and $R_0 = \max\{\eta, 1 - \eta\}$.

REMARK 2.10. Note that Corollary 2.9 extends Theorem 1.2 to the concept of $(\mathcal{M}_\phi, G)$ h -convexity. It also generalizes Proposition 1 from [11] to this type of convexity.

In the following lemma, we present a refined version and reversed of the following inequality.

$$f(\phi^{-1}(\eta\phi(x))) \leq f^{h(\eta)}(x). \quad (2.4)$$

LEMMA 2.11. Let f be a non-negative $(\mathcal{M}_\phi, G)$ h -convex function on I , with $f(c) \leq 1$, $\eta \in [0, 1]$ and $x \in I$. we assume that h is non-negative and both super-multiplicative and super-additive. Then

$$\begin{aligned} & \left(\frac{f^{h(\frac{1}{2})}(x)}{f(\phi^{-1}(\frac{\phi(x)}{2}))} \right)^{h(2r_0)} f(\phi^{-1}(\eta\phi(x))) \\ & \leq f^{h(\eta)}(x) \leq \left(\frac{f^{h(\frac{1}{2})}(x)}{f(\phi^{-1}(\frac{\phi(x)}{2}))} \right)^{h(2R_0)} f(\phi^{-1}(\eta\phi(x))), \end{aligned} \quad (2.5)$$

where $r_0 = \min\{\eta, 1 - \eta\}$ and $R_0 = \max\{\eta, 1 - \eta\}$.

Proof. First, note that inequality (2.4) follows directly from (2.3). By Corollary 2.9, we have for $u, v \in I$

$$f^{h(1-\eta)}(u)f^{h(\eta)}(v) \leq \left(\frac{f^{h(\frac{1}{2})}(u)f^{h(\frac{1}{2})}(v)}{f(\mathcal{M}_\phi(\frac{1}{2}, u, v))} \right)^{h(2R_0)} f(\mathcal{M}_\phi(\eta, u, v)), \quad (2.6)$$

and

$$f^{h(1-\eta)}(u)f^{h(\eta)}(v) \geq \left(\frac{f^{h(\frac{1}{2})}(u)f^{h(\frac{1}{2})}(v)}{f(\mathcal{M}_\phi(\frac{1}{2}, u, v))} \right)^{h(2r_0)} f(\mathcal{M}_\phi(\eta, u, v)). \quad (2.7)$$

Putting $u = c$ and $v = x$ in both (2.6) and (2.7), we get

$$f^{h(\eta)}(x) \leq \left(\frac{f^{h(\frac{1}{2})}(x)}{f(\phi^{-1}(\frac{\phi(x)}{2}))} \right)^{h(2R_0)} f(\phi^{-1}(\eta\phi(x))) f^{(h(2R_0)h(\frac{1}{2})-h(1-\eta))}(c),$$

and

$$f^{h(\eta)}(x) \geq \left(\frac{f^{h(\frac{1}{2})}(x)}{f(\phi^{-1}(\frac{\phi(x)}{2}))} \right)^{h(2r_0)} f(\phi^{-1}(\eta\phi(x))) f^{(h(2r_0)h(\frac{1}{2})-h(1-\eta))}(c).$$

However, since h is both multiplicative and increasing, we have $h(2R_0)h(\frac{1}{2}) - h(1 - \eta) = h(R_0) - h(1 - \eta) \geq 0$, and $h(2r_0)h(\frac{1}{2}) - h(1 - \eta) = h(r_0) - h(1 - \eta) \leq 0$. Since $f(c) \leq 1$ we obtain

$$f^{h(\eta)}(x) \leq \left(\frac{f^{h(\frac{1}{2})}(x)}{f\left(\phi^{-1}\left(\frac{\phi(x)}{2}\right)\right)} \right)^{h(2R_0)} f\left(\phi^{-1}(\eta\phi(x))\right),$$

and

$$f^{h(\eta)}(x) \geq \left(\frac{f^{h(\frac{1}{2})}(x)}{f\left(\phi^{-1}\left(\frac{\phi(x)}{2}\right)\right)} \right)^{h(2r_0)} f\left(\phi^{-1}(\eta\phi(x))\right). \quad \square$$

In the following theorem, we provide an improvement of Theorem 2.4.

THEOREM 2.12. *Let h be a non-negative, multiplicative, and super-additive function, and f be a non-negative $(\mathcal{M}_\phi, G)$ h -convex function on I , with $f(c) \leq 1$. Then*

$$f(\phi^{-1}(\phi(u) + \phi(v))) \leq f(u)f(v) \left(\frac{f\left(\phi^{-1}\left(\frac{\phi(u)}{2}\right)\right)f\left(\phi^{-1}\left(\frac{\phi(v)}{2}\right)\right)}{(f(u)f(v))^{h(\frac{1}{2})}} \right)^{h(2R_0)}.$$

Proof. By observing that

$$\phi(u) + \phi(v) = (1 - \eta)\phi\left(\phi^{-1}\left(\frac{1}{1 - \eta}\phi(u)\right)\right) + \eta\phi\left(\phi^{-1}\left(\frac{1}{\eta}\phi(v)\right)\right),$$

we get by $(\mathcal{M}_\phi, G)$ h -convexity of f and the second inequality in (2.5),

$$\begin{aligned} f(\phi^{-1}(\phi(u) + \phi(v))) &= f\left(\mathcal{M}_\phi\left(\eta, \phi^{-1}\left(\frac{1}{1 - \eta}\phi(u)\right), \phi^{-1}\left(\frac{1}{\eta}\phi(v)\right)\right)\right) \\ &\leq f^{h(1 - \eta)}\left(\phi^{-1}\left(\frac{1}{1 - \eta}\phi(u)\right)\right) f^{h(\eta)}\left(\phi^{-1}\left(\frac{1}{\eta}\phi(v)\right)\right) \\ &\leq f(u)f(v) \left(\frac{f\left(\phi^{-1}\left(\frac{\phi(u)}{2}\right)\right)f\left(\phi^{-1}\left(\frac{\phi(v)}{2}\right)\right)}{(f(u)f(v))^{h(\frac{1}{2})}} \right)^{h(2R_0)}. \end{aligned}$$

The last inequality is obtained by multiplying two inequalities from (2.5) after substituting $x = \phi^{-1}\left(\frac{1}{1 - \eta}\phi(u)\right)$ and $x = \phi^{-1}\left(\frac{1}{\eta}\phi(v)\right)$. $\square$

For a more detailed refinement of Theorem 2.8 and Corollary 2.9 we introduce certain definitions that will be applied in our proofs.

DEFINITION 2.13. Let n be a positive integer. We consider the sequence of functions $(r_n(\eta))$ on the interval $[0, 1]$, defined as follows

$$r_0(\eta) = \min\{\eta, 1 - \eta\},$$

$$r_n(\eta) = \min\{2r_{n-1}(\eta), 1 - 2r_{n-1}(\eta)\}.$$

For every integer n , we have the following explicit formula for the function $r_n(\eta)$, as demonstrated by D. Choi et al. in [2].

LEMMA 2.14. ([2]) Let $l \geq 0$, and $1 \leq k \leq 2^n$ be integers. If $\frac{k-1}{2^n} \leq \eta \leq \frac{k}{2^n}$, then

$$r_n(\eta) = \begin{cases} 2^n \eta - k + 1, & \text{if } \frac{k-1}{2^n} \leq \eta \leq \frac{2k-1}{2^{n+1}}, \\ k - 2^n \eta, & \text{if } \frac{2k-1}{2^{n+1}} \leq \eta \leq \frac{k}{2^n}. \end{cases}$$

We also require the following lemma to complete our refinement of Corollary 2.9.

LEMMA 2.15. Let f be a non-negative function defined on I with $u, v \in I$. For a non-negative integer N , define $\Phi_N(\eta)$ by

$$\Phi_N(\eta) = \frac{f^{h(1-\eta)}(u)f^{h(\eta)}(v)}{\prod_{n=0}^{N-1} \left(\prod_{k=1}^{2^n} \left(\Delta_{f,\phi,h}^{(u,v)}(n, k) \right)^{\chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(\eta)} \right)^{h(2r_n(\eta))}},$$

where

$$\Delta_{f,\phi,h}^{(u,v)}(n, k) = \frac{[f(\mathcal{M}_\phi(\frac{k-1}{2^n}, u, v)) f(\mathcal{M}_\phi(\frac{k}{2^n}, u, v))]^{h(\frac{1}{2})}}{f(\mathcal{M}_\phi(\frac{2k-1}{2^{n+1}}, u, v))},$$

for $\frac{k-1}{2^N} \leq \eta \leq \frac{k}{2^N}$ and $k = 1, \dots, 2^N$. We have

$$\Phi_N(\eta) \geq f^{h(k-2^N\eta)} \left(\mathcal{M}_\phi \left(\frac{k-1}{2^N}, u, v \right) \right) f^{h(2^N\eta-k+1)} \left(\mathcal{M}_\phi \left(\frac{k}{2^N}, u, v \right) \right). \quad (2.8)$$

Proof. The proof is carried out using induction on N . For the base case $N = 1$, if $\eta \in [0, \frac{1}{2}]$, we obtain

$$\begin{aligned} \Phi_1(\eta) &= \frac{f^{h(1-\eta)}(u)f^{h(\eta)}(v)}{\left(\Delta_{f,\phi,h}^{(u,v)}(0, 1) \right)^{\chi_{(0,1)}(\eta)h(2r_0\eta)}} \\ &= \frac{f^{h(1-\eta)}(u)f^{h(\eta)}(v)}{\left(\Delta_{f,\phi,h}^{(u,v)}(0, 1) \right)^{h(2\eta)}} \\ &= f^{h(1-\eta)-h(\frac{1}{2})h(2\eta)}(u)f^{h(\eta)-h(\frac{1}{2})h(2\eta)}(v)f^{h(2\eta)} \left(\mathcal{M}_\phi \left(\frac{1}{2}, u, v \right) \right) \end{aligned}$$

$$\begin{aligned} &\geq f^{h(1-\eta)-h(\frac{1}{2})h(2\eta)}(u)f^{h(2\eta)}\left(\mathcal{M}_\phi\left(\frac{1}{2}, u, v\right)\right) \\ &\geq f^{h(1-2\eta)}(u)f^{h(2\eta)}\left(\mathcal{M}_\phi\left(\frac{1}{2}, u, v\right)\right). \end{aligned}$$

If $\eta \in [\frac{1}{2}, 1]$, then $1 - \eta \in [0, \frac{1}{2}]$. So by changing η by $1 - \eta$ and $h(\eta)$ by $h(1 - \eta)$ the desired inequality for the case $\eta \in [\frac{1}{2}, 1]$ is obtained.

Now, assume that (2.8) holds and let $\frac{m-1}{2^{N+1}} \leq \eta \leq \frac{m}{2^{N+1}}$ for $m = 1, \dots, 2^{N+1}$. If $l = 2k - 1$, then $\frac{k-1}{2^N} \leq \eta \leq \frac{2k-1}{2^{N+1}} < \frac{k}{2^N}$ and

$$\begin{aligned} \Phi_{N+1}(\eta) &= \frac{\Phi_N(\eta)}{\left(\Delta_{f,\phi,h}^{(u,v)}(N, k)\right)^{h(2^{N+1}\eta-2k+2)}\chi_{\left(\frac{k-1}{2^N}, \frac{k}{2^N}\right)}(\eta)} \\ &\geq \frac{f^{h(k-2^N\eta)}\left(\mathcal{M}_\phi\left(\frac{k-1}{2^N}, u, v\right)\right)f^{h(2^N\eta-k+1)}\left(\mathcal{M}_\phi\left(\frac{k}{2^N}, u, v\right)\right)}{\left[\frac{\left[f\left(\mathcal{M}_\phi\left(\frac{k-1}{2^N}, u, v\right)\right)f\left(\mathcal{M}_\phi\left(\frac{k}{2^N}, u, v\right)\right)\right]^{h(\frac{1}{2})}}{f\left(\mathcal{M}_\phi\left(\frac{2k-1}{2^{N+1}}, u, v\right)\right)}\right]^{h(2^{N+1}\eta-2k+2)}} \\ &\geq f^{h(2k-2^{N+1}\eta-1)}\left(\mathcal{M}_\phi\left(\frac{k-1}{2^N}, u, v\right)\right)f^{h(2^{N+1}\eta-2k+2)}\left(\mathcal{M}_\phi\left(\frac{2k-1}{2^{N+1}}, u, v\right)\right) \\ &= f^{h(2k-2^{N+1}\eta-1)}\left(\mathcal{M}_\phi\left(\frac{2k-2}{2^{N+1}}, u, v\right)\right)f^{h(2^{N+1}\eta-2k+2)}\left(\mathcal{M}_\phi\left(\frac{2k-1}{2^{N+1}}, u, v\right)\right) \\ &= f^{h(l-2^{N+1}\eta)}\left(\mathcal{M}_\phi\left(\frac{l-1}{2^{N+1}}, u, v\right)\right)f^{h(2^{N+1}\eta-l+1)}\left(\mathcal{M}_\phi\left(\frac{l}{2^{N+1}}, u, v\right)\right), \end{aligned}$$

by Lemma 2.14. Similarly, if $l = 2k$, then $\frac{k-1}{2^N} < \frac{2k-1}{2^{N+1}} \leq \eta \leq \frac{k}{2^N}$ and

$$\begin{aligned} \Phi_{N+1}(\eta) &= \frac{\Phi_N(\eta)}{\left(\Delta_{f,\phi,h}^{(u,v)}(N, k)\right)^{h(2k-2^{N+1}\eta)}} \\ &\geq \frac{f^{h(k-2^N\eta)}\left(\mathcal{M}_\phi\left(\frac{k-1}{2^N}, u, v\right)\right)f^{h(2^N\eta-k+1)}\left(\mathcal{M}_\phi\left(\frac{k}{2^N}, u, v\right)\right)}{\left[\frac{\left[f\left(\mathcal{M}_\phi\left(\frac{k-1}{2^N}, u, v\right)\right)f\left(\mathcal{M}_\phi\left(\frac{k}{2^N}, u, v\right)\right)\right]^{h(\frac{1}{2})}}{f\left(\mathcal{M}_\phi\left(\frac{2k-1}{2^{N+1}}, u, v\right)\right)}\right]^{h(2k-2^{N+1}\eta)}} \\ &\geq f^{h(2^{N+1}-2k+1)}\left(\mathcal{M}_\phi\left(\frac{k}{2^N}, u, v\right)\right)f^{h(2k-2^{N+1}\eta)}\left(\mathcal{M}_\phi\left(\frac{2k-1}{2^{N+1}}, u, v\right)\right) \\ &= f^{h(2^{N+1}-l+1)}\left(\mathcal{M}_\phi\left(\frac{l}{2^{N+1}}, u, v\right)\right)f^{h(l-2^{N+1}\eta)}\left(\mathcal{M}_\phi\left(\frac{l-1}{2^{N+1}}, u, v\right)\right). \quad \square \end{aligned}$$

As a consequence of the previous lemma, we obtain the following refinement and reversed of Corollary 2.9, featuring multiple multiplicative terms.

THEOREM 2.16. *Let f be a non-negative $(\mathcal{M}_\phi, G)$ h -convex function on I , and $u, v \in I$. For any non-negative integer N , we have the following inequalities.*

$$f^{h(1-\eta)}(u)f^{h(\eta)}(v) \geq f(\mathcal{M}_\phi(\eta, u, v)) \prod_{n=0}^{N-1} \left(\prod_{k=1}^{2^n} \left(\Delta_{f, \phi, h}^{(u, v)}(n, k) \right)^{\chi(\frac{k-1}{2^n}, \frac{k}{2^n})(\eta)} \right)^{h(2r_n(\eta))}, \quad (2.9)$$

and

$$\begin{aligned} & f^{h(1-\eta)}(u)f^{h(\eta)}(v) \\ & \leq \frac{f(u)f(v)}{f(\mathcal{M}_\phi(1-\eta, u, v)) \prod_{n=0}^{N-1} \left(\prod_{k=1}^{2^n} \left(\Delta_{f, \phi, h}^{(u, v)}(n, 2^n - k + 1) \right)^{\chi(\frac{k-1}{2^n}, \frac{k}{2^n})(\eta)} \right)^{h(2r_n(\eta))}}, \end{aligned} \quad (2.10)$$

where

$$\Delta_{f, \phi, h}^{(u, v)}(n, k) = \frac{[f(\mathcal{M}_\phi(\frac{k-1}{2^n}, u, v))f(\mathcal{M}_\phi(\frac{k}{2^n}, u, v))]^{h(\frac{1}{2})}}{f(\mathcal{M}_\phi(\frac{2k-1}{2^{n+1}}, u, v))}.$$

Proof. By Lemma 2.15 and the $(\mathcal{M}_\phi, G)$ h -convexity of the function f we get

$$\begin{aligned} \Phi_N(\eta) & \geq f^{h(k-2^N\eta)} \left(\mathcal{M}_\phi \left(\frac{k-1}{2^N}, u, v \right) \right) f^{h(2^N\eta-k+1)} \left(\mathcal{M}_\phi \left(\frac{k}{2^N}, u, v \right) \right) \\ & \geq f \left(\mathcal{M}_\phi \left(2^N\eta - k + 1, \mathcal{M}_\phi \left(\frac{k-1}{2^N}, u, v \right), \mathcal{M}_\phi \left(\frac{k}{2^N}, u, v \right) \right) \right) \\ & = f \left(\phi^{-1} \left((k-2^N\eta) \left(\frac{2^N-k+1}{2^N} \phi(u) + \frac{k-1}{2^N} \phi(v) \right) \right. \right. \\ & \quad \left. \left. + (2^N\eta - k + 1) \left(\frac{2^N-k}{2^N} \phi(u) + \frac{k}{2^N} \phi(v) \right) \right) \right) \\ & = f(\phi^{-1}((1-\eta)\phi(u) + \eta\phi(v))) = f(\mathcal{M}_\phi(\eta, u, v)). \end{aligned}$$

This concludes the proof of (2.9).

By substituting η with $1-\eta$ in (2.9) and observing that $r_n(\eta) = r_n(1-\eta)$, we obtain

$$\begin{aligned} & f^{h(1-\eta)}(u)f^{h(\eta)}(v) \\ & = \frac{(f(u)f(v))^{h(1-\eta)+h(\eta)}}{f(u)^{h(\eta)}f^{h(1-\eta)}(v)} \\ & \leq \frac{(f(u)f(v))^{h(1)}}{f(\mathcal{M}_\phi(1-\eta, u, v)) \prod_{n=0}^{N-1} \left(\prod_{k=1}^{2^n} \left(\Delta_{f, \phi, h}^{(u, v)}(n, k) \right)^{\chi(\frac{k-1}{2^n}, \frac{k}{2^n})(1-\eta)} \right)^{h(2r_n(1-\eta))}} \end{aligned}$$

$$\leq \frac{(f(u)f(v))}{f(\mathcal{M}_\phi(1-\eta, u, v)) \prod_{n=0}^{N-1} \left(\prod_{k=1}^{2^n} \left(\Delta_{f, \phi, h}^{(u, v)}(n, k) \right)^{\chi(\frac{k-1}{2^n}, \frac{k}{2^n})(1-\eta)} \right)^{h(2r_n(\eta))}}.$$

Now, substituting k with $2^n - k + 1$ in the inner sum and observing that

$$\frac{k-1}{2^n} < 1-\eta < \frac{k}{2^n} \iff 1 - \frac{k}{2^n} < \eta < 1 - \frac{k-1}{2^n},$$

we reach (2.10), thus concluding the proof. $\square$

In the following theorem, we present an improvement of Theorem 2.8 by introducing n multiplicative type refining terms.

THEOREM 2.17. *Let f be a non-negative $(\mathcal{M}_\phi, G)$ h -convex function on I , and $0 < \eta \leq \theta < 1$. Then for all non-negative integer N , and $u, v \in I$ we have*

$$\begin{aligned} f^{h(1-\eta)}(u) f^{h(\eta)}(v) &\geq \left(\frac{f^{h(1-\theta)}(u) f^{h(\theta)}(v)}{f(\mathcal{M}_\phi(\theta, u, v))} \right)^{h(\frac{\eta}{\theta})} f(\mathcal{M}_\phi(\eta, u, v)) \\ &\quad \times \prod_{n=0}^{N-1} \left(\prod_{k=1}^{2^n} \left(\Delta_{g, id, h}^{(0,1)}(n, k) \right)^{\chi(\frac{k-1}{2^n}, \frac{k}{2^n})(\frac{\eta}{\theta})} \right)^{h(2r_n(\frac{\eta}{\theta}))}, \end{aligned}$$

where

$$\Delta_{g, id, h}^{(0,1)}(n, k) = \frac{\left[g\left(\frac{k-1}{2^n}\right) g\left(\frac{k}{2^n}\right) \right]^{h(\frac{1}{2})}}{g\left(\frac{2k-1}{2^{n+1}}\right)},$$

and $g(t) := f(\mathcal{M}_\phi(t\theta, u, v))$.

Proof. For a $(\mathcal{M}_\phi, G)$ h -convex function f on $[u, v]$, define $g : [0, 1] \rightarrow \mathbb{R}$ by $g(t) := f(\mathcal{M}_\phi(t\theta, u, v))$. Then, g is $(\mathcal{M}_{id}, G)$ h -convex on $[0, 1]$. By applying Theorem 2.16 on the function g and by super-multiplicativity and super-additivity of h , we have

$$\begin{aligned} &\frac{f^{h(1-\eta)}(u) f^{h(\eta)}(v)}{\left(\frac{f^{h(1-\theta)}(u) f^{h(\theta)}(v)}{f(\mathcal{M}_\phi(\theta, u, v))} \right)^{h(\frac{\eta}{\theta})}} \\ &= f^{(h(1-\eta)-h(1-\theta)h(\frac{\eta}{\theta}))}(u) f^{(h(\eta)-h(\theta)h(\frac{\eta}{\theta}))}(v) f^{h(\frac{\eta}{\theta})}(\mathcal{M}_\phi(\theta, u, v)) \\ &\geq f(u)^{h(1-\frac{\eta}{\theta})} f(\mathcal{M}_\phi(\theta, u, v))^{h(\frac{\eta}{\theta})} = g(0)^{h(1-\frac{\eta}{\theta})} g(1)^{h(\frac{\eta}{\theta})} \\ &\geq g\left(\frac{\eta}{\theta}\right) \prod_{n=0}^{N-1} \left(\prod_{k=1}^{2^n} \left(\Delta_{g, id, h}^{(0,1)}(n, k) \right)^{\chi(\frac{k-1}{2^n}, \frac{k}{2^n})(\frac{\eta}{\theta})} \right)^{h(2r_n(\frac{\eta}{\theta}))} \\ &= f(\mathcal{M}_\phi(\eta, u, v)) \prod_{n=0}^{N-1} \left(\prod_{k=1}^{2^n} \left(\Delta_{g, id, h}^{(0,1)}(n, k) \right)^{\chi(\frac{k-1}{2^n}, \frac{k}{2^n})(\frac{\eta}{\theta})} \right)^{h(2r_n(\frac{\eta}{\theta}))}. \quad \square \end{aligned}$$

2.3. The case $\eta \notin [0, 1]$

We start this section with following reverse version of inequality (1.3), applicable when $\eta \in \mathbb{R}$. This extends the first inequality in Proposition 1.1 to the concept of $(\mathcal{M}_\phi, G)$ h -convexity.

THEOREM 2.18. *Let $f: \mathbb{R} \rightarrow (0, +\infty)$ be a $(\mathcal{M}_\phi, G)$ h -convex function on I , and $\eta \in \mathbb{R}$.*

If $\eta \geq 0$, we have

$$f^{h(1+\eta)}(u)f^{-h(\eta)}(v) \leq f(\mathcal{M}_\phi(-\eta, u, v)).$$

If $\eta < 0$, we have

$$f^{h(1-\eta)}(u)f^{-h(-\eta)}(v) \leq f(\mathcal{M}_\phi(\eta, u, v)).$$

Proof. Notice first that for $\eta \geq 0$ and $u, v \in \mathbb{R}$ one has

$$\phi(u) = \frac{\eta}{\eta+1}\phi(v) + \frac{1}{\eta+1}\phi(\phi^{-1}((1+\eta)\phi(u) - \eta\phi(v))).$$

So,

$$f(u) = f\left(\mathcal{M}_\phi\left(\frac{1}{1+\eta}, v, \mathcal{M}_\phi(-\eta, u, v)\right)\right).$$

$(\mathcal{M}_\phi, G)$ -convexity of f and the fact that h is a non-negative super-multiplicative function implies

$$\begin{aligned} f(u) &\leq f^{h\left(\frac{\eta}{\eta+1}\right)}(v)f^{h\left(\frac{1}{\eta+1}\right)}(\mathcal{M}_\phi(-\eta, u, v)) \\ &\leq f^{\frac{h(\eta)}{h(\eta+1)}}(v)f^{\frac{1}{h(\eta+1)}}(\mathcal{M}_\phi(-\eta, u, v)), \end{aligned}$$

which implies

$$f^{h(1+\eta)}(u)f^{-h(\eta)}(v) \leq f(\mathcal{M}_\phi(-\eta, u, v)). \quad (2.11)$$

If $\eta < 0$, observe that

$$\phi(u) = \frac{-\eta}{1-\eta}\phi(v) + \frac{1}{1-\eta}\phi(\mathcal{M}_\phi(\eta, u, v)).$$

So,

$$f(u) = f\left(\mathcal{M}_\phi\left(\frac{1}{1-\eta}, v, \mathcal{M}_\phi(\eta, u, v)\right)\right).$$

Then, by $(\mathcal{M}_\phi, G)$ h -convexity of f , we have

$$f(u) \leq f^{h\left(\frac{-\eta}{1-\eta}\right)}(v)f^{h\left(\frac{1}{1-\eta}\right)}(\mathcal{M}_\phi(\eta, u, v)).$$

Since, h is super-multiplicative and super-additive, we have

$$\begin{aligned} f^{h(1-\eta)}(u) &\leq f^{h\left(\frac{-\eta}{1-\eta}\right)h(1-\eta)}(v) f^{h\left(\frac{1}{1-\eta}\right)h(1-\eta)}(\mathcal{M}_\phi(\eta, u, v)) \\ &\leq f^{h(-\eta)}(v) f(\mathcal{M}_\phi(\eta, u, v)), \end{aligned}$$

then we conclude that

$$f^{h(1-\eta)}(u) f^{-h(-\eta)}(v) \leq f(\mathcal{M}_\phi(\eta, u, v)). \quad \square$$

Next, we introduce a refined version of the preceding theorem by adding several multiplicatively structured refining terms in the case where $\eta > 0$.

THEOREM 2.19. *Let ϕ be a strictly increasing function on $\mathbb{R}$, and $f : \mathbb{R} \rightarrow (0, +\infty)$ be $(\mathcal{M}_\phi, G)$ h -convex, $N \in \mathbb{N}^*$ and let $u < v$. If $\eta \geq 0$, then*

$$\frac{f^{h(1+\eta)}(u)}{f^{h(\eta)}(v)} \prod_{k=1}^N \left[\frac{\left(f(u) f\left(\mathcal{M}_\phi\left(\frac{1}{2^{k-1}}, u, v\right)\right) \right)^{h\left(\frac{1}{2}\right)}}{f\left(\mathcal{M}_\phi\left(\frac{1}{2^k}, u, v\right)\right)} \right]^{h(2^k \eta)} \leq f(\mathcal{M}_\phi(-\eta, u, v)). \quad (2.12)$$

Proof. We proceed by induction on N . So, assume that f is $(\mathcal{M}_\phi, G)$ h -convex, h super-multiplicative and super-additive, $u < v$ and $\eta \geq 0$, we have for $N = 1$,

$$\begin{aligned} &\frac{f^{h(1+\eta)}(u)}{f^{h(\eta)}(v)} \left[\frac{(f(u)f(v))^{h\left(\frac{1}{2}\right)}}{f\left(\mathcal{M}_\phi\left(\frac{1}{2}, u, v\right)\right)} \right]^{h(2\eta)} \\ &= \frac{f^{h(1+\eta)+h(2\eta)h\left(\frac{1}{2}\right)}(u) f^{h(2\eta)h\left(\frac{1}{2}\right)-h(\eta)}(v)}{f^{h(2\eta)}(\mathcal{M}_\phi\left(\frac{1}{2}, u, v\right))} \\ &\leq \frac{f^{h(1+2\eta)}(u)}{f^{h(2\eta)}(\mathcal{M}_\phi\left(\frac{1}{2}, u, v\right))} \\ &\leq f\left(\phi^{-1}\left((1+2\eta)\phi(u) - (2\eta)\left(\frac{\phi(u)+\phi(v)}{2}\right)\right)\right) \\ &= f(\mathcal{M}_\phi(-\eta, u, v)). \end{aligned}$$

Here, we applied Theorem 2.18, substituting 2η for η and $\mathcal{M}_\phi\left(\frac{1}{2}, u, v\right)$ for v . Notably, when $u < v$, the strict monotonicity of ϕ implies that $u < \mathcal{M}_\phi\left(\frac{1}{2}, u, v\right)$. Additionally, when $v \geq 0$, we also have $2v \geq 0$, justifying the application of Theorem 2.18.

Now assume that, for some $N \in \mathbb{N}^*$, (2.12) holds whenever $u < v$ and $\eta \geq 0$. We assert the truth of the inequality for $N + 1$. Observe that

$$\begin{aligned}
 K &= \frac{f^{h(1+\eta)}(u)}{f^{h(\eta)}(v)} \prod_{k=1}^{N+1} \left[\frac{\left(f(u) f\left(\mathcal{M}_\phi\left(\frac{1}{2^{k-1}}, u, v \right) \right) \right)^{h(\frac{1}{2})}}{f\left(\mathcal{M}_\phi\left(\frac{1}{2^k}, u, v \right) \right)} \right]^{h(2^k \eta)} \\
 &= \frac{f^{h(1+\eta)}(u)}{f^{h(\eta)}(v)} \left[\frac{(f(u) f(v))^{h(\frac{1}{2})}}{f\left(\mathcal{M}_\phi\left(\frac{1}{2}, u, v \right) \right)} \right]^{h(2\eta)} \prod_{k=2}^{N+1} \left[\frac{\left(f(u) f\left(\mathcal{M}_\phi\left(\frac{1}{2^{k-1}}, u, v \right) \right) \right)^{h(\frac{1}{2})}}{f\left(\mathcal{M}_\phi\left(\frac{1}{2^k}, u, v \right) \right)} \right]^{h(2^k \eta)} \\
 &\leq \frac{f^{h(1+2\eta)}(u)}{f^{h(2\eta)}\left(\mathcal{M}_\phi\left(\frac{1}{2}, u, v \right) \right)} \prod_{k=1}^N \left[\frac{\left(f(u) f\left(\mathcal{M}_\phi\left(\frac{1}{2^k}, u, v \right) \right) \right)^{h(\frac{1}{2})}}{f\left(\mathcal{M}_\phi\left(\frac{1}{2^{k+1}}, u, v \right) \right)} \right]^{h(2^{k+1} \eta)}. \quad (2.13)
 \end{aligned}$$

For simplicity, let $2\eta = s$, $\mathcal{M}_\phi\left(\frac{1}{2}, u, v\right) = v'$, so $\mathcal{M}_\phi\left(\frac{1}{2^k}, u, v\right) = \mathcal{M}_\phi\left(\frac{1}{2^{k-1}}, u, v'\right)$ for all $k = 1, \dots, N$. Then (2.13) becomes

$$\begin{aligned}
 K &\leq \frac{f^{h(1+s)}(u)}{f^{h(s)}(v')} \prod_{k=1}^N \left[\frac{\left(f(u) f^{h(\frac{1}{2})}\left(\mathcal{M}_\phi\left(\frac{1}{2^{k-1}}, u, v' \right) \right) \right)}{f^{h(2^k s)}\left(\mathcal{M}_\phi\left(\frac{1}{2^k}, u, v' \right) \right)} \right] \\
 &\leq f\left(\mathcal{M}_\phi(-s, u, v') \right) \\
 &= f\left(\mathcal{M}_\phi(-\eta, u, v) \right). \quad (2.14)
 \end{aligned}$$

Here, we used the inductive step to derive (2.14). It is important to note that when $u < v$, we have $u < v'$, which justifies the application of the inductive step. $\square$

3. On Jensen's type inequality for $(\mathcal{M}_\phi, G)$ h -convex functions

In this section, we examine the reversed version of Jensen's type inequality for an $(\mathcal{M}_\phi, G)$ h -convex function, offering notable improvements and generalizations of Theorem 2.18.

THEOREM 3.1. *Let $f : \mathbb{R} \mapsto (0, +\infty)$ be a $(\mathcal{M}_\phi, G)$ h -convex function, $u \in \mathbb{R}$ and $\{\eta_k\} > 0$. If $\{u_k\} \subset \mathbb{R}$. Then we have*

$$f^{h(1+\xi)}(u) \prod_{k=1}^n f^{-h(\eta_k)}(v_k) \leq f\left(\phi^{-1}\left((1+\xi)\phi(u) - \sum_{k=1}^n \eta_k \phi(v_k)\right)\right),$$

where $\sum_{k=1}^n \eta_k = \xi$.

Proof. By Theorem 2.18, for $\alpha \geq 0$, we get

$$\frac{f^{h(\alpha+1)}(v)}{f[\phi^{-1}((1+\alpha)\phi(v) - \alpha\phi(u))]} \leq f^{h(\alpha)}(u). \quad (3.1)$$

Now, applying (3.1), we have

$$\begin{aligned} \frac{f^{h(1+\xi)}(u)}{f[\phi^{-1}((1+\xi)\phi(u) - \sum_{k=1}^n \eta_k \phi(v_k))]} &= \frac{f^{h(1+\xi)}(u)}{f[\phi^{-1}((1+\xi)\phi(u) - \xi \sum_{k=1}^n \frac{\eta_k}{\xi} \phi(v_k))]} \\ &\leq f^{h(\xi)}\left(\phi^{-1}\left(\sum_{k=1}^n \frac{\eta_k}{\xi} \phi(v_k)\right)\right) \\ &\leq \left(\prod_{k=1}^n f^{h(\frac{\eta_k}{\xi})}(v_k)\right)^{h(\xi)} \\ &\leq \prod_{k=1}^n f^{h(\eta_k)}(v_k). \quad \square \end{aligned}$$

In the following result, we present a refined version of Theorem 3.1, incorporating a multiplicative term.

THEOREM 3.2. Let $f: \mathbb{R} \mapsto (0, +\infty)$ be a $(\mathcal{M}_\phi, G)$ h -convex function, $u \in \mathbb{R}$ and $\{\eta_k\} > 0$. If $\{u_k\} \subset \mathbb{R}$. Then we have

$$\begin{aligned} &f^{h(1+\xi)}(u) \prod_{k=1}^n f^{-h(\eta_k)}(v_k) \times \left[\frac{(f(u) \prod_{k=1}^n f(v_k))^{h(\frac{1}{n+1})}}{f[\phi^{-1}(\frac{\phi(u) + \phi(\sum_{k=1}^n v_k)}{n+1})]} \right]^{h((n+1)r_0)} \\ &\leq f\left(\phi^{-1}\left((1+\xi)\phi(u) - \sum_{k=1}^n \eta_k \phi(v_k)\right)\right), \end{aligned}$$

where $\sum_{k=1}^n \eta_k = \xi$ and $r_0 = \min\{\eta_k, k = 1, 2, \dots, n\}$.

Proof. Given that h is both super-multiplicative and super-additive, it follows that

$$\begin{aligned} I &:= f^{h(1+\xi)}(u) \prod_{k=1}^n f^{-h(\eta_k)}(v_k) \left[\frac{(f(u) \prod_{k=1}^n f(v_k))^{h(\frac{1}{n+1})}}{f[\phi^{-1}(\frac{\phi(u) + \phi(\sum_{k=1}^n v_k)}{n+1})]} \right]^{h((n+1)r_0)} \\ &\leq f^{(h(1+\xi) + h((n+1)r_0)h(\frac{1}{n+1}))}(u) \\ &\quad \times \prod_{k=1}^n f^{(-h(\eta_k) + h(r_0))}(v_k) f^{-h((n+1)r_0)}\left(\phi^{-1}\left[\left(\frac{\phi(u) + \phi(\sum_{k=1}^n v_k)}{n+1}\right)\right]\right) \\ &\leq f^{h(1+\xi+r_0)}(u) \prod_{k=1}^n f^{-h(\eta_k-r_0)}(v_k) f^{-h((n+1)r_0)}\left(\phi^{-1}\left[\left(\frac{\phi(u) + \phi(\sum_{k=1}^n v_k)}{n+1}\right)\right]\right). \end{aligned}$$

Let $\gamma_{n+1} = (n+1)r_0$ and for $1 \leq k \leq n$, let $\gamma_k = \eta_k - r_0$. Further, denote $\xi + r_0$ by λ . Then

$$\sum_{k=1}^n \gamma_k + \gamma_{n+1} = \xi + r_0 = \lambda.$$

Consequently, applying Theorem 3.1 yields

$$\begin{aligned} I &\leq f \left[\phi^{-1} \left((1 + \xi + r_0) \phi(u) - \sum_{k=1}^n (\eta_k - r_0) \phi(v_k) - (n+1)r_0 \frac{\phi(a) + \phi(\sum_{k=1}^n v_k)}{n+1} \right) \right] \\ &= f \left[\phi^{-1} \left((1 + \xi) \phi(u) - \sum_{k=1}^n \eta_k \phi(v_k) \right) \right]. \quad \square \end{aligned}$$

4. Application to inequalities involving unitarily invariant norms

In this section, we provide concrete examples of $(\mathcal{M}_\phi, G)$ h -convex functions. Then, using the results established in Section 2, we derive new refinements of some classical inequalities related to unitarily invariant norms.

Let $\mathbf{M}_n$ represent the algebra of all complex $n \times n$. A matrix $\mathcal{E} \in \mathbf{M}_n$ is called Hermitian if $\mathcal{E} = \mathcal{E}^*$, where $\mathcal{E}^*$ is the adjoint of $\mathcal{E}$. The notation $\mathcal{E} \geq 0$ ($\mathcal{E} > 0$) is used to mean that $\mathcal{E}$ is positive semi-definite (positive definite), if $\mathcal{E}$ and $\mathcal{F}$ are Hermitian and $\mathcal{E} - \mathcal{F}$ is positive semi-definite, then we write $\mathcal{E} \geq \mathcal{F}$. The set of all positive semi-definite matrices is denoted by $\mathbf{M}_n^+$ while the set of positive definite matrices within $\mathbf{M}_n^+$ is denoted by $\mathbf{M}_n^{++}$. A matrix $\mathcal{E} \in \mathbf{M}_n$ is considered contractive if $\mathcal{E} < I$ and expansive if $\mathcal{E} > I$. The singular values of a matrix $\mathcal{E} \in \mathbf{M}_n$ are the eigenvalues of the positive semi-definite matrix $|\mathcal{E}| := (\mathcal{E}^* \mathcal{E})^{1/2}$, denoted by $s_j(\mathcal{E})$ for $j = 1, 2, 3, \dots, n$; arranged in a non-increasing order. A norms $\|\cdot\|$ on $\mathbf{M}_n$ is called unitarily invariant if it satisfies the property: is a matrix norm that satisfies the invariance property: $\|U\mathcal{E}V\| = \|\mathcal{E}\|$ for every $\mathcal{E} \in \mathbf{M}_n$ and for all unitary matrices $U, V \in \mathbf{M}_n$. The trace norm is given by $\|\mathcal{E}\|_1 := \text{tr}|\mathcal{E}| = \sum_{j=1}^n s_j(\mathcal{E})$, where tr is the usual trace. This norm is unitarily invariant. Another key example of a unitarily invariant norm is the Hilbert-Schmidt norm $\|\cdot\|_2$ defined by

$$\|\mathcal{E}\|_2 := (\text{tr}(\mathcal{E}\mathcal{E}^*))^{1/2} = \left(\sum_{i,j} |a_{i,j}|^2 \right)^{1/2}, \quad (\mathcal{E} = (a_{i,j})).$$

When referring to the eigenvalues of a Hermitian matrix $\mathcal{E}$, we use the notation $\lambda_j(\mathcal{E})$ represents the j -th eigenvalue of $\mathcal{E}$, listed in descending order.

In this section, we consider a function $h: J \rightarrow \mathbb{R}$ such that $h(x) \geq x$ for all $x \in [0, 1]$, and ϕ as an increasing concave function. Examples of such functions include $\phi(x) = \ln(x)$ and $\phi(x) = x^p$ for $p \in (0, 1)$.

LEMMA 4.1. ([12]) *Let $\mathcal{E}, \mathcal{F} \in \mathbf{M}_n$ be expansive operators and let $\mathcal{X} \in \mathbf{M}_n$ be arbitrary. Then the function*

$$f(t) = \|\mathcal{E}^t \mathcal{X} \mathcal{F}^t\|$$

is increasing on $(0, +\infty)$.

PROPOSITION 4.2. Let $\mathcal{E}, \mathcal{F} \in \mathbf{M}_n$ be expansive operators and let $\mathcal{X} \in \mathbf{M}_n$ be arbitrary. Then the function

$$f(t) = \left\| \left\| \mathcal{E}^t \mathcal{X} \mathcal{F}^t \right\| \right\|$$

is $(\mathcal{M}_\phi, G)$ - h -convex on $(0, +\infty)$.

Proof. By utilizing the fact that ϕ is an increasing concave function, we have

$$\mathcal{M}_\phi(\eta, t_1, t_2) \leq (1 - \eta)t_1 + \eta t_2.$$

Thus, by applying Lemma 4.1, we obtain

$$\begin{aligned} f(\mathcal{M}_\phi(\eta, t_1, t_2)) &\leq \left\| \left\| \mathcal{E}^{(1-\eta)t_1 + \eta t_2} \mathcal{X} \mathcal{F}^{(1-\eta)t_1 + \eta t_2} \right\| \right\| \\ &\leq f^{1-\eta}(t_1) f^\eta(t_2) \\ &\leq f^{h(1-\eta)}(t_1) f^{h(\eta)}(t_2), \end{aligned}$$

where the second inequality results from the log-convexity of the function f , and the final inequality follows from the condition that $h(x) \geq x$ for all $x \in [0, 1]$. $\square$

By utilizing the preceding proposition and Corollary 2.9, we derive the following results.

PROPOSITION 4.3. Let $\mathcal{E}, \mathcal{F} \in \mathbf{M}_n$ expansive and $\mathcal{X} \in \mathbf{M}_n$, arbitrary, then for any $\eta \in [0, 1]$ and $t, s \in (0, +\infty)$, we have

$$\left\| \left\| \mathcal{E}^t \mathcal{X} \mathcal{F}^t \right\| \right\|^{h(1-\eta)} \left\| \left\| \mathcal{E}^s \mathcal{X} \mathcal{F}^s \right\| \right\|^{h(\eta)} \leq \left\| \left\| \mathcal{E}^{\mathcal{M}_\phi(\eta, t, s)} \mathcal{X} \mathcal{F}^{\mathcal{M}_\phi(\eta, t, s)} \right\| \right\| \left(\frac{f^{h(\frac{1}{2})}(t) f^{h(\frac{1}{2})}(s)}{f(\mathcal{M}_\phi(\frac{1}{2}, t, s))} \right)^{h(2R_0)},$$

and

$$\left\| \left\| \mathcal{E}^t \mathcal{X} \mathcal{F}^t \right\| \right\|^{h(1-\eta)} \left\| \left\| \mathcal{E}^s \mathcal{X} \mathcal{F}^s \right\| \right\|^{h(\eta)} \geq \left\| \left\| \mathcal{E}^{\mathcal{M}_\phi(\eta, t, s)} \mathcal{X} \mathcal{F}^{\mathcal{M}_\phi(\eta, t, s)} \right\| \right\| \left(\frac{f^{h(\frac{1}{2})}(t) f^{h(\frac{1}{2})}(s)}{f(\mathcal{M}_\phi(\frac{1}{2}, t, s))} \right)^{h(2r_0)},$$

where $r_0 = \min\{\eta, 1 - \eta\}$ and $R_0 = \max\{\eta, 1 - \eta\}$.

PROPOSITION 4.4. Let $\mathcal{E} \in \mathbf{M}_n$ expansive, $\mathcal{F} \in \mathbf{M}_n$ contractive and $\mathcal{X} \in \mathbf{M}_n$ arbitrary, then

$$f_1(t) = \left\| \left\| \mathcal{E}^t \mathcal{X} \mathcal{F}^{1-t} \right\| \right\|$$

are $(\mathcal{M}_\phi, G)$ h -convex functions on $(0, +\infty)$.

Proof. Let's begin by proving the $(\mathcal{M}_\phi, G)$ h -convexity of the function f_1 . Observe that if $\mathcal{F}$ is contractive, then $\mathcal{F}^{-1}$ is expansive. Therefore, by applying Proposition 4.2 with $\mathcal{X}$ substituted by $\mathcal{X}\mathcal{F}$ and $\mathcal{F}$ replaced by $\mathcal{F}^{-1}$, we obtain the result. In a similar manner, we can demonstrate the $(\mathcal{M}_\phi, G)$ h -convexity of the function f_2 . $\square$

PROPOSITION 4.5. Let $\mathcal{F} \in \mathbf{M}_n$ expansive, $\mathcal{E} \in \mathbf{M}_n$ contractive and $\mathcal{X} \in \mathbf{M}_n$ arbitrary,

$$f_2(t) = ||| \mathcal{E}^{1-t} \mathcal{X} \mathcal{F}^t |||$$

are $(\mathcal{M}_\phi, G)$ h -convex functions on $(0, +\infty)$.

In a similar manner, we obtain the following results.

PROPOSITION 4.6. Let $\mathcal{E} \in \mathbf{M}_n$ expansive, $\mathcal{F} \in \mathbf{M}_n$ contractive and $\mathcal{X} \in \mathbf{M}_n$, arbitrary, then for any $\eta \in [0, 1]$ and $t, s \in (0, +\infty)$, we have

$$\begin{aligned} & ||| \mathcal{E}^t \mathcal{X} \mathcal{F}^{1-t} |||^{h(1-\eta)} ||| \mathcal{E}^s \mathcal{X} \mathcal{F}^{1-s} |||^{h(\eta)} \\ & \leq ||| \mathcal{E}^{\mathcal{M}_\phi(\eta, t, s)} \mathcal{X} \mathcal{F}^{1-\mathcal{M}_\phi(\eta, t, s)} ||| \left(\frac{f_1^{h(\frac{1}{2})}(t) f_1^{h(\frac{1}{2})}(s)}{f_1(\mathcal{M}_\phi(\frac{1}{2}, t, s))} \right)^{h(2R_0)}, \end{aligned}$$

and

$$\begin{aligned} & ||| \mathcal{E}^t \mathcal{X} \mathcal{F}^{1-t} |||^{h(1-\eta)} ||| \mathcal{E}^s \mathcal{X} \mathcal{F}^{1-s} |||^{h(\eta)} \\ & \geq ||| \mathcal{E}^{\mathcal{M}_\phi(\eta, t, s)} \mathcal{X} \mathcal{F}^{1-\mathcal{M}_\phi(\eta, t, s)} ||| \left(\frac{f_1^{h(\frac{1}{2})}(t) f_1^{h(\frac{1}{2})}(s)}{f_1(\mathcal{M}_\phi(\frac{1}{2}, t, s))} \right)^{h(2r_0)}, \end{aligned}$$

where $r_0 = \min\{\eta, 1 - \eta\}$ and $R_0 = \max\{\eta, 1 - \eta\}$.

REMARK 4.7. Notice that by selecting certain special $(\mathcal{M}_\phi, G)$ h -convex functions in terms of means and applying our refinement to this class in this work, we derive further improvements of some well-known results involving means.

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