

REFINED UPPER BOUNDS FOR THE NUMERICAL RADIUS OF OPERATORS IN HILBERT SPACES

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Abstract. In this work, we employ some existing refinements of the well-known Cauchy–Schwarz and Buzano inequalities to derive improved upper bounds for the numerical radius of operator matrices on Hilbert spaces. As consequences, we obtain new upper bounds for the numerical radius of sums and differences of Hilbert space operators.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ represent the C^* -algebra of all bounded linear operators on the complex Hilbert space $\mathcal{H}$. Recall that for $T \in \mathcal{B}(\mathcal{H})$, the numerical radius $w(\cdot)$ and the usual norm $\|\cdot\|$ are defined as follows:

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle| \text{ and } \|T\| = \sup_{\|x\|=1} \|Tx\|.$$

It is also worth recalling that $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the standard operator norm. More precisely, the following sharp two-sided inequality holds:

$$\frac{1}{2} \|T\| \leq w(T) \leq \|T\|. \quad (1.1)$$

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{H}^{(2)} = \mathcal{H} \oplus \mathcal{H}$ denote the 2-copies of $\mathcal{H}$. Based on this decomposition every operator $Y \in \mathcal{B}(\mathcal{H}^{(2)})$ has a 2×2 operator matrix representation $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$, with $Y_{ij} \in \mathcal{B}(\mathcal{H})$ where $i, j \in \{1, 2\}$.

The numerical radius of an $n \times n$ block diagonal operator matrix is equal to the maximum of the numerical radii of its diagonal blocks. In contrast, computing the numerical radius of an $n \times n$ block anti-diagonal operator matrix is, in general, a more challenging problem.

In [11], Kittaneh refined the second inequality in (1.1) by showing that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w(A) \leq \frac{1}{2} (\|A\| + \|A^*\|). \quad (1.2)$$

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In [6], Dragomir improved the second inequality in (1.1) and established that

$$w^2(A) \leq \frac{1}{2}(w(A^2) + \|A\|^2) \text{ where } A \in \mathcal{B}(\mathcal{H}). \quad (1.3)$$

In [13], Kittaneh presented a further improvement of the second inequality in (1.1), by showing that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w^2(A) \leq \frac{1}{2}(\|A\|^2 + \|A^*\|^2). \quad (1.4)$$

In [8], El-Haddad and Kettaneh improved more general form the inequality in (1.4), which asserts that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w^{2r}(A) \leq \frac{1}{2}(\|A\|^{2r} + \|A^*\|^{2r}), \quad \text{for every } r \geq 1. \quad (1.5)$$

In [7], Dragomir provided further improvements on upper bounds for the numerical radius. He showed that if $A, B \in \mathcal{B}(\mathcal{H})$, then

$$w^r(A^*B) \leq \frac{1}{2}(\|B\|^{2r} + \|A\|^{2r}), \quad \text{for every } r \geq 1. \quad (1.6)$$

In [14] Hongwei, Guojon and Burittu applied the extension of the Cauchy-Schwarz inequality to refine the inequality (1.4). They proved that if $A \in \mathcal{B}(\mathcal{H})$ and $\alpha \in [0, 1]$, then

$$w^4(A) \leq \frac{(1+\alpha)}{4}(\|A\|^4 + \|A^*\|^4) + \frac{(1-\alpha)}{2}w^2(A^2), \quad (1.7)$$

and

$$w^4(A) \leq \frac{1}{8}(\|A\|^4 + \|A^*\|^4) + \frac{(2+\alpha)}{8}(\|A\|^2 + \|A^*\|^2)w(A^2) + \frac{(1-\alpha)}{4}w^2(A^2). \quad (1.8)$$

And they proved if $P, Q, T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha \in [0, 1]$, then

$$\begin{aligned} w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq 2(2+\alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\ &\quad + 4(2-\alpha)w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right), \end{aligned} \quad (1.9)$$

$$\begin{aligned} w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq (2+\alpha) \max \{ \| |T|^2 + |S^*|^2 \|, \| |S|^2 + |T^*|^2 \| \} \max \{ w(TS), w(ST) \} \\ &\quad + 8 \max \{ w^4(P), w^4(Q) \} + 2(1-\alpha) \max \{ w^2(TS), w^2(ST) \} \\ &\quad + \max \{ \| |T|^4 + |S^*|^4 \|, \| |S|^4 + |T^*|^4 \| \}, \end{aligned} \quad (1.10)$$

and

$$\begin{aligned} w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq (2+\alpha) \max \{ \| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} w \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\ &\quad + 3 \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\ &\quad + 2(3-\alpha)w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right). \end{aligned} \quad (1.11)$$

In this work, we establish a set of new upper bounds for the powers of the numerical radius of a 2×2 block anti-diagonal operator matrix. These bounds are then employed to improve and refine inequalities (1.5)–(1.11).

2. Main results

To achieve our goal, we first recall some well-known lemmas that will be used to establish our results. The first lemma follows directly from the spectral theorem in conjunction with Jensen's inequality.

LEMMA 2.1. [12] *Let $S \in \mathcal{B}(\mathcal{H})$ be a positive operator, and let $x \in \mathcal{H}$ be a unit vector. Then*

$$\langle Sx, x \rangle^r \leq \langle S^r x, x \rangle \quad \text{for any } r \geq 1.$$

The second lemma concerns non-negative convex functions and positive operators.

LEMMA 2.2. [3] *Let $S_1, S_2 \in \mathcal{B}(\mathcal{H})$ be a positive operators. Then*

$$\|(S_1 + S_2)^r\| \leq 2^{r-1} \|S_1^r + S_2^r\| \quad \text{for every } r \geq 1.$$

The next inequality is a special case of “mixed Schwarz” inequality that was established in [10]

LEMMA 2.3. [10] *Let $A \in \mathcal{B}(\mathcal{H})$ and let $x, y \in H$. Then*

$$|\langle Ax, y \rangle|^2 \leq \langle A|x, x \rangle \langle A^*|y, y \rangle.$$

The fourth lemma is Buzanos's extension of the Cauchy-Schwarz inequality detailed in [4].

LEMMA 2.4. [4] *Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} (\|x\| \|y\| + |\langle x, y \rangle|).$$

The next lemma concerns specific 2×2 operator matrices and is presented in [9].

LEMMA 2.5. [9] *Let $X, Y \in \mathcal{B}(\mathcal{H})$. Then*

1. $w\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{w(X), w(Y)\}.$
2. $w\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) = \max\{w(X+Y), w(X-Y)\}.$
3. $w\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = w(Y).$

The following lemma will be applied in Theorems 2.21 and 2.26 to derive a new upper bound for the numerical radii of 2×2 operator matrices.

LEMMA 2.6. [2] Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then

$$|\langle x, e \rangle \langle e, y \rangle|^r \leq \frac{1+\alpha}{2} \|x\|^r \|y\|^r + \frac{1-\alpha}{2} |\langle x, y \rangle|^{2r},$$

for every $\alpha \in [0, 1]$ and $r \geq 1$.

LEMMA 2.7. [1] Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then

$$|\langle x, e \rangle \langle e, y \rangle|^{2r} \leq \frac{1}{4} \|x\|^{2r} \|y\|^{2r} + \frac{(2+\alpha)}{4} \|x\|^r \|y\|^r |\langle x, y \rangle|^r + \frac{1-\alpha}{4} |\langle x, y \rangle|^{2r},$$

for every $\alpha \in [0, 1]$ and $r \geq 1$.

LEMMA 2.8. [5] Let $x, y, z \in \mathcal{H}$. Then

$$|\langle z, x \rangle|^2 + |\langle z, y \rangle|^2 \leq \|z\| \sqrt{|\langle x, x \rangle|^2 + |\langle y, y \rangle|^2 + 2|\langle x, y \rangle|^2}.$$

LEMMA 2.9. [1] Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and let $r \geq 1$. Then

$$\sup_{\|x\|=1} \langle Ax, x \rangle^r \langle Bx, x \rangle^r \leq \frac{1}{4} \|A^{2r} + B^{2r}\| + \frac{1}{2} \min\{w(A^r B^r), w(B^r A^r)\}.$$

We are now prepared to derive several new upper bounds for the numerical radius of a Hilbert space operator which are refinements for some existing ones.

THEOREM 2.10. Let $B, C \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned} w^{4r} \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) &\leq \frac{1}{16} \max\{\| |B|^{4r} + |C^*|^{4r} \|, \| |C|^{4r} + |B^*|^{4r} \|\} \\ &+ \frac{1}{8} \min\{\max\{w(|B|^{2r} |C^*|^{2r}), w(|C|^{2r} |B^*|^{2r})\}, \\ &\max\{w(|B^*|^{2r} |C|^{2r}), w(|C^*|^{2r} |B|^{2r})\}\} \\ &+ \frac{2+\alpha}{8} \max\{\| |B|^{2r} + |C^*|^{2r} \|, \| |C|^{2r} + |B^*|^{2r} \|\} \\ &\times \max\{w^r(BC), w^r(CB)\} \\ &+ \frac{1-\alpha}{4} \max\{w^{2r}(BC), w^{2r}(CB)\} \quad (\text{for every } \alpha \in [0, 1] \text{ and } r \geq 1). \end{aligned}$$

Proof. Let $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Then for any unit vector $y \in \mathcal{H}^{(2)}$, we have

$$\begin{aligned} |\langle My, y \rangle|^{4r} &= (|\langle My, y \rangle|^2)^{2r} \\ &= (|\langle My, y \rangle|^2)^{2r} \\ &= (|\langle My, y \rangle \langle My, y \rangle|)^{2r} \\ &= (|\langle My, y \rangle \langle y, M^* y \rangle|)^{2r} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4} \|My\|^{2r} \|M^*y\|^{2r} + \frac{2+\alpha}{4} \|My\|^r \|M^*y\|^r |\langle My, M^*y \rangle|^r \\
&\quad + \frac{1-\alpha}{4} |\langle My, M^*y \rangle|^{2r} \text{ (by Lemma 2.7)} \\
&= \frac{1}{4} \langle |M|^2y, y \rangle^r \langle |M^*|^2y, y \rangle^r + \frac{2+\alpha}{8} (\|My\|^{2r} + \|M^*y\|^{2r}) |\langle M^2y, y \rangle|^r \\
&\quad + \frac{1-\alpha}{4} |\langle M^2y, y \rangle|^{2r} \text{ (by the arithmetic-geometric mean inequality)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
w^{4r}(M) &= \sup_{\|y\|=1} |\langle My, y \rangle|^{4r} \\
&\leq \sup_{\|y\|=1} \left(\frac{1}{4} \langle |M|^2y, y \rangle^r \langle |M^*|^2y, y \rangle^r + \frac{1-\alpha}{4} |\langle M^2y, y \rangle|^{2r} \right) \\
&\quad + \sup_{\|y\|=1} \left(\frac{2+\alpha}{8} \langle (|M|^{2r} + |M^*|^{2r})y, y \rangle |\langle M^2y, y \rangle|^r \right) \text{ (by Lemma 2.1)} \\
&\leq \frac{1}{4} \sup_{\|y\|=1} \left(\langle |M|^2y, y \rangle^r \langle |M^*|^2y, y \rangle^r \right) + \frac{2+\alpha}{8} \| |M|^{2r} + |M^*|^{2r} \| w^r(M^2) \\
&\quad + \frac{1-\alpha}{4} w^{2r}(M^2) \\
&\leq \frac{1}{16} \| |M|^{4r} + |M^*|^{4r} \| + \frac{1}{8} \min \{ w(|M|^{2r} |M^*|^{2r}), w(|M^*|^{2r} |M|^{2r}) \} \\
&\quad + \frac{1-\alpha}{4} w^{2r}(M^2) + \frac{2+\alpha}{8} \| |M|^{2r} + |M^*|^{2r} \| w^r(M^2) \text{ (by Lemma 2.9)} \\
&= \frac{1}{16} \max \{ \| |B|^{4r} + |C^*|^{4r} \|, \| |C|^{4r} + |B^*|^{4r} \| \} \\
&\quad + \frac{1}{8} \min \{ \max \{ w(|B|^{2r} |C^*|^{2r}), w(|C|^{2r} |B^*|^{2r}) \}, \\
&\quad \max \{ w(|B^*|^{2r} |C|^{2r}), w(|C^*|^{2r} |B|^{2r}) \} \} \\
&\quad + \frac{2+\alpha}{8} \max \{ \| |B|^{2r} + |C^*|^{2r} \|, \| |C|^{2r} + |B^*|^{2r} \| \} \max \{ w^r(BC), w^r(CB) \} \\
&\quad + \frac{1-\alpha}{4} \max \{ w^{2r}(BC), w^{2r}(CB) \}. \quad \square
\end{aligned}$$

Using Lemma 2.2 and the inequality (1.6), we can establish the following refinement of inequality 2.5.

COROLLARY 2.11. *Let $Y \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
w^{4r}(Y) &\leq \frac{1}{16} \| |Y|^{4r} + |Y^*|^{4r} \| + \frac{1}{8} \min \{ w(|Y|^{2r} |Y^*|^{2r}), w(|Y^*|^{2r} |Y|^{2r}) \} \\
&\quad + \frac{1-\alpha}{4} w^{2r}(Y^2) + \frac{2+\alpha}{8} \| |Y|^{2r} + |Y^*|^{2r} \| w^r(Y^2) \\
&\leq \frac{1}{2} \| |Y|^{4r} + |Y^*|^{4r} \|.
\end{aligned}$$

Proof.

$$\begin{aligned}
 w^{4r}(Y) &\leq \frac{1}{16} \| |Y|^{4r} + |Y^*|^{4r} \| + \frac{1}{8} \min \{ w(|Y|^{2r} |Y^*|^{2r}), w(|Y^*|^{2r} |Y|^{2r}) \} \\
 &\quad + \frac{1-\alpha}{4} w^{2r}(Y^2) + \frac{2+\alpha}{8} \| |Y|^{2r} + |Y^*|^{2r} \| w^r(Y^2) \\
 &\quad \text{(by Theorem 2.10 with } B = C = Y) \\
 &\leq \frac{1}{16} \| |Y|^{4r} + |Y^*|^{4r} \| + \frac{1}{8} \cdot \frac{1}{2} \| |Y|^{4r} + |Y^*|^{4r} \| + \frac{2+\alpha}{16} \| (|Y|^{2r} + |Y^*|^{2r})^2 \| \\
 &\quad + \frac{1-\alpha}{8} \| |Y|^{4r} + |Y^*|^{4r} \| \quad \text{(by inequality (1.6))} \\
 &\leq \frac{2}{16} \| |Y|^{4r} + |Y^*|^{4r} \| + 2^{2-1} \cdot \frac{2+\alpha}{16} \| |Y|^{4r} + |Y^*|^{4r} \| \\
 &\quad + \frac{1-\alpha}{8} \| |Y|^{4r} + |Y^*|^{4r} \| \quad \text{(by Lemma 2.2)} \\
 &= \frac{1}{2} \| |Y|^{4r} + |Y^*|^{4r} \|. \quad \square
 \end{aligned}$$

The following result provides a refinement of the upper bound in inequality (1.10).

COROLLARY 2.12. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
 w^{4r} \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &\leq 2^{4r-1} \max \{ w^{4r}(A), w^{4r}(D) \} \\
 &\quad + 2^{4r-5} \max \{ \| |B|^{4r} + |C^*|^{4r} \|, \| |C|^{4r} + |B^*|^{4r} \| \} \\
 &\quad + 2^{4r-4} \min \{ \max \{ w((|B||C^*|)^{2r}), w((|C||B^*|)^{2r}) \}, \\
 &\quad \quad \max \{ w((|B^*||C|)^{2r}), w((|C^*||B|)^{2r}) \} \} \\
 &\quad + (2+\alpha) 2^{4r-4} \max \{ \| |B|^{2r} + |C^*|^{2r} \|, \| |C|^{2r} + |B^*|^{2r} \| \} \\
 &\quad \times \max \{ w^r(BC), w^r(CB) \} + (1-\alpha) 2^{4r-3} \max \{ w^{2r}(BC), w^{2r}(CB) \} \\
 &\quad \text{(for every } \alpha \in [0, 1] \text{ and } r \geq 1).
 \end{aligned}$$

Proof.

$$\begin{aligned}
 w^{4r} \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &= w^{4r} \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \\
 &\leq \left(w \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) + w \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right)^{4r} \\
 &= \left(2 \cdot \frac{1}{2} w \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) + 2 \cdot \frac{1}{2} w \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right)^{4r} \\
 &= 2^{4r} \left(\frac{1}{2} w \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) + \frac{1}{2} w \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right)^{4r}
 \end{aligned}$$

$$\begin{aligned}
&= 2^{4r} \left(\frac{1}{2} w^{4r} \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) + \frac{1}{2} w^{4r} \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right) \\
&\quad (\text{by the convexity of the function } f(t) = t^r, r \geq 1) \\
&\leq 2^{4r-1} \max \{ w^{4r}(A), w^{4r}(D) \} \\
&\quad + 2^{4r-5} \max \{ \| |B|^{4r} + |C^*|^{4r} \|, \| |C|^{4r} + |B^*|^{4r} \| \} \\
&\quad + 2^{4r-4} \min \{ \max \{ w((|B||C^*|)^{2r}), w((|C||B^*|)^{2r}) \}, \\
&\quad \max \{ w((|B^*||C|)^{2r}), w((|C^*||B|)^{2r}) \} \} \\
&\quad + (2 + \alpha) 2^{4r-4} \max \{ \| |B|^{2r} + |C^*|^{2r} \|, \| |C|^{2r} + |B^*|^{2r} \| \} \\
&\quad \times \max \{ w^r(BC), w^r(CB) \} + (1 - \alpha) 2^{4r-3} \max \{ w^{2r}(BC), w^{2r}(CB) \} \\
&\quad (\text{by Theorem 2.10}). \quad \square
\end{aligned}$$

REMARK 2.13. The upper bound in corollary 2.12 refines the upper bound of the inequality (1.10). To justify this, note that

$$\begin{aligned}
w^4 \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &\leq 8 \max \{ w^4(A), w^4(D) \} + \frac{1}{2} \max \{ \| |B|^4 + |C^*|^4 \|, \| |C|^4 + |B^*|^4 \| \} \\
&\quad + \min \{ \max \{ w((|B||C^*|)^2), w((|C||B^*|)^2) \}, \\
&\quad \max \{ w((|B^*||C|^2), w((|C^*||B|^2)) \} \} \\
&\quad + (2 + \alpha) \max \{ \| |B|^2 + |C^*|^2 \|, \| |C|^2 + |B^*|^2 \| \} \max \{ w(BC), w(CB) \} \\
&\quad + 2(1 - \alpha) \max \{ w^2(BC), w^2(CB) \} \quad (\text{by Corollary 2.12}) \\
&\leq 8 \max \{ w^4(A), w^4(D) \} + \frac{1}{2} \max \{ \| |B|^4 + |C^*|^4 \|, \| |C|^4 + |B^*|^4 \| \} \\
&\quad + \min \left\{ \max \left\{ \frac{1}{2} \| |B|^4 + |C^*|^4 \|, \frac{1}{2} \| |C|^4 + |B^*|^4 \| \right\}, \right. \\
&\quad \left. \max \left\{ \frac{1}{2} \| |B^*|^4 + |C|^4 \|, \frac{1}{2} \| |C^*|^4 + |B|^4 \| \right\} \right\} \\
&\quad + (2 + \alpha) \max \{ \| |B|^2 + |C^*|^2 \|, \| |C|^2 + |B^*|^2 \| \} \max \{ w(BC), w(CB) \} \\
&\quad + 2(1 - \alpha) \max \{ w^2(BC), w^2(CB) \} \quad (\text{by inequality (1.6)}) \\
&= 8 \max \{ w^4(A), w^4(D) \} + \max \{ \| |B|^4 + |C^*|^4 \|, \| |C|^4 + |B^*|^4 \| \} \\
&\quad + (2 + \alpha) \max \{ \| |B|^2 + |C^*|^2 \|, \| |C|^2 + |B^*|^2 \| \} \max \{ w(BC), w(CB) \} \\
&\quad + 2(1 - \alpha) \max \{ w^2(BC), w^2(CB) \}.
\end{aligned}$$

The following example demonstrates that the upper bound given by Corollary 2.12 can be strictly smaller than the corresponding upper bound in (1.10).

EXAMPLE 2.14. Consider the Hilbert space $\mathbb{C}^2$, and Let $A, B, C, D \in \mathcal{B}(\mathbb{C}^2)$ be given by

$$A = D = 0, \quad \text{and} \quad B = C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, the upper bound obtained from Corollary 2.12 is equal to $\frac{1}{2}$, whereas the corresponding upper bound in (1.10) is equal to 1, which demonstrates that the upper bound in Corollary 2.12 is strictly smaller than the previously known upper bound in (1.10).

In the following example, we fix the value of the parameter α to demonstrate the strict improvement provided by Corollary 2.12 over the upper bound in (1.10).

EXAMPLE 2.15. Consider the Hilbert space $\mathbb{C}^2$, and Let $A, B, C, D \in \mathcal{B}(\mathbb{C}^2)$ be given by $A = D = 0$, $B = 3I$, and $C = I$ where I is the identity operator. Then, by Corollary 2.12, with $\alpha = \frac{1}{2}$, we obtain

$$w^4 \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 134.$$

On the other hand, inequality (1.10), with $\alpha = \frac{1}{2}$, gives

$$w^4 \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 166.$$

Thus, Corollary 2.12 provides a strict and quantitatively significant improvement over the upper bound in (1.10), even for the choice $\alpha = \frac{1}{2}$.

The next theorem provides a new upper bound for the numerical radius of general 2×2 operator matrices.

THEOREM 2.16. Let $P, T, S, Q \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned} w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq 2 \max \{ w^2(|P|^2 + |T^*|^2), w^2(|Q|^2 + |S^*|^2) \} + 4w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\ &\quad + \frac{1}{2} \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\ &\quad + \min \{ \max \{ w((|P||T^*|)^2), w((|Q||S^*|)^2) \}, \\ &\quad \max \{ w((|T^*||P|)^2), w((|S^*||Q|)^2) \} \} \\ &\quad + (2 + \alpha) \max \{ \| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} \cdot w \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\ &\quad + 2(1 - \alpha)w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right), \quad (\text{for every } \alpha \in [0, 1]). \end{aligned}$$

Proof. Let $A = \begin{bmatrix} P & T \\ S & Q \end{bmatrix}$, $R_1 = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ and $R_2 = \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}$. Then for any unit vector $x \in \mathcal{H}^{(2)}$, we have

$$\begin{aligned} |\langle Ax, x \rangle|^2 &= |\langle R_1 x, x \rangle + \langle R_2 x, x \rangle|^2 \\ &\leq (|\langle R_1 x, x \rangle| + |\langle R_2 x, x \rangle|)^2 \quad (\text{by the triangle inequality}) \\ &= |\langle R_1 x, x \rangle|^2 + |\langle R_2 x, x \rangle|^2 + 2|\langle R_1 x, x \rangle| |\langle R_2 x, x \rangle| \end{aligned}$$

$$\begin{aligned}
&= |\langle x, R_1 x \rangle|^2 + |\langle x, R_2^* x \rangle|^2 + 2|\langle R_1 x, x \rangle| |\langle R_2 x, x \rangle| \\
&\leq \|x\|^2 \sqrt{|\langle R_1 x, R_1 x \rangle|^2 + |\langle R_2^* x, R_2^* x \rangle|^2 + 2|\langle R_1 x, R_2^* x \rangle|^2 + 2|\langle R_1 x, x \rangle| |\langle R_2 x, x \rangle|} \\
&\quad \text{(by Lemma 2.8)} \\
&= \sqrt{|\langle |R_1|^2 x, x \rangle|^2 + |\langle |R_2^*|^2 x, x \rangle|^2 + 2|\langle R_2 R_1 x, x \rangle|^2 + 2|\langle R_1 x, x \rangle| |\langle R_2 x, x \rangle|} \\
&\leq \sqrt{2} \left[\langle |R_1|^2 x, x \rangle^2 + \langle |R_2^*|^2 x, x \rangle^2 + 2|\langle R_2 R_1 x, x \rangle|^2 + 4|\langle R_1 x, x \rangle|^2 |\langle R_2 x, x \rangle|^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\langle Ax, x \rangle|^4 &\leq 2 \left[\langle |R_1|^2 x, x \rangle^2 + \langle |R_2^*|^2 x, x \rangle^2 \right] + 4|\langle R_2 R_1 x, x \rangle|^2 + 8|\langle R_1 x, x \rangle|^2 |\langle x, R_2^* x \rangle|^2 \\
&= 2 \left[\langle |R_1|^2 x, x \rangle^2 + i \langle |R_2^*|^2 x, x \rangle^2 \right] + 4|\langle R_2 R_1 x, x \rangle|^2 + 8|\langle R_1 x, x \rangle|^2 |\langle x, R_2^* x \rangle|^2 \\
&\quad \text{(2.1)} \\
&\leq 2 \left| \langle (|R_1|^2 + i|R_2^*|^2)x, x \rangle \right|^2 + 4|\langle R_2 R_1 x, x \rangle|^2 + 2\|R_1 x\|^2 \|R_2^* x\|^2 \\
&\quad + 2(2 + \alpha) \|R_1 x\| \|R_2^* x\| |\langle R_1 x, R_2^* x \rangle| + 2(1 - \alpha) |\langle R_1 x, R_2^* x \rangle|^2 \\
&\quad \text{(by Lemma 1.7)} \\
&\leq 2 \left| \langle (|R_1|^2 + i|R_2^*|^2)x, x \rangle \right|^2 + 4|\langle R_2 R_1 x, x \rangle|^2 + 2\|R_1 x\|^2 \|R_2^* x\|^2 \\
&\quad + (2 + \alpha) [\|R_1 x\|^2 + \|R_2^* x\|^2] |\langle R_2 R_1 x, x \rangle| + 2(1 - \alpha) |\langle R_2 R_1 x, x \rangle|^2 \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= 2 \left| \langle (|R_1|^2 + i|R_2^*|^2)x, x \rangle \right|^2 + 4|\langle R_2 R_1 x, x \rangle|^2 + 2\|R_1 x\|^2 \|R_2^* x\|^2 \\
&\quad + (2 + \alpha) [\langle (|R_1|^2 + |R_2^*|^2)x, x \rangle] |\langle R_2 R_1 x, x \rangle| + 2(1 - \alpha) |\langle R_2 R_1 x, x \rangle|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
w^4(A) &\leq 2w^2(|R_1|^2 + i|R_2^*|^2) + 4w^2(R_2 R_1) + 2 \sup_{\|x\|=1} (\langle |R_1|^2 x, x \rangle \langle |R_2^*|^2 x, x \rangle) \\
&\quad + (2 + \alpha) \| |R_1|^2 + |R_2^*|^2 \| w(R_2 R_1) + 2(1 - \alpha) w^2(R_2 R_1) \\
&\leq 2w^2(|R_1|^2 + i|R_2^*|^2) + 4w^2(R_2 R_1) + \frac{1}{2} \| |R_1|^4 + |R_2^*|^4 \| \\
&\quad + \min \{ w(|R_1|^2 |R_2^*|^2), w(|R_2^*|^2 |R_1|^2) \} \\
&\quad + (2 + \alpha) \| |R_1|^2 + |R_2^*|^2 \| w(R_2 R_1) + 2(1 - \alpha) w^2(R_2 R_1) \quad \text{(by Lemma 1.9)} \\
&= 2 \max \{ w(|P|^2 + i|T^*|^2), w(|Q|^2 + i|S^*|^2) \} + 4w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
&\quad + \frac{1}{2} \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
&\quad + \min \{ \max \{ w((|P||T^*|)^2), w((|Q||S^*|)^2) \}, \\
&\quad \quad \max \{ w((|T^*||P|)^2), w((|S^*||Q|)^2) \} \} \\
&\quad + (2 + \alpha) \max \{ \| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} \cdot w \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
&\quad + 2(1 - \alpha) w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right), \text{ for every } \alpha \in [0, 1]. \quad \square
\end{aligned}$$

REMARK 2.17. The upper bound in Theorem 2.16 refines the upper bound in inequality (1.11). To justify this, note that

$$\begin{aligned}
 w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq (2 + \alpha) \max \{ \| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} \cdot w \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 &\quad + 2 \max \{ w^2(|P|^2 + i|T^*|^2), w^2(|Q|^2 + i|S^*|^2) \} \\
 &\quad + \frac{1}{2} \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 &\quad + \min \{ \max \{ w((|P||T^*|)^2), w((|Q||S^*|)^2) \}, \\
 &\quad \quad \max \{ w((|T^*||P|)^2), w((|S^*||Q|)^2) \} \} \\
 &\quad + 2(3 - \alpha) w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \quad (\text{by Theorem 2.16}) \\
 &\leq (2 + \alpha) \max \{ \| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} \cdot w \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 &\quad + 2 \max \{ w^2(|P|^2 + i|T^*|^2), w^2(|Q|^2 + i|S^*|^2) \} \\
 &\quad + \frac{1}{2} \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 &\quad + \frac{1}{2} \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 &\quad + 2(3 - \alpha) w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \quad (\text{by inequality (1.6)}).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq (2 + \alpha) \max \{ \| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} \cdot w \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 &\quad + 2 \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 &\quad + \frac{1}{2} \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 &\quad + \frac{1}{2} \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 &\quad + 2(3 - \alpha) w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 &\quad (\text{by the fact, if } A, B \in \mathcal{B}(\mathcal{H}) \text{ are Hermitian,} \\
 &\quad \text{then } w^2(A + iB) \leq \|A^2 + B^2\|) \\
 &= (2 + \alpha) \max \{ \| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} \cdot w \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 &\quad + 3 \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 &\quad + 2(3 - \alpha) w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right).
 \end{aligned}$$

EXAMPLE 2.18. The following example demonstrates that the upper bound given by Theorem 2.16 can be strictly smaller than the corresponding upper bound in (1.11). To see this, consider the Hilbert space $\mathbb{C}^2$, and let $P, Q, T, S \in \mathcal{B}(\mathbb{C}^2)$ be given by $P = Q = 0$, and $T = S = I$ where I is the identity operator, and let $\alpha = \frac{1}{2}$. Then, Theorem (2.16) gives $w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \leq \frac{5}{2}$. On the other hand, inequality (1.11) gives $w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \leq 3$. Therefore, Theorem 2.16 provides a strict improvement over the upper bound in (1.11).

As a direct consequence of Theorem 2.16 for a special case $P = Q = X$ and $S = T = Y$, we have the following corollary.

COROLLARY 2.19. *Let $X, Y \in \mathcal{B}(\mathcal{H})$. Then for every $\alpha \in [0, 1]$,*

$$\begin{aligned} \max \{w^4(X+Y), w^4(X-Y)\} &= w^4 \left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \right) \\ &\leq 2w^2(|X|^2 + i|Y^*|^2) + 2(3 - \alpha)w^2(YX) \\ &\quad + (2 + \alpha) \||X|^2 + |Y^*|^2\| w(YX) \\ &\quad + \min\{w((|X||Y^*|)^2), w((|Y^*||X|)^2)\} \\ &\quad + \frac{1}{2} \||X|^4 + |Y^*|^4\| \\ &\leq 3 \||X|^4 + |Y^*|^4\| + 2(3 - \alpha)w^2(YX) \\ &\quad + (2 + \alpha) \||X|^2 + |Y^*|^2\| w(YX). \end{aligned}$$

Proof.

$$\begin{aligned} w^4 \left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \right) &\leq 2w^2(|X|^2 + i|Y^*|^2) + 2(3 - \alpha)w^2(YX) \\ &\quad + (2 + \alpha) \||X|^2 + |Y^*|^2\| w(YX) \\ &\quad + \min\{w((|X||Y^*|)^2), w((|Y^*||X|)^2)\} \\ &\quad + \frac{1}{2} \||X|^4 + |Y^*|^4\| \\ &\quad \text{(by Theorem 2.16 } P = Q = X \text{ and } S = T = Y) \\ &\leq 2w^2(|X|^2 + i|Y^*|^2) + 2(3 - \alpha)w^2(YX) \\ &\quad + \frac{1}{2} \||X|^4 + |Y^*|^4\| + \frac{1}{2} \||X|^4 + |Y^*|^4\| \\ &\quad + (2 + \alpha) \||X|^2 + |Y^*|^2\| w(YX) \\ &\quad \text{(by inequality (1.6))} \\ &= 2w^2(|X|^2 + i|Y^*|^2) + 2(3 - \alpha)w^2(YX) \\ &\quad + \||X|^4 + |Y^*|^4\| + (2 + \alpha) \||X|^2 + |Y^*|^2\| w(YX) \end{aligned}$$

$$\leq 3 \| |X|^4 + |Y^*|^4 \| + 2(3 - \alpha) w^2(YX) \\ + (2 + \alpha) \| |X|^2 + |Y^*|^2 \| w(YX). \quad \square$$

As a direct consequence of Corollary 2.19 for a special case $X = Y = M$, we have the following corollary.

COROLLARY 2.20. *Let $M \in \mathcal{B}(\mathcal{H})$. Then for every $\alpha \in [0, 1]$,*

$$w^4(M) \leq \frac{1}{8} w^2(|M|^2 + i|M^*|^2) + \frac{3-\alpha}{8} w^2(M^2) \\ + \frac{1}{16} \| |M|^4 + |M^*|^4 \| + \frac{2+\alpha}{16} \| |M|^2 + |M^*|^2 \| w(M^2) \\ \leq \frac{3}{16} \| |M|^4 + |M^*|^4 \| + \frac{3-\alpha}{8} w^2(M^2) \\ + \frac{2+\alpha}{16} \| |M|^2 + |M^*|^2 \| w(M^2).$$

Proof. It follows from Lemma 2.5 and Corollary 2.19 that

$$16w^4(M) = w^4(2M) \leq 2w^2(|M|^2 + i|M^*|^2) + 2(3 - \alpha)w^2(M^2) \\ + \| |M|^4 + |M^*|^4 \| + (2 + \alpha) \| |M|^2 + |M^*|^2 \| w(M^2) \\ \text{(by taking } X = Y = M \text{ in Corollary 2.19).}$$

Thus,

$$w^4(M) \leq \frac{1}{8} w^2(|M|^2 + i|M^*|^2) + \frac{3-\alpha}{8} w^2(M^2) \\ + \frac{1}{16} \| |M|^4 + |M^*|^4 \| + \frac{2+\alpha}{16} \| |M|^2 + |M^*|^2 \| w(M^2) \\ \leq \frac{3-\alpha}{8} w^2(M^2) + \frac{3}{16} \| |M|^4 + |M^*|^4 \| \\ + \frac{2+\alpha}{16} \| |M|^2 + |M^*|^2 \| w(M^2). \quad \square$$

The next Theorem provides a new upper bound that will be used to refine the inequality (1.7).

THEOREM 2.21. *Let $Y_1, Y_2 \in \mathcal{B}(\mathcal{H})$. Then for any $\alpha \in [0, 1]$ and $r \geq 1$,*

$$w^{4r} \left(\begin{bmatrix} 0 & Y_1 \\ Y_2 & 0 \end{bmatrix} \right) \leq \frac{1+\alpha}{8} \max \{ \| |Y_1|^{4r} + |Y_2^*|^{4r} \|, \| |Y_2|^{4r} + |Y_1^*|^{4r} \| \} \\ + \frac{1+\alpha}{4} \min \{ \max \{ w((|Y_1||Y_2^*|)^{2r}), w((|Y_2||Y_1^*|)^{2r}) \}, \\ \max \{ w((|Y_1^*||Y_2|)^{2r}), w((|Y_2^*||Y_1|)^{2r}) \} \} \\ + \frac{1-\alpha}{2} \max \{ w^{2r}(Y_1 Y_2), w^{2r}(Y_2 Y_1) \}.$$

Proof. Let $M = \begin{bmatrix} 0 & Y_1 \\ Y_2 & 0 \end{bmatrix}$ and let $y \in \mathcal{H}^{(2)}$ be any unit vector. Then

$$\begin{aligned} |\langle My, y \rangle|^{4r} &= (|\langle My, y \rangle|^2)^{2r} \\ &= |\langle My, y \rangle^2|^{2r} \\ &= |\langle My, y \rangle \langle y, M^* y \rangle|^{2r} \\ &\leq \frac{1+\alpha}{2} \|My\|^{2r} \|M^* y\|^{2r} + \frac{1-\alpha}{2} |\langle My, M^* y \rangle|^{2r} \quad (\text{by Lemma 2.6}) \\ &= \frac{1+\alpha}{2} \langle |M|^2 y, y \rangle^r \langle |M^*|^2 y, y \rangle^r + \frac{1-\alpha}{2} |\langle M^2 y, y \rangle|^{2r}. \end{aligned}$$

Therefore,

$$\begin{aligned} w^{4r}(M) &= \sup_{\|y\|=1} |\langle My, y \rangle|^{4r} \\ &\leq \frac{1+\alpha}{2} \sup_{\|y\|=1} (\langle |M|^2 y, y \rangle^r \langle |M^*|^2 y, y \rangle^r) + \frac{1-\alpha}{2} w^{2r}(M^2) \\ &\leq \frac{1+\alpha}{2} \left[\frac{1}{4} \| |M|^{4r} + |M^*|^{4r} \| \right] + \frac{1}{2} \min \{ w(|M|^{2r} |M^*|^{2r}), w(|M^*|^{2r} |M|^{2r}) \} \\ &\quad + \frac{1-\alpha}{2} w^{2r}(M^2) \\ &\leq \frac{1+\alpha}{8} \max \{ \| |Y_1|^{4r} + |Y_2^*|^{4r} \|, \| |Y_2|^{4r} + |Y_1^*|^{4r} \| \} \\ &\quad + \frac{1+\alpha}{4} \min \{ \max \{ w((|Y_1| |Y_2^*|)^{2r}), w((|Y_2| |Y_1^*|)^{2r}) \}, \\ &\quad \max \{ w((|Y_1^*| |Y_2|)^{2r}), w((|Y_2^*| |Y_1|)^{2r}) \} \} \\ &\quad + \frac{1-\alpha}{2} \max \{ w^{2r}(Y_1 Y_2), w^{2r}(Y_2 Y_1) \}. \quad \square \end{aligned}$$

COROLLARY 2.22. *Let $Y \in \mathcal{B}(\mathcal{H})$. Then for every $\alpha \in [0, 1]$ and $r \geq 1$, we have*

$$\begin{aligned} w^{4r}(Y) &\leq \frac{1+\alpha}{8} \| |Y|^{4r} + |Y^*|^{4r} \| + \frac{1+\alpha}{4} \min \{ w(|Y|^{2r} |Y^*|^{2r}), w(|Y^*|^{2r} |Y|^{2r}) \} \\ &\quad + \frac{1-\alpha}{2} w^{2r}(Y^2) \\ &\leq \frac{1}{2} \| |Y|^{4r} + |Y^*|^{4r} \|. \end{aligned}$$

Proof.

$$\begin{aligned} w^{4r}(Y) &= w^{4r} \left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix} \right) \\ &\leq \frac{1+\alpha}{8} \| |Y|^{4r} + |Y^*|^{4r} \| + \frac{1+\alpha}{4} \min \{ w(|Y|^{2r} |Y^*|^{2r}), w(|Y^*|^{2r} |Y|^{2r}) \} \\ &\quad + \frac{1-\alpha}{2} w^{2r}(Y^2) \quad (\text{by Theorem 2.21 } Y_1 = Y_2 = Y) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1+\alpha}{8} \| |Y|^{4r} + |Y^*|^{4r} \| + \frac{1+\alpha}{8} \| |Y|^{4r} + |Y^*|^{4r} \| \\
&\quad + \frac{1-\alpha}{2} \cdot \frac{1}{2} \| |Y|^{4r} + |Y^*|^{4r} \| \quad (\text{by inequality (1.6)}) \\
&= \frac{1}{2} \| |Y|^{4r} + |Y^*|^{4r} \|. \quad \square
\end{aligned}$$

COROLLARY 2.23. Let $Y \in \mathcal{B}(\mathcal{H})$ and let $\alpha \in [0, 1]$. Then

$$\begin{aligned}
w^4(Y) &\leq \frac{1+\alpha}{8} \| |Y|^4 + |Y^*|^4 \| + \frac{1+\alpha}{4} \min\{w((|Y||Y^*|)^2), w((|Y^*||Y|)^2)\} \\
&\quad + \frac{1-\alpha}{2} w^2(Y^2) \\
&\leq \frac{1+\alpha}{4} \| |Y|^4 + |Y^*|^4 \| + \frac{1-\alpha}{2} w^2(Y^2) \quad (\text{by the inequality (1.7)}).
\end{aligned}$$

COROLLARY 2.24. Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then for any $\alpha \in [0, 1]$ and $r \geq 1$, we have

$$\begin{aligned}
w^{4r} \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &\leq 2^{4r-1} \max\{w^{4r}(A), w^{4r}(D)\} + (1-\alpha)2^{4r-2} \max\{w^{2r}(BC), w^{2r}(CB)\} \\
&\quad + (1+\alpha)2^{4r-4} \max\{\| |B|^{4r} + |C^*|^{4r} \|, \| |C|^{4r} + |B^*|^{4r} \| \} \\
&\quad + (1+\alpha)2^{4r-3} \min\{ \max\{w((|B||C^*|)^{2r}), w((|C||B^*|)^{2r})\}, \\
&\quad \max\{w((|B^*||C|)^{2r}), w((|C^*||B|)^{2r})\} \}.
\end{aligned}$$

Proof. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $M_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Then

$$\begin{aligned}
w^{4r}(M) &= w^{4r}(M_1 + M_2) \\
&\leq (w(M_1) + w(M_2))^{4r} \\
&\leq 2^{4r-1} w^{4r}(M_1) + 2^{4r-1} w^{4r}(M_2) \\
&\quad (\text{by the convexity of the function } f(t) = t^{4r}, r \geq 1) \\
&\leq 2^{4r-1} \max\{w^{4r}(A), w^{4r}(D)\} + (1-\alpha)2^{4r-2} \max\{w^{2r}(BC), w^{2r}(CB)\} \\
&\quad + (1+\alpha)2^{4r-4} \max\{\| |B|^{4r} + |C^*|^{4r} \|, \| |C|^{4r} + |B^*|^{4r} \| \} \\
&\quad + (1+\alpha)2^{4r-3} \min\{ \max\{w((|B||C^*|)^{2r}), w((|C||B^*|)^{2r})\}, \\
&\quad \max\{w((|B^*||C|)^{2r}), w((|C^*||B|)^{2r})\} \} \\
&\quad (\text{by Theorem 2.21}). \quad \square
\end{aligned}$$

By Corollary 2.24 for a special case $A = D$ and $B = C$, we have the following Corollary.

COROLLARY 2.25. Let $A, B \in \mathcal{B}(\mathcal{H})$. Then for any $\alpha \in [0, 1]$ and $r \geq 1$, we have

$$\begin{aligned} \max \{w^{4r}(A+B), w^{4r}(A-B)\} &= w^{4r} \left(\begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) \\ &\leq 2^{4r-1} w^{4r}(A) + (1+\alpha) 2^{4r-4} \| |B|^{4r} + |B^*|^{4r} \| \\ &\quad + (1+\alpha) 2^{4r-3} \min \{w((|B||B^*|)^{2r}), w((|B^*||B|)^{2r})\} \\ &\quad + (1-\alpha) 2^{4r-2} w^{2r}(B^2). \end{aligned}$$

The next Theorem is the last result in this work that provides a new upper bound which will be used to refine the upper bound in inequality (1.9).

THEOREM 2.26. Let $P, T, S, Q \in \mathcal{B}(\mathcal{H})$, and let $\alpha \in [0, 1]$. Then

$$\begin{aligned} w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq 2 \max \{w^2(|P|^2 + i|T^*|^2), w^2(|Q|^2 + i|S^*|^2)\} \\ &\quad + 4(2-\alpha) w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\ &\quad + (1+\alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\ &\quad + 2(1+\alpha) \min \{ \max \{w((|P||T^*|)^2), w((|Q||S^*|)^2)\}, \\ &\quad \max \{w((|T^*||P|)^2), w((|S^*||Q|)^2)\} \}. \end{aligned}$$

Proof. Let $A = \begin{bmatrix} P & T \\ S & Q \end{bmatrix}$, $R_1 = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ and $R_2 = \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}$. Then for any unit vector $x \in \mathcal{H}^{(2)}$, we have

$$\begin{aligned} |\langle Ax, x \rangle|^2 &\leq 2|\langle |R_1|^2 x, x \rangle + i\langle |R_2^*|^2 x, x \rangle|^2 + 4|\langle R_2 R_1 x, x \rangle|^2 + 8|\langle R_1 x, x \rangle \langle x, R_2^* x \rangle|^2 \\ &\quad \text{(by inequality (2.1))} \\ &\leq 2|\langle (|R_1|^2 + i|R_2^*|^2)x, x \rangle|^2 + 4|\langle R_2 R_1 x, x \rangle|^2 + 4(1+\alpha)\|R_1 x\|^2\|R_2^* x\|^2 \\ &\quad + 4(1-\alpha)|\langle R_1 x, R_2^* x \rangle|^2 \text{ (by Lemma 2.6)} \\ &= 2|\langle (|R_1|^2 + i|R_2^*|^2)x, x \rangle|^2 + 4(2-\alpha)|\langle R_2 R_1 x, x \rangle|^2 \\ &\quad + 4(1+\alpha)\langle |R_1|^2 x, x \rangle \langle |R_2^*|^2 x, x \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} w^4(A) &= \sup_{\|x\|=1} |\langle Ax, x \rangle|^4 \\ &\leq 2w^2(|R_1|^2 + i|R_2^*|^2) + 4(2-\alpha)w^2(R_2 R_1) \\ &\quad + 4(1+\alpha) \sup_{\|x\|=1} (\langle |R_1|^2 x, x \rangle \langle |R_2^*|^2 x, x \rangle) \\ &\leq 2w^2(|R_1|^2 + i|R_2^*|^2) + 4(2-\alpha)w^2(R_2 R_1) + (1+\alpha) \| |R_1|^4 + |R_2^*|^4 \| \\ &\quad + 2(1+\alpha) \min \{w(|R_1|^2 |R_2^*|^2), w(|R_2^*|^2 |R_1|^2)\} \\ &\quad \text{(by Lemma 2.9).} \end{aligned}$$

Thus,

$$\begin{aligned}
 w^4(A) \leq & 2 \max \{w^2(|P|^2 + i|T^*|^2), w^2(|Q|^2 + i|S^*|^2)\} + 4(2 - \alpha)w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 & + (1 + \alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 & + 2(1 + \alpha) \min \{ \max \{ w((|P||T^*|)^2), w((|Q||S^*|)^2) \}, \\
 & \max \{ w((|T^*||P|)^2), w((|S^*||Q|)^2) \} \}. \quad \square
 \end{aligned}$$

REMARK 2.27. The upper bound of the above theorem refines the upper bound given in inequality (1.9). To see this, note that

$$\begin{aligned}
 w^4 \left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) & \leq 4(2 - \alpha)w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 & + 2 \max \{ w^2(|P|^2 + i|T^*|^2), w^2(|Q|^2 + i|S^*|^2) \} \\
 & + (1 + \alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 & + 2(1 + \alpha) \min \{ \max \{ w((|P||T^*|)^2), w((|Q||S^*|)^2) \}, \\
 & \max \{ w((|T^*||P|)^2), w((|S^*||Q|)^2) \} \}
 \end{aligned}$$

(by Theorem 2.26)

$$\begin{aligned}
 & \leq 4(2 - \alpha)w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 & + 2 \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 & + (1 + \alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 & + 2(1 + \alpha) \min \{ \max \{ w((|P||T^*|)^2), w((|Q||S^*|)^2) \}, \\
 & \max \{ w((|T^*||P|)^2), w((|S^*||Q|)^2) \} \}
 \end{aligned}$$

(by the fact:

if $A, B \in B(H)$ are Hermitian, then $w^2(A + iB) \leq \|A^2 + B^2\|$)

$$\begin{aligned}
 & \leq 4(2 - \alpha)w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 & + (3 + \alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 & + (1 + \alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 & \text{(by inequality (1.6)).}
 \end{aligned}$$

$$\begin{aligned}
 & = 4(2 - \alpha)w^2 \left(\begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 & + 2(2 + \alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \}.
 \end{aligned}$$

EXAMPLE 2.28. The improvement obtained by Theorem 2.26 over inequality (1.9) can be illustrated by the following example. Let $P = Q = I$, $T = S = 0$, and $\alpha = 1$. Then the upper bound obtained from Theorem 2.26 is equal to 4, whereas the upper bound given by inequality (1.9) is equal to 6. Therefore, this example demonstrates that the upper bound obtained from Theorem 2.26 is strictly sharper than the upper bound given in inequality (1.9).

COROLLARY 2.29. Let $X, Y \in \mathcal{B}(\mathcal{H})$. Then for every $\alpha \in [0, 1]$,

$$\begin{aligned} & \max \{w^4(X+Y), w^4(X-Y)\} \\ &= w^4 \left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \right) \\ &\leq 2w^2(|X|^2 + |Y^*|^2) + 4(2 - \alpha)w^2(YX) + (1 + \alpha) \left(\|X\|^4 + \|Y^*\|^4 \right) \\ &\quad + 2(1 + \alpha) \min \{w((|X||Y^*|)^2), w((|Y^*||X|)^2)\} \\ &\quad \text{(by Theorem 2.26 with } P = Q = X \text{ and } S = T = Y). \end{aligned}$$

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