

BOUNDS FOR THE EUCLIDEAN $(A-M)$ -NUMERICAL RADIUS OF A PAIR OF OPERATORS IN HILBERT SPACES WITH APPLICATIONS

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Abstract. We define the p -arithmetic mean $(A-M)$ -numerical radius for the pair of operators $(A, B) \in \mathcal{B}(H)$ as

$$w_{p,v}(A, B) := \sup_{\|x\|=1} ((1-v)|\langle Ax, x \rangle|^p + v|\langle Bx, x \rangle|^p)^{1/p},$$

where $v \in [0, 1]$ and $p \geq 1$. For $p = 2$ we put $w_{e,v}(A, B) := w_{2,v}(A, B)$ and call it the *Euclidean $(A-M)$ -numerical radius*.

In this paper we obtain among others that, if $\alpha, v \in [0, 1]$, then the following upper bound

$$\begin{aligned} w_{e,v}^2(A, B) &\leq \max\{v, 1-v\} w^2(A-B) + \left\| (1-v)|A|^{2\alpha} + v|B|^{2\alpha} \right\| \\ &\quad \times \left\| (1-v)|A^*|^{2(1-\alpha)} + v|B^*|^{2(1-\alpha)} \right\|. \end{aligned}$$

and lower bound

$$\begin{aligned} w_{e,v}^2(A, B) &\geq \max\left\{ w\left[(1-v)^2 A^2 + v^2 B^2\right], (1-v)vw(AB+BA) \right\} \\ &\quad + \max\{(1-v)w(A), vw(B)\} \\ &\quad \times |w[(1-v)A + vB] - w[(1-v)A - vB]| \end{aligned}$$

are valid.

Applications for the numerical radius and Davis-Wielandt radius of an operator are also provided. Some integral inequalities of Hermite-Hadamard type are given as well.

1. Introduction

Let H be a complex Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and a corresponding norm $\|\cdot\|$. We denote the C^* -algebra of bounded linear operators on H as $\mathcal{B}(H)$. For any operator $T \in \mathcal{B}(H)$, the adjoint of T is denoted as T^* , and $|T| = (T^*T)^{\frac{1}{2}}$ represents the positive square root of T^*T . The real part and imaginary part of T are defined as $\operatorname{Re}(T) = \frac{1}{2}(T + T^*)$ and $\operatorname{Im}(T) = \frac{1}{2i}(T - T^*)$, respectively. We define the numerical range of T , denoted by $W(T)$, as the set of values $\{\langle Tx, x \rangle : x \in H, \|x\| = 1\}$.

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Let $\|T\|$ and $w(T)$ denote the *operator norm* and the *numerical radius* of T , respectively. The operator norm is given by

$$\|T\| = \sup \{ |\langle Tx, y \rangle| ; x, y \in H, \|x\| = \|y\| = 1 \},$$

while the numerical radius is defined as

$$w(T) = \sup \{ |\langle Tx, x \rangle| : x \in H, \|x\| = 1 \}.$$

It is known that the numerical radius $w(\cdot)$ defines a norm on $\mathcal{B}(H)$ and is equivalent to the operator norm $\|\cdot\|$. In fact, the following double inequality holds:

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \quad (1.1)$$

These inequalities are sharp. The first inequality becomes an equality if $T^2 = 0$, while the second inequality becomes an equality if and only if T is a normal operator. An improvement to these inequalities was established by Kittaneh [7], who showed that

$$\frac{1}{2}\sqrt{\| |T|^2 + |T^*|^2 \|} \leq w(T) \leq \frac{\sqrt{2}}{2}\sqrt{\| |T|^2 + |T^*|^2 \|}. \quad (1.2)$$

The *Davis-Wielandt radius* of $T \in \mathcal{B}(H)$, denoted by $dw(T)$, is defined as [3], [12]

$$dw(T) := \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + \|Tx\|^4 \right)^{1/2}.$$

Observe that $dw(T) \geq 0$ and $dw(T) = 0$ if and only if $T = 0$. For $\lambda \in \mathbb{C}$, we have that $dw(\lambda T) > |\lambda| dw(T)$ if $|\lambda| > 1$, $dw(\lambda T) < |\lambda| dw(T)$ if $|\lambda| < 1$ and $dw(\lambda T) = |\lambda| dw(T)$ for $|\lambda| = 1$. Note that the triangle inequality $dw(T+S) \leq dw(T) + dw(S)$ does not always hold for arbitrary $T, S \in \mathcal{B}(H)$. However this triangle inequality holds when $\operatorname{Re}(T^*S) = 0$, see [1, Corollary 2.2]. It is easy to see that

$$\max \{ w(T), \|T\|^2 \} \leq dw(T) \leq \left(w^2(T) + \|T\|^4 \right)^{1/2}$$

and they are sharp, [1, Corollary 2.2].

Zamani and Shebrawi [13, Theorem 2.1] proved that

$$dw(T) \leq \left[w^2(T - |T|^2) + 2\|T\|^2 w(T) \right]^{1/2}. \quad (1.3)$$

Further in [13, Theorems 2.13, 2.14 and 2.17], the authors also proved that

$$dw^2(T) \leq \|T\|^2 \max \{ 1, \|T\|^2 \} + \sqrt{2}w \left(|T|^2 T \right), \quad (1.4)$$

$$dw^2(T) \leq \frac{1}{2} \left(\left\| |T|^4 + |T|^2 \right\| + \left\| |T|^4 - |T|^2 \right\| + \sqrt{2}w \left(|T|^2 T \right) \right) \quad (1.5)$$

and

$$dw^2(T) \leq \|T\| \max \left\{ w(T), \|T\|^2 \right\} \left[1 + \|T\|^2 + 2w(T) \right]^{1/2} \quad (1.6)$$

for $T \in \mathcal{B}(H)$.

Recently, Bhunia et al. in [1, Theorem 2.4] obtained the following upper bound as well

$$dw(T) \leq \left\| |T|^4 + |T|^2 \right\|^{1/2} \quad (1.7)$$

for $T \in \mathcal{B}(H)$.

Let (C, D) be a pair of bounded linear operators on H . The *Euclidean operator radius* is defined by:

$$w_e(C, D) := \sup_{\|x\|=1} \left(|\langle Cx, x \rangle|^2 + |\langle Dx, x \rangle|^2 \right)^{1/2}. \quad (1.8)$$

As pointed out in [11], $w_e : B^2(H) \rightarrow [0, \infty)$ is a norm and the following inequality holds:

$$\frac{\sqrt{2}}{4} \left\| |C|^2 + |D|^2 \right\|^{1/2} \leq w_e(C, D) \leq \left\| |C|^2 + |D|^2 \right\|^{1/2}, \quad (1.9)$$

for $(C, D) \in B^2(H)$, where the constants $\frac{\sqrt{2}}{4}$ and 1 are best possible in (1.9).

We observe that for $C = T$ and $D = |T|^2$, we have that

$$\begin{aligned} w_e(T, |T|^2) &:= \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + \left| \langle |T|^2 x, x \rangle \right|^2 \right)^{1/2} \\ &= \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + \|Tx\|^4 \right)^{1/2} = dw(T). \end{aligned}$$

If we take $C = T$ and $D = |T|^2$ in (1.9), then we get

$$\frac{\sqrt{2}}{4} \left\| |T|^2 + |T|^4 \right\|^{1/2} \leq dw(T) \leq \left\| |T|^2 + |T|^4 \right\|^{1/2}, \quad (1.10)$$

which gives the upper bound from (1.7) and a similar lower bound.

In [4], for $r \geq 1$, the generalization of the numerical radius for a pair of operators $B, C \in \mathcal{B}(H)$, referred to as the *generalized s - r -numerical radius*, is defined by

$$w_r(B, C) := \sup_{\|x\|=1} (|\langle x, Bx \rangle|^r + |\langle x, Cx \rangle|^r)^{\frac{1}{r}}.$$

In 2017, Moslehian et. al. [10] obtained the following fundamental inequalities for the s - r -numerical radius of the pair of operators $B, C \in \mathcal{B}(H)$

$$w_p(B, C) \leq w_q(B, C) \leq 2^{\frac{1}{q} - \frac{1}{p}} w_p(B, C) \text{ for } p \geq q \geq 1,$$

$$2^{\frac{1}{p} - 2} \left\| |B^*|^2 + |C^*|^2 \right\| \leq w_p(B, C), \quad p \geq 2,$$

$$2^{\frac{1}{p}-1} \max \{w(B+C), w(B-C)\} \leq w_p(B, C), \text{ and}$$

$$2^{\frac{1}{p}-1} \max \{w(B), w(C)\} \leq w_p(B, C), \quad p \geq 1,$$

and

$$2^{\frac{1}{p}-1} w^{\frac{1}{2}}(B^2 + C^2) \leq w_p(B, C), \quad p \geq 1,$$

to mention only a few. For $p = 2$, the above inequalities provided various bounds for the Euclidean numerical radius. The authors of [10] also applied their results for the *Cartesian decomposition* of an operator A .

We define the p -arithmetic mean (A - M)-numerical radius for the pair of operators $(A, B) \in \mathcal{B}(H)$ as

$$w_{p,v}(A, B) := \sup_{\|x\|=1} ((1-v)|\langle Ax, x \rangle|^p + v|\langle Bx, x \rangle|^p)^{1/p},$$

where $v \in [0, 1]$ and $p \geq 1$.

We observe that $w_{p,v}(A, B) \geq 0$ for $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$ and $w_{p,v}(A, B) = 0$ only for $(A, B) = (0, 0)$. Also for $\alpha \in \mathbb{C}$ we have that $w_{p,v}(\alpha A, \alpha B) = |\alpha| w_{p,v}(A, B)$ and by the *weighted Minkowski's inequality*

$$w_{e,v}(A+C, B+D) \leq w_{e,v}(A, B) + w_{e,v}(C, D),$$

which shows that $w_{p,v}(\cdot, \cdot)$ is a *norm* on $\mathcal{B}(H) \times \mathcal{B}(H)$.

We observe that, by the convexity of power function $g(t) = t^p$, $p \geq 1$ on $[0, \infty)$ we have that

$$\begin{aligned} ((1-v)|\langle Ax, x \rangle|^p + v|\langle Bx, x \rangle|^p)^{1/p} &\geq (1-v)|\langle Ax, x \rangle| + v|\langle Bx, x \rangle| \\ &\geq |\langle [(1-v)A + vB]x, x \rangle| \end{aligned}$$

for all $x \in H$. By taking the supremum over $x \in H$, $\|x\| = 1$ we obtain the following lower bound for the p -(A - M)-numerical radius

$$w_{p,v}(A, B) \geq w[(1-v)A + vB] \quad (1.11)$$

for $v \in [0, 1]$ and $p \geq 1$. When $B = A$ we obtain equality.

For $v = 1/2$ we have

$$w_{p,1/2}(A, B) = \frac{1}{2^{1/p}} w_p(A, B) \text{ for } p \geq 1.$$

This gives for $p = 2$ that

$$w_{2,1/2}^2(A, B) = \frac{1}{2} w_e^2(A, B)$$

For $p = 1$, we put

$$w_v(A, B) := w_{1,v}(A, B) = \sup_{\|x\|=1} ((1-v)|\langle Ax, x \rangle| + v|\langle Bx, x \rangle|)$$

and for $p = 2$,

$$w_{e,v}(A, B) := w_{2,v}(A, B) = \sup_{\|x\|=1} \left((1-v) |\langle Ax, x \rangle|^2 + v |\langle Bx, x \rangle|^2 \right)^{1/2}.$$

We use the following double inequality obtained by Kittaneh and Manasrah [8], [9] that provide a refinement and an additive reverse for *Young's inequality* as follows:

$$r \left(\sqrt{a} - \sqrt{b} \right)^2 \leq (1-v)a + vb - a^{1-v}b^v \leq R \left(\sqrt{a} - \sqrt{b} \right)^2 \quad (1.12)$$

where $a, b \geq 0$, $v \in [0, 1]$, $r = \min \{1-v, v\}$ and $R = \max \{1-v, v\}$.

In 1952, Kato [5] proved the following celebrated generalization of *Schwarz inequality* for any bounded linear operator T on H :

$$|\langle Tx, y \rangle|^2 \leq \langle (T^*T)^\alpha x, x \rangle \langle (TT^*)^{1-\alpha} y, y \rangle, \quad (1.13)$$

for any $x, y \in H$, $\alpha \in [0, 1]$. Utilizing the modulus notation introduced before, we can write (1.13) as follows

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2(1-\alpha)} y, y \rangle \quad (1.14)$$

for any $x, y \in H$, $\alpha \in [0, 1]$.

Motivated by the above results, in this paper we obtained among others that, if $\alpha, v \in [0, 1]$, then the following upper bound

$$\begin{aligned} w_{e,v}^2(A, B) &\leq R w^2(A - B) + \left\| (1-v) |A|^{2\alpha} + v |B|^{2\alpha} \right\| \\ &\quad \times \left\| (1-v) |A^*|^{2(1-\alpha)} + v |B^*|^{2(1-\alpha)} \right\| \end{aligned}$$

and the lower bound

$$\begin{aligned} w_{e,v}^2(A, B) &\geq \max \left\{ w \left[(1-v)^2 A^2 + v^2 B^2 \right], (1-v) v w(AB + BA) \right\} \\ &\quad + \max \{ (1-v) w(A), v w(B) \} \\ &\quad \times |w[(1-v)A + vB] - w[(1-v)A - vB]| \end{aligned}$$

are valid

In particular, for $v = 1/2$, we have

$$w_e^2(A, B) \leq w^2(A - B) + \frac{1}{2} \left\| |A|^{2\alpha} + |B|^{2\alpha} \right\| \left\| |A^*|^{2(1-\alpha)} + |B^*|^{2(1-\alpha)} \right\|$$

and

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} \max \{ w(A^2 + B^2), w(AB + BA) \} \\ &\quad + \frac{1}{2} \max \{ w(A), w(B) \} |w(A + B) - w(A - B)|. \end{aligned}$$

This shows that all the inequalities for $w_{e,v}(\cdot, \cdot)$ are weighted generalizations for the case of Euclidean numerical radius $w_e(\cdot, \cdot)$ that was investigated by other authors as mentioned above.

Applications for the numerical radius and Davis-Wielandt radius of an operator are also provided. Some integral inequalities of Hermite-Hadamard type are given as well.

In order to improve readability, we summarize in the following the notations used in this paper

$$\begin{aligned} dw(T) &:= \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + \|Tx\|^4 \right)^{1/2}, \\ w_r(B, C) &:= \sup_{\|x\|=1} (|\langle x, Bx \rangle|^r + |\langle x, Cx \rangle|^r)^{\frac{1}{r}}, \\ w_e(C, D) &:= \sup_{\|x\|=1} \left(|\langle Cx, x \rangle|^2 + |\langle Dx, x \rangle|^2 \right)^{1/2}, \\ w_{p,v}(A, B) &:= \sup_{\|x\|=1} ((1-v)|\langle Ax, x \rangle|^p + v|\langle Bx, x \rangle|^p)^{1/p}, \\ w_{e,i}(A, B) &:= \int_0^1 \sup_{\|x\|=1} \left((1-v)|\langle Ax, x \rangle|^2 + v|\langle Bx, x \rangle|^2 \right)^{1/2} dv, \\ \|x, y\|_{p-HH} &:= \begin{cases} \left(\int_0^1 \|(1-t)x + ty\|^p dt \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{t \in [0,1]} \|(1-t)x + ty\| & \text{if } p = \infty. \end{cases} \end{aligned}$$

Some more particular notations for one operator are also listed below:

$$\begin{aligned} \eta_{p,v}(T) &:= \sup_{\|x\|=1} ((1-v)|\langle \operatorname{Re} Tx, x \rangle|^p + v|\langle \operatorname{Im} Tx, x \rangle|^p)^{1/p}, \\ \eta_p(T) &:= \sup_{\|x\|=1} (|\langle \operatorname{Re} Tx, x \rangle|^p + |\langle \operatorname{Im} Tx, x \rangle|^p)^{1/p}, \\ d_{p,v}w(T) &:= \sup_{\|x\|=1} \left((1-v)|\langle Tx, x \rangle|^p + v\|Tx\|^{2p} \right)^{1/p}. \end{aligned}$$

2. Upper bounds

In the sequel $\|T\|$ denotes the operator norm of $T \in \mathcal{B}(H)$. For $v \in [0, 1]$, we also consider $R := \max\{1-v, v\}$.

Our first main result is as follows:

THEOREM 1. *Let $A, B \in \mathcal{B}(H)$. For all $v \in [0, 1]$, we have that*

$$w_{e,v}^2(A, B) \leq R w^2(A - B) + [w(A)]^{2(1-v)} [w(B)]^{2v}. \quad (2.1)$$

In particular, for $v = 1/2$ we get

$$w_e^2(A, B) \leq w^2(A - B) + 2w(A)w(B).$$

If $\alpha \in [0, 1]$, then also

$$w_{e,v}^2(A, B) \leq R w^2(A - B) + \left\| (1 - v) |A|^{2\alpha} + v |B|^{2\alpha} \right\| \times \left\| (1 - v) |A^*|^{2(1-\alpha)} + v |B^*|^{2(1-\alpha)} \right\|. \quad (2.2)$$

In particular, for $v = 1/2$ we get

$$w_e^2(A, B) \leq w^2(A - B) + \frac{1}{2} \left\| |A|^{2\alpha} + |B|^{2\alpha} \right\| \left\| |A^*|^{2(1-\alpha)} + |B^*|^{2(1-\alpha)} \right\|.$$

Also, we have

$$w_{e,v}^2(A, B) \leq R w^2(A - B) + \left\| (1 - v) |A|^{2\alpha} + v |B^*|^{2(1-\alpha)} \right\| \times \left\| (1 - v) |A^*|^{2(1-\alpha)} + v |B|^{2\alpha} \right\| \quad (2.3)$$

and, in particular,

$$w_e^2(A, B) \leq w^2(A - B) + \frac{1}{2} \left\| |A|^{2\alpha} + |B^*|^{2(1-\alpha)} \right\| \left\| |A^*|^{2(1-\alpha)} + |B|^{2\alpha} \right\|.$$

Proof. From (1.12) we get

$$\begin{aligned} & (1 - v) |\langle Ax, x \rangle|^2 + v |\langle Bx, x \rangle|^2 \\ & \leq |\langle Ax, x \rangle|^{2(1-v)} |\langle Bx, x \rangle|^{2v} + R (|\langle Ax, x \rangle| - |\langle Bx, x \rangle|)^2 \\ & \leq |\langle Ax, x \rangle|^{2(1-v)} |\langle Bx, x \rangle|^{2v} + R |\langle Ax, x \rangle - \langle Bx, x \rangle|^2 \\ & = |\langle Ax, x \rangle|^{2(1-v)} |\langle Bx, x \rangle|^{2v} + R |\langle (A - B)x, x \rangle|^2 \end{aligned} \quad (2.4)$$

for $x \in H$.

For $x \in H$ with $\|x\| = 1$ we have that

$$|\langle Ax, x \rangle|^{2(1-v)} |\langle Bx, x \rangle|^{2v} \leq [w(A)]^{2(1-v)} [w(B)]^{2v}$$

and

$$|\langle (A - B)x, x \rangle|^2 \leq w^2(A - B).$$

Therefore

$$(1 - v) |\langle Ax, x \rangle|^2 + v |\langle Bx, x \rangle|^2 \leq [w(A)]^{2(1-v)} [w(B)]^{2v} + R w^2(A - B)$$

for all $x \in H$ with $\|x\| = 1$. By taking the supremum over $\|x\| = 1$ we obtain (2.1).

Now, by Kato's inequality we have that

$$|\langle Ax, x \rangle|^{2(1-v)} \leq \left\langle |A|^{2\alpha} x, x \right\rangle^{1-v} \left\langle |A^*|^{2(1-\alpha)} x, x \right\rangle^{1-v}$$

and

$$|\langle Bx, x \rangle|^{2\nu} \leq \left\langle |B|^{2\alpha} x, x \right\rangle^\nu \left\langle |B^*|^{2(1-\alpha)} x, x \right\rangle^\nu,$$

which implies that

$$\begin{aligned} & |\langle Ax, x \rangle|^{2(1-\nu)} |\langle Bx, x \rangle|^{2\nu} \\ & \leq \left\langle |A|^{2\alpha} x, x \right\rangle^{1-\nu} \left\langle |B|^{2\alpha} x, x \right\rangle^\nu \left\langle |A^*|^{2(1-\alpha)} x, x \right\rangle^{1-\nu} \left\langle |B^*|^{2(1-\alpha)} x, x \right\rangle^\nu \end{aligned} \quad (2.5)$$

for all $x \in H$.

Now, by the *A-G inequality* we have that

$$\begin{aligned} \left\langle |A|^{2\alpha} x, x \right\rangle^{1-\nu} \left\langle |B|^{2\alpha} x, x \right\rangle^\nu & \leq (1-\nu) \left\langle |A|^{2\alpha} x, x \right\rangle + \nu \left\langle |B|^{2\alpha} x, x \right\rangle \\ & = \left\langle \left[(1-\nu) |A|^{2\alpha} + \nu |B|^{2\alpha} \right] x, x \right\rangle \end{aligned}$$

and

$$\begin{aligned} & \left\langle |A^*|^{2(1-\alpha)} x, x \right\rangle^{(1-\nu)} \left\langle |B^*|^{2(1-\alpha)} x, x \right\rangle^\nu \\ & \leq \left\langle (1-\nu) |A^*|^{2(1-\alpha)} x, x \right\rangle + \nu \left\langle |B^*|^{2(1-\alpha)} x, x \right\rangle \\ & = \left\langle \left[(1-\nu) |A^*|^{2(1-\alpha)} + \nu |B^*|^{2(1-\alpha)} \right] x, x \right\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} & \left\langle |A|^{2\alpha} x, x \right\rangle^{1-\nu} \left\langle |B|^{2\alpha} x, x \right\rangle^\nu \left\langle |A^*|^{2(1-\alpha)} x, x \right\rangle^{(1-\nu)} \left\langle |B^*|^{2(1-\alpha)} x, x \right\rangle^\nu \\ & \leq \left\langle \left[(1-\nu) |A|^{2\alpha} + \nu |B|^{2\alpha} \right] x, x \right\rangle \\ & \quad \times \left\langle \left[(1-\nu) |A^*|^{2(1-\alpha)} + \nu |B^*|^{2(1-\alpha)} \right] x, x \right\rangle \end{aligned} \quad (2.6)$$

for all $x \in H$

From (2.4)–(2.5) we get that

$$\begin{aligned} & (1-\nu) |\langle Ax, x \rangle|^2 + \nu |\langle Bx, x \rangle|^2 \\ & \leq R |\langle (A-B)x, x \rangle|^2 + \left\langle \left[(1-\nu) |A|^{2\alpha} + \nu |B|^{2\alpha} \right] x, x \right\rangle \\ & \quad \times \left\langle \left[(1-\nu) |A^*|^{2(1-\alpha)} + \nu |B^*|^{2(1-\alpha)} \right] x, x \right\rangle \\ & \leq R w^2 (A-B) + \left\| (1-\nu) |A|^{2\alpha} + \nu |B|^{2\alpha} \right\| \\ & \quad \times \left\| (1-\nu) |A^*|^{2(1-\alpha)} + \nu |B^*|^{2(1-\alpha)} \right\| \end{aligned}$$

for all $x \in H$ with $\|x\| = 1$. By taking the supremum over $\|x\| = 1$ we obtain (2.2).

Also, we have that

$$\begin{aligned} & \left\langle |A|^{2\alpha} x, x \right\rangle^{1-\nu} \left\langle |B^*|^{2(1-\alpha)} x, x \right\rangle^{\nu} \left\langle |A^*|^{2(1-\alpha)} x, x \right\rangle^{1-\nu} \left\langle |B|^{2\alpha} x, x \right\rangle^{\nu} \\ & \leq \left[(1-\nu) \left\langle |A|^{2\alpha} x, x \right\rangle + \nu \left\langle |B^*|^{2(1-\alpha)} x, x \right\rangle \right] \\ & \quad \times \left[(1-\nu) \left\langle |A^*|^{2(1-\alpha)} x, x \right\rangle + \nu \left\langle |B|^{2\alpha} x, x \right\rangle \right] \\ & = \left\langle \left[(1-\nu) |A|^{2\alpha} + \nu |B^*|^{2(1-\alpha)} \right] x, x \right\rangle \\ & \quad \times \left\langle \left[(1-\nu) |A^*|^{2(1-\alpha)} + \nu |B|^{2\alpha} \right] x, x \right\rangle \end{aligned}$$

for all $x \in H$, which, as above, gives the desired result (2.3). $\square$

REMARK 1. Let $\nu \in [0, 1]$. From (2.2) we get for $\alpha = 1/2$ that

$$w_{e,\nu}^2(A, B) \leq R w^2(A - B) + \|(1-\nu)|A| + \nu|B|\| \|(1-\nu)|A^*| + \nu|B^*|\|.$$

In particular, for $\nu = 1/2$ we obtain

$$w_e^2(A, B) \leq w^2(A - B) + \frac{1}{2} \||A| + |B|\| \||A^*| + |B^*|\|.$$

From (2.3) we also get

$$w_{e,\nu}^2(A, B) \leq R w^2(A - B) + \|(1-\nu)|A| + \nu|B^*|\| \|(1-\nu)|A^*| + \nu|B|\|$$

and for $\nu = 1/2$ we obtain

$$w_e^2(A, B) \leq w^2(A - B) + \frac{1}{2} \||A| + |B^*|\| \||A^*| + |B|\|.$$

3. Lower bounds

The following result provides lower bounds for the p -($A-M$)-numerical radius:

THEOREM 2. Let $A, B \in \mathcal{B}(H)$. For all $\nu \in [0, 1]$ we have that

$$\begin{aligned} w_{e,\nu}^2(A, B) & \geq \max \left\{ w \left[(1-\nu)^2 A^2 + \nu^2 B^2 \right], (1-\nu) \nu w(AB + BA) \right\} \\ & \quad + \max \{ (1-\nu) w(A), \nu w(B) \} \\ & \quad \times |w[(1-\nu)A + \nu B] - w[(1-\nu)A - \nu B]|. \end{aligned} \quad (3.1)$$

In particular, for $\nu = 1/2$ we obtain:

$$\begin{aligned} w_e^2(A, B) & \geq \frac{1}{2} \max \{ w(A^2 + B^2), w(AB + BA) \} \\ & \quad + \frac{1}{2} \max \{ w(A), w(B) \} |w(A + B) - w(A - B)|. \end{aligned} \quad (3.2)$$

Proof. From (1.11) we have

$$w_{e,v}^2(A, B) \geq w^2[(1-v)A + vB]$$

and

$$w_{e,v}^2(A, B) \geq w^2[(1-v)A - vB]$$

for $v \in [0, 1]$.

This implies that

$$\begin{aligned} w_{e,v}^2(A, B) &\geq \max \{w^2[(1-v)A + vB], w^2[(1-v)A - vB]\} \\ &= \frac{1}{2} (w^2[(1-v)A + vB] + w^2[(1-v)A - vB]) \\ &\quad + \frac{1}{2} |w^2[(1-v)A + vB] - w^2[(1-v)A - vB]| \\ &= \frac{1}{2} (w^2[(1-v)A + vB] + w^2[(1-v)A - vB]) \\ &\quad + \frac{1}{2} [w[(1-v)A + vB] + w[(1-v)A - vB]] \\ &\quad \times |w[(1-v)A + vB] - w[(1-v)A - vB]| \end{aligned} \quad (3.3)$$

for $v \in [0, 1]$.

Observe that

$$\begin{aligned} &w^2[(1-v)A + vB] + w^2[(1-v)A - vB] \\ &\geq w\left([(1-v)A + vB]^2\right) + w\left([(1-v)A - vB]^2\right) \\ &\geq \max \left\{ w\left([(1-v)A + vB]^2 + [(1-v)A - vB]^2\right), \right. \\ &\quad \left. w\left([(1-v)A + vB]^2 - [(1-v)A - vB]^2\right) \right\}. \end{aligned} \quad (3.4)$$

Since

$$\begin{aligned} &w\left([(1-v)A + vB]^2 + [(1-v)A - vB]^2\right) \\ &= w\left[(1-v)^2 A^2 + (1-v)v(AB + BA) + v^2 B^2\right. \\ &\quad \left.+ (1-v)^2 A^2 - (1-v)v(AB + BA) + v^2 B^2\right] \\ &= 2w\left[(1-v)^2 A^2 + v^2 B^2\right] \end{aligned}$$

and

$$\begin{aligned} &w\left([(1-v)A + vB]^2 - [(1-v)A - vB]^2\right) \\ &= w\left[(1-v)^2 A^2 + (1-v)v(AB + BA) + v^2 B^2\right. \\ &\quad \left.- (1-v)^2 A^2 + (1-v)v(AB + BA) - v^2 B^2\right] \\ &= 2(1-v)vw(AB + BA), \end{aligned}$$

then by (3.4) we get for $v \in [0, 1]$ that

$$\begin{aligned} & w^2[(1-v)A + vB] + w^2[(1-v)A - vB] \\ & \geq 2 \max \left\{ w \left[(1-v)^2 A^2 + v^2 B^2 \right], (1-v)vw(AB + BA) \right\}. \end{aligned} \quad (3.5)$$

Also we have that

$$w[(1-v)A + vB] + w[(1-v)A - vB] \geq 2 \max \{ w[(1-v)A], w(vB) \} \quad (3.6)$$

for $v \in [0, 1]$.

By (3.3), (3.5) and (3.6) we then get (3.1). $\square$

COROLLARY 1. *Let $C, D \in \mathcal{B}(H)$. For all $v \in [0, 1]$ we have that*

$$\begin{aligned} & w_e^2(C, D) + |1 - 2v|w(C)w(D) \\ & \geq w_{e,v}^2(C + D, C - D) \\ & \geq \max \left\{ w \left[\left[(1-v)^2 + v^2 \right] (C^2 + D^2) + \left((1-v)^2 - v^2 \right) (CD + DC) \right], \right. \\ & \quad \left. 2(1-v)vw(C^2 - D^2) \right\} \\ & \quad + \max \{ (1-v)w(C + D), vw(C - D) \} \\ & \quad \times |w[C + (1 - 2v)D] - w[(1 - 2v)C + D]|. \end{aligned} \quad (3.7)$$

In particular, for $v = 1/2$, we get

$$\begin{aligned} w_e^2(C, D) & \geq \frac{1}{2} \max \{ w(C^2 + D^2), w(C^2 - D^2) \} \\ & \quad + \frac{1}{2} \max \{ w(C + D), w(C - D) \} |w(C) - w(D)|. \end{aligned} \quad (3.8)$$

Proof. If we replace A by $C + D$ and B by $C - D$ in (3.1) then we get

$$\begin{aligned} & w_{2,v}^2(C + D, C - D) \\ & \geq \max \left\{ w \left[(1-v)^2 (C + D)^2 + v^2 (C - D)^2 \right], \right. \\ & \quad \left. (1-v)vw((C + D)(C - D) + (C - D)(C + D)) \right\} \\ & \quad + \max \{ (1-v)w(C + D), vw(C - D) \} \\ & \quad \times |w[(1-v)(C + D) + v(C - D)] - w[(1-v)(C + D) - v(C - D)]| \end{aligned} \quad (3.9)$$

for $v \in [0, 1]$.

Observe that

$$\begin{aligned} & (1-v)^2 (C + D)^2 + v^2 (C - D)^2 \\ & = (1-v)^2 (C^2 + CD + DC + D^2) + v^2 (C^2 - CD - DC + D^2) \\ & = \left[(1-v)^2 + v^2 \right] (C^2 + D^2) + \left((1-v)^2 - v^2 \right) (CD + DC), \end{aligned}$$

and

$$\begin{aligned} & (C+D)(C-D) + (C-D)(C+D) \\ &= C^2 + DC - CD - D^2 + C^2 - DC + CD - D^2 = 2(C^2 - D^2). \end{aligned}$$

By utilising (3.9) we obtain the second inequality in (3.7).

Observe that

$$\begin{aligned} & w_{2,v}^2(C+D, C-D) \\ &= \sup_{\|x\|=1} \left((1-v) |\langle (C+D)x, x \rangle|^2 + v |\langle (C-D)x, x \rangle|^2 \right) \\ &= \sup_{\|x\|=1} \left((1-v) |\langle Cx, x \rangle + \langle Dx, x \rangle|^2 + v |\langle Cx, x \rangle - \langle Dx, x \rangle|^2 \right). \end{aligned}$$

Also, for $x \in H$, $v \in [0, 1]$ we have that

$$\begin{aligned} & (1-v) |\langle Cx, x \rangle + \langle Dx, x \rangle|^2 + v |\langle Cx, x \rangle - \langle Dx, x \rangle|^2 \\ &= (1-v) \left(|\langle Cx, x \rangle|^2 + 2\operatorname{Re} [\langle Cx, x \rangle \overline{\langle Dx, x \rangle}] + |\langle Dx, x \rangle|^2 \right) \\ &\quad + v \left(|\langle Cx, x \rangle|^2 - 2\operatorname{Re} [\langle Cx, x \rangle \overline{\langle Dx, x \rangle}] + |\langle Dx, x \rangle|^2 \right) \\ &= |\langle Cx, x \rangle|^2 + |\langle Dx, x \rangle|^2 + 2\operatorname{Re} [(1-2v) \langle Cx, x \rangle \overline{\langle Dx, x \rangle}]. \end{aligned}$$

Since, for $x \in H$, $\|x\| = 1$,

$$\begin{aligned} \operatorname{Re} [(1-2v) \langle Cx, x \rangle \overline{\langle Dx, x \rangle}] &\leq \left| (1-2v) \langle Cx, x \rangle \overline{\langle Dx, x \rangle} \right| \\ &\leq |1-2v| |\langle Cx, x \rangle| |\langle Dx, x \rangle| \\ &\leq |1-2v| w(C) w(D), \end{aligned}$$

then

$$\begin{aligned} & (1-v) |\langle Cx, x \rangle + \langle Dx, x \rangle|^2 + v |\langle Cx, x \rangle - \langle Dx, x \rangle|^2 \\ &\leq |\langle Cx, x \rangle|^2 + |\langle Dx, x \rangle|^2 + |1-2v| w(C) w(D) \end{aligned}$$

for $x \in H$, $\|x\| = 1$, which, by taking the supremum over $x \in H$, $\|x\| = 1$ gives the first inequality in (3.7). $\square$

4. Some integral inequalities

We define the *integral numerical radius* by

$$\begin{aligned} w_{e,i}(A, B) &:= \int_0^1 w_{e,v}(A, B) dv \\ &= \int_0^1 \sup_{\|x\|=1} \left((1-v) |\langle Ax, x \rangle|^2 + v |\langle Bx, x \rangle|^2 \right)^{1/2} dv. \end{aligned}$$

We observe that $w_{e,i}(A, B) \geq 0$ for any $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$. If $w_{e,i}(A, B) = 0$ then $w_{e,v}(A, B) = 0$ for almost every $v \in [0, 1]$ and since $w_{2,v}(A, B)$ is a norm, then $(A, B) = (A, B)$. Also $w_{e,i}(\alpha A, \alpha B) = |\alpha| w_{e,i}(A, B)$ and

$$\begin{aligned} w_{e,i}(A + C, B + D) &= \int_0^1 w_{e,v}(A + C, B + D) dv \\ &\leq \int_0^1 (w_{e,v}(A, B) + w_{e,v}(C, D)) dv = w_{e,i}(A, B) + w_{e,i}(C, D), \end{aligned}$$

which shows that $w_{e,i}(\cdot, \cdot)$ is also a *norm* on $\mathcal{B}(H) \times \mathcal{B}(H)$.

In the sequel, by utilising some convexity properties and the celebrated Hermite-Hadamard inequalities, we can provide other bounds as follows:

PROPOSITION 1. For $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$ we have for $v_1, v_2 \in [0, 1]$ with $v_1 \neq v_2$ that

$$w_{e,(1-t)v_1+tv_2}(A, B) \leq (1-t)^{1/2} w_{e,v_1}(A, B) + t^{1/2} w_{e,v_2}(A, B) \quad (4.1)$$

for all $t \in [0, 1]$.

In particular, for $t = 1/2$ we get

$$\sqrt{2} w_{e, \frac{v_1+v_2}{2}}(A, B) \leq w_{e,v_1}(A, B) + w_{e,v_2}(A, B). \quad (4.2)$$

We also have that

$$\begin{aligned} \frac{\sqrt{2}}{2} w_{e, \frac{v_1+v_2}{2}}(A, B) &\leq \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} w_{e,v}(A, B) dv \\ &\leq \frac{2}{3} [w_{e,v_1}(A, B) + w_{e,v_2}(A, B)]. \end{aligned} \quad (4.3)$$

In particular, for $v_1 = 0, v_2 = 1$ we obtain

$$\frac{1}{2} w_e(A, B) \leq w_{e,i}(A, B) \leq \frac{2}{3} [w(A) + w(B)].$$

Proof. Consider the function

$$f_x(v) := \left((1-v) |\langle Ax, x \rangle|^2 + v |\langle Bx, x \rangle|^2 \right)^{1/2}, \quad v \in [0, 1].$$

Let $v_1, v_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned} &f_x(\alpha v_1 + \beta v_2) \\ &= \left((1 - \alpha v_1 - \beta v_2) |\langle Ax, x \rangle|^2 + (\alpha v_1 + \beta v_2) |\langle Bx, x \rangle|^2 \right)^{1/2} \\ &= \left(\alpha \left[(1 - v_1) |\langle Ax, x \rangle|^2 + v_1 |\langle Bx, x \rangle|^2 \right] \right. \\ &\quad \left. + \beta \left[(1 - v_2) |\langle Ax, x \rangle|^2 + v_2 |\langle Bx, x \rangle|^2 \right] \right)^{1/2}. \end{aligned}$$

By making use of the property

$$(c+d)^{1/2} \leq c^{1/2} + d^{1/2}, \quad c, d \geq 0$$

we have that

$$\begin{aligned} & \left(\alpha \left[(1-v_1) |\langle Ax, x \rangle|^2 + v_1 |\langle Bx, x \rangle|^2 \right] \right. \\ & \quad \left. + \beta \left[(1-v_2) |\langle Ax, x \rangle|^2 + v_2 |\langle Bx, x \rangle|^2 \right] \right)^{1/2} \\ & \leq \alpha^{1/2} \left[(1-v_1) |\langle Ax, x \rangle|^2 + v_1 |\langle Bx, x \rangle|^2 \right]^{1/2} \\ & \quad + \beta^{1/2} \left[(1-v_2) |\langle Ax, x \rangle|^2 + v_2 |\langle Bx, x \rangle|^2 \right]^{1/2} \\ & = \alpha^{1/2} f_x(v_1) + \beta^{1/2} f_x(v_2). \end{aligned}$$

These imply that

$$f_x(\alpha v_1 + \beta v_2) \leq \alpha^{1/2} f_x(v_1) + \beta^{1/2} f_x(v_2).$$

By taking the supremum over $x \in H$, $\|x\| = 1$, we get that

$$\begin{aligned} w_{e, \alpha v_1 + \beta v_2}(A, B) &= \sup_{\|x\|=1} f_x(\alpha v_1 + \beta v_2) \\ &\leq \sup_{\|x\|=1} \left[\alpha^{1/2} f_x(v_1) + \beta^{1/2} f_x(v_2) \right] \\ &\leq \alpha^{1/2} \sup_{\|x\|=1} f_x(v_1) + \beta^{1/2} \sup_{\|x\|=1} f_x(v_2) \\ &= \alpha^{1/2} w_{e, v_1}(A, B) + \beta^{1/2} w_{e, v_2}(A, B), \end{aligned}$$

which proves (4.1).

If we take the integral over $t \in [0, 1]$ in (4.1), then we get

$$\begin{aligned} & \int_0^1 w_{e, (1-t)v_1 + tv_2}(A, B) dt \\ & \leq \int_0^1 (1-t)^{1/2} dt w_{e, v_1}(A, B) + \int_0^1 t^{1/2} dt w_{e, v_2}(A, B) \end{aligned} \tag{4.4}$$

and since

$$\int_0^1 w_{e, (1-t)v_1 + tv_2}(A, B) dt = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} w_{e, v}(A, B) dv$$

and

$$\int_0^1 (1-t)^{1/2} dt = \int_0^1 t^{1/2} dt = \frac{2}{3},$$

then by (4.4) we get the second inequality in (4.3).

Further, by (4.2) we obtain

$$\sqrt{2}w_{e, \frac{v_1+v_2}{2}}(A, B) \leq w_{e, (1-t)v_1+tv_2}(A, B) + w_{e, tv_1+(1-t)v_2}(A, B)$$

for all $t \in [0, 1]$.

If we take the integral over $t \in [0, 1]$, then we get

$$\begin{aligned} & \sqrt{2}w_{e, \frac{v_1+v_2}{2}}(A, B) \\ & \leq \int_0^1 w_{e, (1-t)v_1+tv_2}(A, B) dt + \int_0^1 w_{e, tv_1+(1-t)v_2}(A, B) dt \end{aligned} \quad (4.5)$$

and since

$$\begin{aligned} \int_0^1 w_{e, (1-t)v_1+tv_2}(A, B) dt &= \int_0^1 w_{e, tv_1+(1-t)v_2}(A, B) dt \\ &= \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} w_{e, v}(A, B) dv, \end{aligned}$$

then by (4.5) we deduce the first inequality in (4.3). $\square$

COROLLARY 2. Let $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$. We have for $v \in (0, 1)$ that

$$\begin{aligned} & \frac{\sqrt{2}}{2} \left[vw_{e, \frac{v}{2}}(A, B) + (1-v)w_{e, \frac{v+1}{2}}(A, B) \right] \\ & \leq w_{e, i}(A, B) \leq \frac{2}{3} [vw(A) + w_{e, v}(A, B) + (1-v)w(B)]. \end{aligned} \quad (4.6)$$

In particular

$$\begin{aligned} & \frac{\sqrt{2}}{4} \left[w_{e, \frac{1}{4}}(A, B) + w_{e, \frac{3}{4}}(A, B) \right] \\ & \leq w_{e, i}(A, B) \leq \frac{1}{3} \left[w(A) + \sqrt{2}w_e(A, B) + w(B) \right]. \end{aligned} \quad (4.7)$$

Proof. If we take in (4.3) $v_1 = 0$ and $v \in (0, 1)$ then

$$\frac{\sqrt{2}}{2}w_{e, \frac{v}{2}}(A, B) \leq \frac{1}{v} \int_0^v w_{e, v}(A, B) dv \leq \frac{2}{3} [w_{e, 0}(A, B) + w_{e, v}(A, B)],$$

namely

$$\frac{\sqrt{2}}{2}vw_{e, \frac{v}{2}}(A, B) \leq \int_0^v w_{e, v}(A, B) dv \leq \frac{2}{3}v[w(A) + w_{e, v}(A, B)]. \quad (4.8)$$

If we take in (4.3) $v_1 = v \in (0, 1)$ and $v_2 = 1$ then

$$\frac{\sqrt{2}}{2}w_{e, \frac{v+1}{2}}(A, B) \leq \frac{1}{1-v} \int_v^1 w_{e, v}(A, B) dv \leq \frac{2}{3} [w_{e, v}(A, B) + w_{e, 1}(A, B)]$$

namely

$$\begin{aligned} & \frac{\sqrt{2}}{2} (1 - \nu) w_{e, \frac{\nu+1}{2}}(A, B) \\ & \leq \int_{\nu}^1 w_{e, \nu}(A, B) d\nu \leq \frac{2}{3} (1 - \nu) [w_{e, \nu}(A, B) + w(B)]. \end{aligned} \quad (4.9)$$

Now, if we add (4.8) and (4.9) then we get

$$\begin{aligned} & \frac{\sqrt{2}}{2} \nu w_{e, \frac{\nu}{2}}(A, B) + \frac{\sqrt{2}}{2} (1 - \nu) w_{e, \frac{\nu+1}{2}}(A, B) \\ & \leq \int_0^1 w_{e, \nu}(A, B) d\nu \\ & \leq \frac{2}{3} \nu [w(A) + w_{e, \nu}(A, B)] + \frac{2}{3} (1 - \nu) [w_{e, \nu}(A, B) + w(B)], \end{aligned}$$

which is equivalent to (4.6). $\square$

Let $(X, \|\cdot\|)$ be a normed space. In [6] Kikianty and Dragomir introduced the concept of p -HH-norm on the Cartesian product of two copies of a normed space as follows:

$$\|(x, y)\|_{p-HH} := \begin{cases} \left(\int_0^1 \|(1-t)x + ty\|^p dt \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{t \in [0, 1]} \|(1-t)x + ty\| & \text{if } p = \infty. \end{cases}$$

By the classical Hermite-Hadamard inequality for convex functions $f : [a, b] \rightarrow \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(s) ds \leq \frac{f(a) + f(b)}{2},$$

we conclude that

$$\left\| \frac{x+y}{2} \right\| \leq \|(x, y)\|_{p-HH} \leq \left(\frac{\|x\|^p + \|y\|^p}{2} \right)^{1/p}$$

for $1 \leq p < \infty$ and $(x, y) \in X \times X$.

PROPOSITION 2. For $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$ we have for $\nu_1, \nu_2 \in [0, 1]$ with $\nu_1 \neq \nu_2$ that

$$\begin{aligned} & w\left(\left(1 - \frac{\nu_1 + \nu_2}{2}\right)A + \frac{\nu_1 + \nu_2}{2}B\right) \\ & \leq \frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} w((1 - \nu)A + \nu B) d\nu \\ & \leq \frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} w_{e, \nu}(A, B) d\nu \leq \left[\frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} w_{e, \nu}^2(A, B) d\nu \right]^{1/2} \\ & \leq \left[\left(1 - \frac{\nu_1 + \nu_2}{2}\right)w^2(A) + \frac{\nu_1 + \nu_2}{2}w^2(B) \right]^{1/2}. \end{aligned} \quad (4.10)$$

In particular, for $v_1 = 0$ and $v_2 = 1$ we obtain

$$\begin{aligned} w\left(\frac{A+B}{2}\right) &\leq \int_0^1 w((1-v)A + vB) dv \\ &\leq w_{e,i}(A, B) \leq \left[\int_0^1 w_{e,v}^2(A, B) dv \right]^{1/2} \\ &\leq \frac{\sqrt{2}}{2} [w^2(A) + w^2(B)]^{1/2}. \end{aligned} \quad (4.11)$$

Proof. By the elementary inequality of Schwarz type we have for $v \in [0, 1]$ and $x \in H$ that

$$(1-v)|\langle Ax, x \rangle| + v|\langle Bx, x \rangle| \leq \left((1-v)|\langle Ax, x \rangle|^2 + v|\langle Bx, x \rangle|^2 \right)^{1/2}$$

and since

$$|\langle [(1-v)A + vB]x, x \rangle| \leq (1-v)|\langle Ax, x \rangle| + v|\langle Bx, x \rangle|$$

then we get

$$|\langle [(1-v)A + vB]x, x \rangle| \leq \left((1-v)|\langle Ax, x \rangle|^2 + v|\langle Bx, x \rangle|^2 \right)^{1/2}$$

for $v \in [0, 1]$ and $x \in H$.

By taking the supremum over $x \in H$, $\|x\| = 1$, we get that

$$w((1-v)A + vB) \leq w_{e,v}(A, B)$$

for all $v \in [0, 1]$.

Now, if we take the integral mean $\frac{1}{v_2-v_1} \int_{v_1}^{v_2}$ we get that

$$\frac{1}{v_2-v_1} \int_{v_1}^{v_2} w((1-v)A + vB) dv \leq \frac{1}{v_2-v_1} \int_{v_1}^{v_2} w_{e,v}(A, B) dv.$$

If we use the properties of integral and norms, since w is a norm, then we get

$$\begin{aligned} &\frac{1}{v_2-v_1} \int_{v_1}^{v_2} w((1-v)A + vB) dv \\ &\geq w\left(\frac{1}{v_2-v_1} \int_{v_1}^{v_2} [(1-v)A + vB] dv\right) \\ &= w\left(\left(1 - \frac{v_1+v_2}{2}\right)A + \frac{v_1+v_2}{2}B\right). \end{aligned}$$

By the Schwarz integral inequality we have that

$$\begin{aligned}
 & \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} w_{e,v}(A, B) dv \\
 & \leq \left[\frac{1}{v_2 - v_1} \int_{v_1}^{v_2} w_{e,v}^2(A, B) dv \right]^{1/2} \\
 & = \left[\frac{1}{v_2 - v_1} \int_{v_1}^{v_2} \sup_{\|x\|=1} \left((1-v) |\langle Ax, x \rangle|^2 + v |\langle Bx, x \rangle|^2 \right) dv \right]^{1/2} \\
 & \leq \left[\frac{1}{v_2 - v_1} \int_{v_1}^{v_2} \left((1-v) \sup_{\|x\|=1} |\langle Ax, x \rangle|^2 + v \sup_{\|x\|=1} |\langle Bx, x \rangle|^2 \right) dv \right]^{1/2} \\
 & = \left[\frac{1}{v_2 - v_1} \int_{v_1}^{v_2} ((1-v) w^2(A) + v w^2(B)) dv \right]^{1/2} \\
 & = \left[\left(1 - \frac{v_1 + v_2}{2} \right) w^2(A) + \frac{v_1 + v_2}{2} w^2(B) \right]^{1/2}
 \end{aligned}$$

and the proof is completed. $\square$

We have the following upper bounds:

PROPOSITION 3. *Let $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$, then*

$$\int_0^1 w_{e,v}^2(A, B) dv \leq \frac{3}{4} w^2(A - B) + w^2(A) L(w^2(A), w^2(B)), \quad (4.12)$$

where the logarithmic mean $L(\cdot, \cdot)$ is defined by

$$L(x, y) := \begin{cases} \frac{x-y}{\ln x - \ln y} & \text{if } x \neq y, \\ x & \text{if } x = y \end{cases}$$

and

$$\begin{aligned}
 \int_0^1 w_{e,v}^2(A, B) dv & \leq \frac{3}{4} w^2(A - B) \\
 & + \left\| \left(|A|^{2\alpha}, |B|^{2\alpha} \right) \right\|_{2-HH} \left\| \left(|A^*|^{2(1-\alpha)}, |B^*|^{2(1-\alpha)} \right) \right\|_{2-HH}
 \end{aligned} \quad (4.13)$$

for $\alpha \in (0, 1)$. In particular

$$\int_0^1 w_{e,v}^2(A, B) dv \leq \frac{3}{4} w^2(A - B) + \|(|A|, |B|)\|_{2-HH} \|(|A^*|, |B^*|)\|_{2-HH}.$$

Proof. From (2.1) we have by integration that

$$\begin{aligned}
 & \int_0^1 w_{e,v}^2(A, B) dv \\
 & \leq w^2(A - B) \int_0^1 \max\{1 - v, v\} dv + \int_0^1 [w(A)]^{2(1-v)} [w(B)]^{2v} dv \\
 & = w^2(A - B) \int_0^1 \left(\frac{1}{2} + \left| v - \frac{1}{2} \right| \right) dv + w^2(A) \int_0^1 \left(\frac{w^2(B)}{w^2(A)} \right)^v dv \\
 & = \frac{3}{4} w^2(A - B) + w^2(A) \times \begin{cases} \frac{w^2(B) - w^2(A)}{\ln w^2(B) - \ln w^2(A)} & \text{if } w(A) \neq w(B) \\ w^2(A) & \text{if } w(A) = w(B), \end{cases}
 \end{aligned}$$

which proves (4.12).

From (2.2) we also have

$$\begin{aligned}
 & \int_0^1 w_{e,v}^2(A, B) dv \leq \frac{3}{4} w^2(A - B) + \int_0^1 \left\| (1 - v) |A|^{2\alpha} + v |B|^{2\alpha} \right\| \\
 & \quad \times \left\| (1 - v) |A^*|^{2(1-\alpha)} + v |B^*|^{2(1-\alpha)} \right\| dv \\
 & \leq \frac{3}{4} w^2(A - B) + \left(\int_0^1 \left\| (1 - v) |A|^{2\alpha} + v |B|^{2\alpha} \right\|^2 dv \right)^{1/2} \\
 & \quad \times \left(\int_0^1 \left\| (1 - v) |A^*|^{2(1-\alpha)} + v |B^*|^{2(1-\alpha)} \right\|^2 dv \right)^{1/2} \\
 & = \frac{3}{4} w^2(A - B) + \left\| \left(|A|^{2\alpha}, |B|^{2\alpha} \right) \right\|_{2-HH} \\
 & \quad \times \left\| \left(|A^*|^{2(1-\alpha)}, |B^*|^{2(1-\alpha)} \right) \right\|_{2-HH},
 \end{aligned}$$

which proves (4.13). $\square$

Also we can get the following lower bounds:

PROPOSITION 4. Let $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$, then

$$\int_0^1 w_{e,v}^2(A, B) dv \geq \frac{1}{3} \max \left\{ w(A^2 + B^2), w \left(\frac{AB + BA}{2} \right) \right\}. \quad (4.14)$$

Proof. From (3.1) we have

$$w_{e,v}^2(A, B) \geq w \left[(1 - v)^2 A^2 + v^2 B^2 \right]$$

and

$$w_{e,v}^2(A, B) \geq (1 - v) v w(AB + BA)$$

for all $v \in [0, 1]$.

By taking the integral, we get that

$$\begin{aligned} \int_0^1 w_{e,v}^2(A, B) dv &\geq \int_0^1 w \left[(1-v)^2 A^2 + v^2 B^2 \right] dv \\ &\geq w \left[\int_0^1 \left((1-v)^2 A^2 + v^2 B^2 \right) dv \right] = \frac{1}{3} w(A^2 + B^2) \end{aligned}$$

and

$$\int_0^1 w_{e,v}^2(A, B) dv \geq w(AB + BA) \int_0^1 (1-v)v dv = \frac{1}{6} w(AB + BA),$$

which proves (4.14). $\square$

5. Applications for one operator

If we take $B = A^*$, then

$$w_{e,v}(A, A^*) = \sup_{\|x\|=1} \left((1-v)|\langle Ax, x \rangle|^2 + v|\langle A^*x, x \rangle|^2 \right)^{1/2} = \sup_{\|x\|=1} |\langle Ax, x \rangle| = w(A)$$

for all $v \in [0, 1]$ and by Theorem 1, inequality (2.2), we obtain

$$\begin{aligned} w^2(A) &\leq R \|A - A^*\|^2 + \left\| (1-v)|A|^{2\alpha} + v|A^*|^{2\alpha} \right\| \\ &\quad \times \left\| (1-v)|A^*|^{2(1-\alpha)} + v|A|^{2(1-\alpha)} \right\| \end{aligned} \quad (5.1)$$

for all $v \in [0, 1]$.

In particular, for $v = 1/2$ we get

$$\begin{aligned} w^2(A) &\leq \frac{1}{2} \|A - A^*\|^2 + \frac{1}{4} \left\| |A|^{2\alpha} + |A^*|^{2\alpha} \right\| \\ &\quad \times \left\| |A^*|^{2(1-\alpha)} + |A|^{2(1-\alpha)} \right\|. \end{aligned} \quad (5.2)$$

Now, if we take $\alpha = 1/2$ in (5.1), then we get

$$\begin{aligned} w^2(A) &\leq R \|A - A^*\|^2 + \left\| (1-v)|A| + v|A^*| \right\| \\ &\quad \times \left\| (1-v)|A^*| + v|A| \right\|, \end{aligned}$$

while from (5.2),

$$w^2(A) \leq \frac{1}{2} \|A - A^*\|^2 + \frac{1}{4} \left\| |A| + |A^*| \right\|^2.$$

Also, from (2.3) we have

$$\begin{aligned} w^2(A) &\leq R \|A - A^*\|^2 + \left\| (1-v)|A|^{2\alpha} + v|A|^{2(1-\alpha)} \right\| \\ &\quad \times \left\| (1-v)|A^*|^{2(1-\alpha)} + v|A^*|^{2\alpha} \right\| \end{aligned} \quad (5.3)$$

and, in particular,

$$w^2(A) \leq \frac{1}{2} \|A - A^*\|^2 + \frac{1}{4} \left\| |A|^{2\alpha} + |A|^{2(1-\alpha)} \right\| \times \left\| |A^*|^{2(1-\alpha)} + |A^*|^{2\alpha} \right\|. \quad (5.4)$$

If we take $B = A^*$ in (3.1), then we get

$$w^2(A) \geq \max \left\{ w \left[(1-\nu)^2 A^2 + \nu^2 (A^*)^2 \right], (1-\nu) \nu \left\| |A|^2 + |A^*|^2 \right\| \right\} \\ + \max \{1-\nu, \nu\} w(A) \\ \times |w[(1-\nu)A + \nu A^*] - w[(1-\nu)A - \nu A^*]| \quad (5.5)$$

for all $\nu \in [0, 1]$.

In particular, for $\nu = 1/2$ we get

$$w^2(A) \geq \frac{1}{4} \max \left\{ \left\| A^2 + (A^*)^2 \right\|, \left\| |A|^2 + |A^*|^2 \right\| \right\} \\ + \frac{1}{4} w(A) \left| \|A + A^*\| - \|A - A^*\| \right|. \quad (5.6)$$

From (3.8) we get for $(C, D) = (A, A^*)$ that

$$w^2(A) \geq \frac{1}{4} \max \left\| A^2 + (A^*)^2 \right\|, \left\| A^2 - (A^*)^2 \right\|. \quad (5.7)$$

For an operator $T \in \mathcal{B}(H)$ we consider

$$\eta_{p,\nu}(T) := w_{p,\nu}(\operatorname{Re} T, \operatorname{Im} T) = \sup_{\|x\|=1} ((1-\nu) |\langle \operatorname{Re} T x, x \rangle|^p + \nu |\langle \operatorname{Im} T x, x \rangle|^p)^{1/p} \quad (5.8)$$

for $\nu, \alpha \in [0, 1]$.

For $\nu = 1/2$ we have

$$\eta_{p,1/2}(T) = \frac{1}{2^{1/p}} \sup_{\|x\|=1} (|\langle \operatorname{Re} T x, x \rangle|^p + |\langle \operatorname{Im} T x, x \rangle|^p)^{1/p} = \frac{1}{2^{1/p}} \eta_p(T),$$

where

$$\eta_p(T) = \sup_{\|x\|=1} (|\langle \operatorname{Re} T x, x \rangle|^p + |\langle \operatorname{Im} T x, x \rangle|^p)^{1/p}.$$

We also observe that, for $p = 2$, we have

$$\eta_2(T) = \sup_{\|x\|=1} \left(|\langle \operatorname{Re} T x, x \rangle|^2 + |\langle \operatorname{Im} T x, x \rangle|^2 \right)^{1/2} = w(T).$$

We observe that $\eta_{p,\nu}(\cdot)$ is a *real norm* on $\mathcal{B}(H)$, namely $\eta_{p,\nu}(T) \geq 0$ for all $T \in \mathcal{B}(H)$, $\eta_{p,\nu}(T) = 0$ only for $T = 0$, $\eta_{p,\nu}(T+V) \leq \eta_{p,\nu}(T) + \eta_{p,\nu}(V)$ for all $T, V \in \mathcal{B}(H)$ and $\eta_{p,\nu}(aT) = |a| \eta_{p,\nu}(T)$ for all $T \in \mathcal{B}(H)$ and any real number a .

If we take $(A, B) = (\operatorname{Re} T, \operatorname{Im} T)$ in (2.1), then we get

$$\eta_{e,v}^2(T) \leq R \|\operatorname{Re} T - \operatorname{Im} T\|^2 + \|\operatorname{Re} T\|^{2(1-v)} \|\operatorname{Im} T\|^{2v} \quad (5.9)$$

for $v \in [0, 1]$.

In particular, for $v = 1/2$ we get

$$w^2(T) \leq \|\operatorname{Re} T - \operatorname{Im} T\|^2 + 2 \|\operatorname{Re} T\| \|\operatorname{Im} T\|. \quad (5.10)$$

If we take $(A, B) = (\operatorname{Re} T, \operatorname{Im} T)$ in (2.2), then we get

$$\begin{aligned} \eta_{e,v}^2(T) &\leq R \|\operatorname{Re} T - \operatorname{Im} T\|^2 + \left\| (1-v) |\operatorname{Re} T|^{2\alpha} + v |\operatorname{Im} T|^{2\alpha} \right\| \\ &\quad \times \left\| (1-v) |\operatorname{Re} T|^{2(1-\alpha)} + v |\operatorname{Im} T|^{2(1-\alpha)} \right\| \end{aligned} \quad (5.11)$$

for $v, \alpha \in [0, 1]$ and for $v = 1/2$ we get

$$\begin{aligned} w^2(T) &\leq \|\operatorname{Re} T - \operatorname{Im} T\|^2 + \frac{1}{2} \left\| |\operatorname{Re} T|^{2\alpha} + |\operatorname{Im} T|^{2\alpha} \right\| \\ &\quad \times \left\| |\operatorname{Re} T|^{2(1-\alpha)} + |\operatorname{Im} T|^{2(1-\alpha)} \right\|. \end{aligned} \quad (5.12)$$

For $\alpha = 1/2$ in (5.11) we obtain

$$\eta_{e,v}^2(T) \leq R \|\operatorname{Re} T - \operatorname{Im} T\|^2 + \|(1-v) |\operatorname{Re} T| + v |\operatorname{Im} T|\|^2 \quad (5.13)$$

while from (5.12)

$$w^2(T) \leq \|\operatorname{Re} T - \operatorname{Im} T\|^2 + \frac{1}{2} \| |\operatorname{Re} T| + |\operatorname{Im} T| \|^2. \quad (5.14)$$

From (2.3) we obtain

$$\begin{aligned} \eta_{e,v}^2(T) &\leq R \|\operatorname{Re} T - \operatorname{Im} T\|^2 + \left\| (1-v) |\operatorname{Re} T|^{2\alpha} + v |\operatorname{Im} T|^{2(1-\alpha)} \right\| \\ &\quad \times \left\| (1-v) |\operatorname{Re} T|^{2(1-\alpha)} + v |\operatorname{Im} T|^{2\alpha} \right\| \end{aligned} \quad (5.15)$$

and, in particular

$$\begin{aligned} w^2(T) &\leq \|\operatorname{Re} T - \operatorname{Im} T\|^2 + \frac{1}{2} \left\| |\operatorname{Re} T|^{2\alpha} + |\operatorname{Im} T|^{2(1-\alpha)} \right\| \\ &\quad \times \left\| |\operatorname{Re} T|^{2(1-\alpha)} + |\operatorname{Im} T|^{2\alpha} \right\|. \end{aligned} \quad (5.16)$$

The case $\alpha = 1/2$ gives the previous inequalities (5.13) and (5.14).

Now, if we take $(A, B) = (\operatorname{Re} T, \operatorname{Im} T)$ in (3.1), then we get

$$\begin{aligned} \eta_{e,v}^2(T) &\geq \max \left\{ \left\| (1-v)^2 \operatorname{Re}^2 T + v^2 \operatorname{Im}^2 T \right\|, (1-v) v \|\operatorname{Re} T \operatorname{Im} T + \operatorname{Im} T \operatorname{Re} T\| \right\} \\ &\quad + \max \{ (1-v) \|\operatorname{Re} T\|, v \|\operatorname{Im} T\| \} \\ &\quad \times \| (1-v) \operatorname{Re} T + v \operatorname{Im} T \| - \| (1-v) \operatorname{Re} T - v \operatorname{Im} T \|. \end{aligned}$$

In particular, for $\nu = 1/2$ we obtain:

$$\begin{aligned} w^2(T) &\geq \frac{1}{2} \max \{ \|\operatorname{Re}^2 T + \operatorname{Im}^2 T\|, \|\operatorname{Re} T \operatorname{Im} T + \operatorname{Im} T \operatorname{Re} T\| \} \\ &\quad + \frac{1}{2} \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \} \|\operatorname{Re} T + \operatorname{Im} T\| - \|\operatorname{Re} T - \operatorname{Im} T\|. \end{aligned}$$

From (3.8) for $(C, D) = (\operatorname{Re} T, \operatorname{Im} T)$ we get that

$$\begin{aligned} w^2(T) &\geq \frac{1}{2} \max \{ \|\operatorname{Re}^2 T + \operatorname{Im}^2 T\|, \|\operatorname{Re}^2 T - \operatorname{Im}^2 T\| \} \\ &\quad + \frac{1}{2} \max \{ \|\operatorname{Re} T + \operatorname{Im} T\|, \|\operatorname{Re} T - \operatorname{Im} T\| \} \|\operatorname{Re} T\| - \|\operatorname{Im} T\|. \end{aligned}$$

6. Applications for Davis-Wielandt radius

We define for $p \geq 1$ and $\nu \in [0, 1]$ the p -(A - M) Davis-Wielandt radius of $T \in \mathcal{B}(H)$ as

$$d_{p,\nu} w(T) := w_{p,\nu} \left(T, |T|^2 \right) = \sup_{\|x\|=1} \left((1-\nu) |\langle Tx, x \rangle|^p + \nu \|Tx\|^{2p} \right)^{1/p}.$$

By (1.11) we obtain the following lower bound

$$d_{p,\nu} w(T) \geq w \left[(1-\nu) T + \nu |T|^2 \right] \quad (6.1)$$

for all $p \geq 1$ and $\nu \in [0, 1]$.

For $\nu = 1/2$ we have

$$d_{p,1/2} w(T) = \frac{1}{2^{1/p}} d_p w(T), \quad p \geq 1,$$

where

$$d_p w(T) := \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^p + \left| \left\langle |T|^2 x, x \right\rangle \right|^p \right)^{1/p}.$$

For $p = 2$ we recapture Davis-Wielandt radius dw of T .

For $p = 2$ and $\nu \in [0, 1]$ we put

$$d_{e,\nu} w(T) := w_{e,\nu} \left(T, |T|^2 \right) = \sup_{\|x\|=1} \left((1-\nu) |\langle Tx, x \rangle|^2 + \nu \|Tx\|^4 \right)^{1/2},$$

which for $\nu = 1/2$ gives

$$d_{e,1/2} w(T) = \frac{\sqrt{2}}{2} dw(T).$$

For $p = 1$ we have that

$$d_{\nu} w(T) := w_{\nu} \left(T, |T|^2 \right) = \sup_{\|x\|=1} \left((1-\nu) |\langle Tx, x \rangle| + \nu \|Tx\|^2 \right)$$

and for $\nu = 1/2$,

$$d_{1/2}w(T) := w_{1/2}(T, |T|^2) = \frac{1}{2} \sup_{\|x\|=1} (|\langle Tx, x \rangle| + \|Tx\|^2) = \frac{1}{2} d_1 w(T).$$

From the inequality (2.1) we have for $(A, B) = (T, |T|^2)$ that

$$d_{e,\nu}w^2(T) \leq R w^2(T - |T|^2) + [w(T)]^{2(1-\nu)} \|T\|^{4\nu}. \quad (6.2)$$

In particular, for $\nu = 1/2$ we get

$$dw^2(T) \leq w^2(T - |T|^2) + 2w(T) \|T\|^2.$$

From (2.3) we obtain that

$$\begin{aligned} d_{e,\nu}w^2(T) &\leq R w^2(T - |T|^2) + \left\| (1-\nu)|T|^{2\alpha} + \nu|T|^{4(1-\alpha)} \right\| \\ &\quad \times \left\| (1-\nu)|T^*|^{2(1-\alpha)} + \nu|T|^{4\alpha} \right\| \end{aligned} \quad (6.3)$$

and, in particular,

$$dw^2(T) \leq w^2(T - |T|^2) + \frac{1}{2} \left\| |T|^{2\alpha} + |T|^{4(1-\alpha)} \right\| \left\| |T^*|^{2(1-\alpha)} + |T|^{4\alpha} \right\|.$$

For $\alpha = 1/2$ we obtain

$$\begin{aligned} d_{e,\nu}w^2(T) &\leq R w^2(T - |T|^2) + \left\| (1-\nu)|T| + \nu|T|^2 \right\| \\ &\quad \times \left\| (1-\nu)|T^*| + \nu|T|^2 \right\| \end{aligned}$$

and

$$dw^2(T) \leq w^2(T - |T|^2) + \frac{1}{2} \left\| |T| + |T|^2 \right\| \left\| |T^*| + |T|^2 \right\|.$$

From the inequality (3.1) we have for $(A, B) = (T, |T|^2)$ that

$$\begin{aligned} d_{e,\nu}w^2(T) &\geq \max \left\{ w \left[(1-\nu)T^2 + \nu^2|T|^4 \right], (1-\nu)\nu w(T|T|^2 + |T|^2T) \right\} \\ &\quad + \max \left\{ (1-\nu)w(T), \nu\|T\|^2 \right\} \\ &\quad \times \left| w \left[(1-\nu)T + \nu|T|^2 \right] - w \left[(1-\nu)T - \nu|T|^2 \right] \right|. \end{aligned}$$

In particular, for $\nu = 1/2$ we obtain:

$$\begin{aligned} dw^2(T) &\geq \frac{1}{2} \max \left\{ w(T^2 + |T|^4), w(T|T|^2 + |T|^2T) \right\} \\ &\quad + \frac{1}{2} \max \left\{ w(A), \|T\|^2 \right\} \left| w(T + |T|^2) - w(T - |T|^2) \right|. \end{aligned}$$

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