

EXPLORING TORAL $[m, \mathcal{C}]$ -ISOMETRIES IN THE CONTEXT OF SEVERAL OPERATORS THEORY

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Abstract. The family of commuting m -isometric tuples has attracted significant attention in operator theory. In this work, we propose and analyze a new class, termed toral $[m, \mathcal{C}]$ -isometric tuples of commuting operators, which extends and unifies several existing frameworks. We develop a comprehensive structural theory for this class, including fundamental properties, and stability under natural operator-theoretic constructions.

1. Introduction and preliminaries

Let $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. We denote by $\mathcal{B}[\mathcal{K}]$ the Banach algebra of all bounded linear operators on $\mathcal{K}$. Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$.

Following [18], a conjugation operator on $\mathcal{K}$ is a conjugate-linear, isometric involution $C : \mathcal{K} \rightarrow \mathcal{K}$; that is,

$$C^2 = I \quad \text{and} \quad \langle C\psi, C\varphi \rangle = \langle \varphi, \psi \rangle \quad \text{for all } \psi, \varphi \in \mathcal{K}.$$

In addition, $\mathcal{C}$ satisfies

$$\begin{cases} C\mathcal{W}C \in \mathcal{B}[\mathcal{K}] \quad \text{for every } \mathcal{W} \in \mathcal{B}[\mathcal{K}], \\ (C\mathcal{W}C)^r = C\mathcal{W}^rC \quad \text{for all } r \in \mathbb{Z}_+, \\ (C\mathcal{W}C)^* = C\mathcal{W}^*C. \end{cases}$$

The theory of m -isometric operators on a Hilbert space $\mathcal{K}$ indeed has significant contributions from mathematicians like Agler and Stankus. [3, 4, 5]. To provide a bit of context for a fixed positive $m \in \mathbb{N}$, a member $\mathcal{W} \in \mathcal{B}[\mathcal{K}]$ is called an m -isometric operator if it satisfies

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \mathcal{W}^{*k} \mathcal{W}^k = 0. \quad (1.1)$$

In recent years, this class of operators has attracted considerable attention from researchers, owing to the significant developments and important scientific contributions

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that have been presented in a number of published research papers ([8, 9, 10, 23, 24, 25, 26, 27]). Recently, The authors in [12], expanded the concept of m -isometry within the context of Hilbert spaces, introducing the notion of a $(m, \mathcal{C})$ -isometric operator. To better understand this extension. For convenience, we present the next definition.. Let $m \in \mathbb{N}$. An operator $\mathcal{W} \in \mathcal{B}[\mathcal{K}]$ is an $(m, \mathcal{C})$ -isometric if there exists $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$ $\mathcal{K}$ for which

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \mathcal{W}^{*k} \mathcal{C} \mathcal{W}^k \mathcal{C} = 0. \quad (1.2)$$

Here, $\mathcal{C}[\mathcal{K}]$ is the family of conjugation operators on $\mathcal{K}$.

An alternative variant of m -isometry, related to a conjugation $\mathcal{C}$, was introduced by the authors in [14] as: $\mathcal{W} \in \mathcal{B}[\mathcal{K}]$ is called an $[m, \mathcal{C}]$ -isometry for some $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$ if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \mathcal{C} \mathcal{W}^k \mathcal{C} \mathcal{W}^k = 0. \quad (1.3)$$

An extended form of $[m; \mathcal{C}]$ -isometry, named n -quasi- $[m, \mathcal{C}]$ -isometry was studied by J. Shen in [29],

$$\mathcal{W}^{*n} \left(\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \mathcal{C} \mathcal{W}^k \mathcal{C} \mathcal{W}^k \right) \mathcal{W}^n = 0. \quad (1.4)$$

where they expanded the concept to incorporate additional structure and explored its implications in various context.

For a detailed and insightful account of the various concepts of operators involving conjugation, see [1, 12, 13, 14, 20, 22, 24, 26, 29], where a thorough analysis is provided.

In the following, we present key concepts associated with multivariable operator theory, starting with a set of important notations to establish the necessary groundwork. We denote by $\mathbb{Z}_+^d$ the d -fold Cartesian product $\mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+$ and let $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}_+^d$, We denote $|\gamma| := \sum_{1 \leq k \leq d} |\gamma_k|$ and $\gamma! := \gamma_1! \cdots \gamma_d!$. Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}(\mathcal{K})^d$

be a commuting d -tuple, i.e., $\mathcal{W}_k \mathcal{W}_j = \mathcal{W}_j \mathcal{W}_k$ for $k, j = 1, \dots, d$. For $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}_+^d$, we define

$$\mathcal{W}^\gamma := \mathcal{W}_1^{\gamma_1} \cdots \mathcal{W}_d^{\gamma_d}, \quad \mathcal{W}^* := (\mathcal{W}_1^*, \dots, \mathcal{W}_d^*), \quad \mathcal{W}^{*\gamma} := \mathcal{W}_1^{*\gamma_1} \cdots \mathcal{W}_d^{*\gamma_d}.$$

We denote by $\mathcal{B}_c[\mathcal{K}]^d$ the set of all commuting elements of $\mathcal{B}[\mathcal{K}]^d$.

For $\gamma = (\gamma_1, \dots, \gamma_d)$ and $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d$, we write $\gamma \leq \beta$ (equivalently, $\beta \geq \gamma$) if $\gamma_i \leq \beta_i$ for all $1 \leq i \leq d$. In this case, we define

$$\binom{\beta}{\gamma} := \binom{\beta_1}{\gamma_1} \cdots \binom{\beta_d}{\gamma_d}.$$

Gleason et al. [17] generalized the classical notion of m -isometries to elements of $\mathcal{B}_c[\mathcal{K}]^d$. This extension leads to a more general framework in which an m -isometric

tuple is defined by means of the following condition:

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\gamma|=k} \frac{k!}{\gamma!} \mathcal{W}^{*\gamma} \mathcal{W}^\gamma \right) = 0 \quad (1.5)$$

for $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}[\mathcal{W}]^d$.

Following the same idea, the notion of $[m; \mathcal{C}]$ -isometry was extended by the author H. Ahmed to the framework of multivariable operator theory. We term $\mathcal{W} := (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}_c[\mathcal{K}]^d$ is an $[m, \mathcal{C}]$ -isometry if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathcal{C} \mathcal{W}^\alpha \mathcal{C} \mathcal{W}^\alpha \right) = 0. \quad (1.6)$$

Adopting the notation presented in [2], we proceed to define

$$\Psi_l(\mathcal{W}, \mathcal{C}) := \sum_{0 \leq k \leq l} (-1)^{l-k} \binom{l}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathcal{C} \mathcal{W}^\alpha \mathcal{C} \mathcal{W}^\alpha \right), \quad l = 0, 1, \dots,$$

where $\Psi_0(\mathcal{W}, \mathcal{C}) = I$. The following identity was rigorously established in [2, Theorem 2.10].

$$\Psi_{l+1}(\mathcal{W}, \mathcal{C}) = \sum_{1 \leq k \leq d} (\mathcal{C} \mathcal{W}_k \mathcal{C}) \Psi_l(\mathcal{W}, \mathcal{C}) \mathcal{W}_k - \Psi_l(\mathcal{W}, \mathcal{C}) \quad (1.7)$$

for $l = 0, 1, \dots$.

By (1.7), any $[m, \mathcal{C}]$ -isometric tuple $\mathcal{W}$ is automatically $[n, \mathcal{C}]$ -isometric for every $n \geq m$.

The concept of toral m -isometric multi-operators has been widely studied in [7, 28]. In this context, a commuting d -tuple $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d)$ is defined to be toral m -isometric if it satisfies the condition:

$$\Delta_\beta(\mathcal{W}) := \sum_{\substack{0 \leq \alpha_1 \leq \beta_1 \\ \vdots \\ 0 \leq \alpha_d \leq \beta_d}} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{W}^{*\alpha} \mathcal{W}^\alpha = 0, \quad (1.8)$$

for every multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d$ satisfying $\beta_1 + \dots + \beta_d = m$. Furthermore, in [19], T. Lee developed the notion of toral (A, m) -isometric tuples associated with an $A \in \mathcal{B}[\mathcal{K}]$. Within this framework, a commuting d -tuple $\mathcal{W}$ is said to be toral (A, m) -isometric provided that it meets the condition:

$$\sum_{\substack{0 \leq \gamma_1 \leq \beta_1 \\ \vdots \\ 0 \leq \gamma_d \leq \beta_d}} (-1)^{|\gamma|} \binom{\beta_1, \dots, \beta_d}{\gamma_1, \dots, \gamma_d} \mathcal{W}^{*\gamma} A \mathcal{W}^\gamma = 0$$

for all multi-indices $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d$ satisfying $|\beta| = m$.

For a comprehensive and insightful discussion of the various concepts in multivariable operator theory, the reader is encouraged to refer to [1, 7, 11, 15, 16, 19, 21, 22].

In this paper, closely related to the problem at hand, we define and study the class of toral $[m; \mathcal{C}]$ -isometric in $\mathcal{B}_c[\mathcal{K}]$ associated with a fixed conjugation $\mathcal{C} \in \mathcal{C}(\mathcal{K})$. We investigate its fundamental properties and establish a series of results that extend and generalize several existing works.

2. Toral $[m, \mathcal{C}]$ -isometries multioperators

This section is devoted to the formal definition of toral $[m, \mathcal{C}]$ -isometric operators and to their structural aspects.

DEFINITION 2.1. Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}_c[\mathcal{K}]^d$. We say that $\mathcal{W}$ is a toral $[m, \mathcal{C}]$ -isometric if there exists $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$ so that

$$\sum_{\substack{0 \leq \gamma \leq \beta \\ \gamma \in \mathbb{Z}_+^d}} (-1)^{|\gamma|} \binom{\beta}{\gamma} \mathcal{C} \mathcal{W}^\gamma \mathcal{C} \mathcal{W}^\gamma = 0, \quad (2.1)$$

for every $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d$ with $|\beta| = m$, where $m \in \mathbb{N}$.

We denote by $\{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$ the class of toral $[m, \mathcal{C}]$ -isometries in $\mathcal{B}_c[\mathcal{K}]^d$.

For $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d)$ and $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$, we set

$$\Psi_\beta(\mathcal{W}, \mathcal{C}) = \sum_{\substack{\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}_+^d \\ 0 \leq \gamma \leq \beta}} (-1)^{|\gamma|} \binom{\beta}{\gamma} \mathcal{C} \mathcal{W}^\gamma \mathcal{C} \mathcal{W}^\gamma. \quad (2.2)$$

REMARK 2.2. (1) When $m = 1$, $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d)$ is in $\{\mathbf{T}\text{-}[1, \mathcal{C}]\text{-Iso}\}$ if and only if

$$\begin{cases} \mathcal{C} \mathcal{W}_1 \mathcal{C} \mathcal{W}_1 - I = 0 \\ \dots \\ \mathcal{C} \mathcal{W}_d \mathcal{C} \mathcal{W}_d - I = 0. \end{cases}$$

We observe that a toral $[1, \mathcal{C}]$ -isometries are just commuting multioperators $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d)$ such that each $\mathcal{W}_k$ is an $[1, \mathcal{C}]$ -isometry, that is, $\mathcal{C} \mathcal{W}_k \mathcal{C} \mathcal{W}_k = I$.

(2) When $d = 2$ and $m = 2$, a pair $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$ is in $\{\mathbf{T}\text{-}[2, \mathcal{C}]\text{-Iso}\}$ if

$$\begin{cases} \mathcal{C} \mathcal{W}_1^2 \mathcal{C} \mathcal{W}_1^2 - 2 \mathcal{C} \mathcal{W}_1 \mathcal{C} \mathcal{W}_1 + I = 0 \\ \mathcal{C} \mathcal{W}_2^2 \mathcal{C} \mathcal{W}_2^2 - 2 \mathcal{C} \mathcal{W}_2 \mathcal{C} \mathcal{W}_2 + I = 0 \\ \mathcal{C} \mathcal{W}_1 \mathcal{W}_2 \mathcal{C} \mathcal{W}_1 \mathcal{W}_2 - \mathcal{C} \mathcal{W}_1 \mathcal{C} \mathcal{W}_1 - \mathcal{C} \mathcal{W}_2 \mathcal{C} \mathcal{W}_2 + I = 0. \end{cases}$$

(3) When $d = 2$ and $m = 3$, a pair $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$ is in $\{\mathbf{T}\text{-}[3, \mathcal{C}]\text{-Iso}\}$ if

$$\left\{ \begin{array}{l} C\mathcal{W}_2C\mathcal{W}_2 + 2C\mathcal{W}_1C\mathcal{W}_1 - 2C\mathcal{W}_1\mathcal{W}_2C\mathcal{W}_1\mathcal{W}_2 - C\mathcal{W}_1C\mathcal{W}_1^2 + C_1\mathcal{W}_2^2\mathcal{W}_2C\mathcal{W}_1^2\mathcal{W}_2 - I = 0 \\ C\mathcal{W}_1C\mathcal{W}_1 + 2C\mathcal{W}_2C\mathcal{W}_2 - 2C\mathcal{W}_1\mathcal{W}_2C\mathcal{W}_1\mathcal{W}_2 - C\mathcal{W}_2^2C\mathcal{W}_2^2 + C\mathcal{W}_1\mathcal{W}_2^2C\mathcal{W}_1\mathcal{W}_2^2 - I = 0 \\ C\mathcal{W}_1^3C\mathcal{W}_1^3 - 3C\mathcal{W}_1^2C\mathcal{W}_1^2 + 3C\mathcal{W}_1C\mathcal{W}_1 - I = 0 \\ C\mathcal{W}_2^3C\mathcal{W}_2^3 - 3C\mathcal{W}_2^2C\mathcal{W}_2^2 + 3C\mathcal{W}_2C\mathcal{W}_2 - I = 0. \end{array} \right.$$

(4) A d -tuple $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \{\mathbf{T}\text{-}[2, \mathcal{C}]\text{-Iso}\}$ if and only if

$$\left\{ \begin{array}{l} C\mathcal{W}_j^2C\mathcal{W}_j^2 - 2C\mathcal{W}_jC\mathcal{W}_j + I = 0, \quad \text{for } j = 1, \dots, d, \\ C\mathcal{W}_k\mathcal{W}_jC\mathcal{W}_k\mathcal{W}_j - C\mathcal{W}_kC\mathcal{W}_k - C\mathcal{W}_jC\mathcal{W}_j + I = 0, \quad \text{for } k, j = 1, \dots, n \ (k \neq j). \end{array} \right.$$

REMARK 2.3. Each $\mathcal{W} \in \{\mathbf{T}\text{-}[1, \mathcal{C}]\text{-Iso}\}$ is also in $\{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$.

In the following example, we present a tuple of operators that is $[m, \mathcal{C}]$ -isometric but not toral $[m, \mathcal{C}]$ -isometric. This serves to illustrate the distinction between these two concepts, justifying the need for the introduction of the toral $[m, \mathcal{C}]$ -isometric tuple as a new class in the theory.

EXAMPLE 2.4. Let $\mathcal{C} \in \mathcal{C}[\mathbb{C}^2]$ defined by $\mathcal{C}[x_1, x_2] = [\overline{x_2}, \overline{x_1}]$. Consider

$$\mathcal{W}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathcal{B}[\mathbb{C}^2] \quad \text{and} \quad \mathcal{W}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -1 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathcal{B}[\mathbb{C}^2].$$

It was proven in [2] that $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$ is a $[1, \mathcal{C}]$ -isometric 2-tuple. However $C\mathcal{W}_1C\mathcal{W}_1 - I \neq 0$. Thus $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$ is not in $\{\mathbf{T}\text{-}[1, \mathcal{C}]\text{-Iso}\}$.

REMARK 2.5. Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$. Then each component $\mathcal{W}_j$ is an $[m, \mathcal{C}]$ -isometric operator for $j = 1, \dots, d$. To see this, choose $\beta = (0, \dots, m, \dots, 0) \in \mathbb{Z}_+^d$, where m appears in the j th position. In this case, (2.1) reduces to

$$\sum_{k=0}^m (-1)^k \binom{m}{k} C\mathcal{W}_j^k C\mathcal{W}_j^k = 0,$$

from which it follows that $\mathcal{W}_j$ is an $[m, \mathcal{C}]$ -isometry. Conversely, the above implication is not valid in general. An illustrative example shows that even if each $\mathcal{W}_j$ is $[m, \mathcal{C}]$ -isometric, the tuple $\mathcal{W}$ need not be toral $[m, \mathcal{C}]$ -isometric.

EXAMPLE 2.6. Consider $\mathcal{K} = \mathbb{C}^3$ and a conjugation $\mathcal{C} \in \mathcal{C}(\mathcal{K})$ given by means of $\mathcal{C}(\psi, \omega, \varphi) = (\overline{\varphi}, \overline{\omega}, \overline{\psi})$. Define $\mathcal{W}_1 \in \mathcal{B}[\mathcal{K}]$ and $\mathcal{W}_2 \in \mathcal{B}[\mathcal{K}]$ by

$$\mathcal{W}_1 = \begin{pmatrix} 1 & i & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{W}_2 = \begin{pmatrix} 1 & -i & -i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\sum_{0 \leq k \leq 3} (-1)^k \binom{3}{k} C\mathcal{W}_r^k C\mathcal{W}_r^k = 0, \quad r = 1, 2.$$

Therefore $\mathcal{W}_1$ and $\mathcal{W}_2$ are $[3, C]$ -isometric operators. A quick calculation demonstrates that, for $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$, we have $\mathcal{W}_1\mathcal{W}_2 \neq \mathcal{W}_2\mathcal{W}_1$. Consequently, $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$ is not an eligible candidate for a toral $[3, C]$ -isometric multi-operators.

PROPOSITION 2.7. *Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}_c[\mathcal{K}]^d$ and let $C \in \mathcal{C}[\mathcal{K}]$. Then the following results hold:*

1. $\mathcal{W}$ is in $\{\mathbf{T}\text{-}[m, C]\text{-Iso}\}$ if and only if the conjugated tuple $C\mathcal{W}C := (C\mathcal{W}_1C, \dots, C\mathcal{W}_dC)$ is also in $\{\mathbf{T}\text{-}[m, C]\text{-Iso}\}$.
2. If $C\mathcal{W}_k = \mathcal{W}_kC$ for all $k = 1, \dots, d$, then $\mathcal{W}$ is in $\{\mathbf{T}\text{-}[m, C]\text{-Iso}\}$ if and only if it is a toral m -isometry.

Proof. (1) Taking into account equation (2.1), we have

$$\begin{aligned} \Psi_\beta(\mathcal{W}, C) &= 0, \quad \forall \beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d \text{ subject to } |\beta| = m \\ \Leftrightarrow \sum_{\substack{0 \leq \gamma \leq \beta \\ \gamma \in \mathbb{Z}_+^d}} (-1)^{|\gamma|} \binom{\beta}{\gamma} C\mathcal{W}^\gamma C\mathcal{W}^\gamma &= 0, \quad \forall \beta \in \mathbb{Z}_+^d, |\beta| = m \\ \Leftrightarrow C \left(\sum_{\substack{0 \leq \gamma \leq \beta \\ \gamma \in \mathbb{Z}_+^d}} (-1)^{|\gamma|} \binom{\beta}{\gamma} (C\mathcal{W}C)^\gamma C(C\mathcal{W}C)^\gamma \right) &= 0, \quad \forall \beta \in \mathbb{Z}_+^d, \text{ subject to } |\beta| = m \\ \Leftrightarrow \sum_{\substack{0 \leq \gamma \leq \beta \\ \gamma \in \mathbb{Z}_+^d}} (-1)^{|\gamma|} \binom{\beta}{\gamma} C(C\mathcal{W}C)^\gamma C(C\mathcal{W}C)^\gamma &= 0, \quad \forall \beta \in \mathbb{Z}_+^d, \text{ subject to } |\beta| = m \\ \Leftrightarrow \Psi_\beta(C\mathcal{W}C, C) &= 0, \quad \forall \beta \in \mathbb{Z}_+^d, \text{ subject to } |\beta| = m. \end{aligned}$$

Which establishes the equivalence stated in (1).

(2) Assume that $\mathcal{W}_kC = C\mathcal{W}_k$ for $k = 1, \dots, d$. In this case, for $|\beta| = m$, we have $\Psi_\beta(\mathcal{W}, C) = C\Delta_\beta(\mathcal{W})C$. This yields that the equivalence stated in (2) holds. $\square$

PROPOSITION 2.8. *Let $\mathcal{W} \in \mathcal{B}[\mathcal{K}]$ be an $[m, C]$ -isometric operator for some conjugation $C \in \mathcal{C}[\mathcal{K}]$. Then the diagonal d -tuple $\widetilde{\mathcal{W}} := (\mathcal{W}, \dots, \mathcal{W}) \in \mathcal{B}(\mathcal{K})^d$ is in $\{\mathbf{T}\text{-}[m, C]\text{-Iso}\}$.*

Proof. Let $\beta \in \mathbb{Z}_+^d$ subject to $|\beta| = m$. Then, we obtain

$$\begin{aligned}
 \Phi_\beta(\mathcal{W}, \mathcal{C}) &= \sum_{\substack{0 \leq \alpha_1 \leq \beta_1 \\ \vdots \\ 0 \leq \alpha_d \leq \beta_d}} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C} \mathcal{W}^\alpha \mathcal{C} \mathcal{W}^\alpha \\
 &= \sum_{\substack{0 \leq \alpha_1 \leq \beta_1 \\ \vdots \\ 0 \leq \alpha_d \leq \beta_d}} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C} \mathcal{W}^\alpha \mathcal{C} \mathcal{W}^\alpha \\
 &= \prod_{1 \leq k \leq d} \left(I - \mathcal{C} \mathcal{W} \mathcal{C} \mathcal{W} \right)^{\beta_k} \\
 &= \left(\mathcal{C} \mathcal{W} \mathcal{C} \mathcal{W} - I \right)^{|\beta|} \\
 &= \left(\mathcal{C} \mathcal{W} \mathcal{C} \mathcal{W} - I \right)^m \\
 &= \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \mathcal{C} \mathcal{W}^k \mathcal{C} \mathcal{W}^k \\
 &= 0.
 \end{aligned}$$

Consequently, $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$. $\square$

LEMMA 2.9. Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d)$ and $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_d)$ in $\mathcal{B}_c(\mathcal{K})^d$. Suppose that there exists a unitary operator $\mathcal{U} \in \mathcal{B}[\mathcal{K}]$ such that $\mathcal{T}_k = \mathcal{U}^* \mathcal{W}_k \mathcal{U}$, $k = 1, \dots, d$. If $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$, then

$$\Phi_\beta(\mathcal{T}, \mathcal{U}^* \mathcal{C} \mathcal{U}) = \mathcal{U}^* \Phi_\beta(\mathcal{W}, \mathcal{C}) \mathcal{U}, \quad \beta \in \mathbb{Z}_+^d. \quad (2.3)$$

In particular, $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$ if and only if $\mathcal{T} \in \{\mathbf{T}\text{-}[m, \mathcal{U}^* \mathcal{C} \mathcal{U}]\text{-Iso}\}$.

Proof. Keep in mind that if $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$ then $\mathcal{U}^* \mathcal{C} \mathcal{U}$ will have the same properties as $\mathcal{C}$. Moreover for all $\alpha \in \mathbb{Z}_+^d$ we have

$$(\mathcal{U}^* \mathcal{C} \mathcal{U}) \mathcal{T}^\alpha (\mathcal{U}^* \mathcal{C} \mathcal{U}) \mathcal{T}^\alpha = \mathcal{U}^* (\mathcal{C} \mathcal{W}^\alpha \mathcal{C} \mathcal{W}^\alpha) \mathcal{U}.$$

$$\begin{aligned}
 \Phi_\beta(\mathcal{T}, \mathcal{U}^* \mathcal{C} \mathcal{U}) &= \sum_{\substack{0 \leq \alpha \leq \beta \\ \alpha \in \mathbb{Z}_+^d}} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{U}^* \mathcal{C} \mathcal{U} \mathcal{T}^\alpha \mathcal{U}^* \mathcal{C} \mathcal{U} \mathcal{T}^\alpha \\
 &= \mathcal{U}^* \left(\sum_{\substack{0 \leq \alpha \leq \beta \\ \alpha \in \mathbb{Z}_+^d}} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C} \mathcal{W}^\alpha \mathcal{C} \mathcal{W}^\alpha \right) \mathcal{U} \\
 &= \mathcal{U}^* \Phi_\beta(\mathcal{W}, \mathcal{C}) \mathcal{U}.
 \end{aligned}$$

By (2.3), we deduce that

$$\begin{aligned} \Phi_{\beta}(\mathcal{W}, \mathcal{C}) &= 0 \text{ for } \beta \in \mathbb{Z}_+^d \text{ subject to } |\beta| = m \\ \iff \Phi_{\beta}(\mathcal{T}, \mathcal{U}^* \mathcal{C} \mathcal{U}) &= 0 \text{ for } \beta \in \mathbb{Z}_+^d \text{ subject to } |\beta| = m. \quad \square \end{aligned}$$

PROPOSITION 2.10. *Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d)$ and $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_d)$ in $\mathcal{B}_c[\mathcal{K}]^d$ so that $\mathcal{W}_j \mathcal{T}_k = \mathcal{T}_k \mathcal{W}_j$ for $j, k = 1, \dots, d$. Assume that $\mathcal{T} \in \{\mathbf{T}\text{-}[1, \mathcal{C}]\text{-Iso}\}$ for some $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$. Then the componentwise product $\mathcal{T}\mathcal{W} := (\mathcal{T}_1 \mathcal{W}_1, \dots, \mathcal{T}_d \mathcal{W}_d)$ is in $\{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$ if and only if $\mathcal{W}$ is in $\{\mathbf{T}\text{-}[m, \mathcal{U}^* \mathcal{C} \mathcal{U}]\text{-Iso}\}$.*

Proof. As it is the case that $\mathcal{T} \in \{\mathbf{T}\text{-}[1, \mathcal{U}^* \mathcal{C} \mathcal{U}]\text{-Iso}\}$, thus $\mathcal{C} \mathcal{T}^{\alpha} \mathcal{C} \mathcal{T}^{\alpha} = I$. Using the commutation relations $[\mathcal{W}_j, \mathcal{T}_k] = 0$ for each $j, k = 1, \dots, d$, a straightforward computation shows that

$$\begin{aligned} \Phi_{\beta}(\mathcal{W} \mathcal{T}, \mathcal{C}) &= \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C} (\mathcal{W} \mathcal{T})^{\alpha} \mathcal{C} (\mathcal{W} \mathcal{T})^{\alpha} \\ &= \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C} \mathcal{W}^{\alpha} \mathcal{T}^{\alpha} \mathcal{C} \mathcal{T}^{\alpha} \mathcal{W}^{\alpha} \\ &= \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C} \mathcal{W}^{\alpha} \mathcal{C} [\mathcal{C} \mathcal{T}^{\alpha} \mathcal{C} \mathcal{T}^{\alpha}] \mathcal{W}^{\alpha} \\ &= \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C} \mathcal{W}^{\alpha} \mathcal{C} \mathcal{W}^{\alpha} \\ &= \Phi_{\beta}(\mathcal{W}, \mathcal{C}), \end{aligned}$$

which confirms the anticipated result. $\square$

PROPOSITION 2.11. *Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}_c[\mathcal{K}]^d$, and let $\mathcal{C} \in \mathcal{C}(\mathcal{K})$. Then, for every $\beta \in \mathbb{Z}_+^d$ and $k = 1, \dots, d$, the following identities hold:*

$$\Phi_{\beta + \varepsilon(k)}(\mathcal{W}, \mathcal{C}) = \Phi_{\beta}(\mathcal{W}, \mathcal{C}) - (\mathcal{C} \mathcal{W}_k \mathcal{C}) \Phi_{\beta}(\mathcal{W}, \mathcal{C}) \mathcal{W}_k, \quad (2.4)$$

where $\varepsilon(k) = (0, \dots, 1, 0, \dots, 0)$ with 1 in the k -th position.

$$\sum_{k=1}^d \Phi_{\beta + \varepsilon(k)}(\mathcal{W}, \mathcal{C}) = d \Phi_{\beta}(\mathcal{W}, \mathcal{C}) - \sum_{k=1}^d (\mathcal{C} \mathcal{W}_k \mathcal{C}) \Phi_{\beta}(\mathcal{W}, \mathcal{C}) \mathcal{W}_k. \quad (2.5)$$

In particular, if $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$, then it is also in $\{\mathbf{T}\text{-}[m+1, \mathcal{C}]\text{-Iso}\}$, and

$$\sum_{k=1}^d \Phi_{\beta + \varepsilon(k)}(\mathcal{W}, \mathcal{C}) = 0.$$

Proof. For $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}_+^d$ and $1 \leq k \leq d$, we denote by γ'_k the $(d-1)$ -tuple obtained from γ by removing its k -th component, that is, $\gamma'_k = (\gamma_1, \dots, \gamma_{k-1}, \gamma_{k+1}, \dots, \gamma_d)$.

A calculation yields

$$\begin{aligned}
 & \Phi_{\beta+\varepsilon(k)}(\mathcal{W}, C) \\
 = & \sum_{\substack{0 \leq \alpha'_k \leq \beta'_k \\ 0 \leq \alpha_k \leq \beta_k + 1}} (-1)^{|\alpha|} \binom{\beta_1}{\alpha_1} \dots \binom{\beta_k + 1}{\alpha_k} \dots \binom{\beta_d}{\alpha_d} C \mathcal{W}^{\alpha'_k} C \mathcal{W}^{\alpha'_k} \\
 = & \sum_{0 \leq \alpha'_k \leq \beta'_k} (-1)^{|\alpha'_k|} \binom{\beta'_k}{\alpha'_k} C \mathcal{W}^\alpha C \mathcal{W}^\alpha \\
 & + \sum_{\substack{0 \leq \alpha'_k \leq \beta'_k \\ 1 \leq \alpha_k \leq \beta_k}} (-1)^{|\alpha|} \binom{\beta_1}{\alpha_1} \dots \binom{\beta_k + 1}{\alpha_k} \dots \binom{\beta_d}{\alpha_d} C \mathcal{W}^\alpha C \mathcal{W}^\alpha \\
 & + \sum_{0 \leq \alpha'_k \leq \beta'_k} (-1)^{\beta_k + 1 + |\alpha|} \binom{\beta'_k}{\alpha'_k} C \mathcal{W}^{\alpha'_k} \mathcal{W}_k^{\beta_k + 1} C \mathcal{W}^{\alpha'_k} \mathcal{W}_k^{\beta_k + 1} \\
 = & \sum_{0 \leq \alpha'_k \leq \beta'_k} (-1)^{|\alpha'_k|} \binom{\beta'_k}{\alpha'_k} C \mathcal{W}^\alpha C \mathcal{W}^\alpha \\
 & + \sum_{\substack{0 \leq \alpha'_k \leq \beta'_k \\ 1 \leq \alpha_k \leq \beta_k}} (-1)^{|\alpha|} \binom{\beta_1}{\alpha_1} \dots \left[\binom{\beta_k}{\alpha_k} + \binom{\beta_k}{\alpha_k - 1} \right] \dots \binom{\beta_d}{\alpha_d} C \mathcal{W}^\alpha C \mathcal{W}^\alpha \\
 & + \sum_{0 \leq \alpha'_k \leq \beta'_k} (-1)^{\beta_k + 1 + |\alpha'_k|} \binom{\beta'_k}{\alpha'_k} C \mathcal{W}_k^{\beta_k + 1} \mathcal{W}^{\alpha'_k} C \mathcal{W}^{\alpha'_k} \mathcal{W}_k^{\beta_k + 1} C \\
 = & \left(\sum_{0 \leq \alpha'_k \leq \beta'_k} (-1)^{|\alpha'_k|} \binom{\beta'_k}{\alpha'_k} C \mathcal{W}^{\alpha'_k} C \mathcal{W}^{\alpha'_k} + \sum_{\substack{0 \leq \alpha'_k \leq \beta'_k \\ 1 \leq \alpha_k \leq \beta_k}} (-1)^{|\alpha|} \binom{\beta}{\alpha} C \mathcal{W}^\alpha C \mathcal{W}^\alpha \right) \\
 & + \left(\sum_{\substack{0 \leq \alpha'_k \leq \beta'_k \\ 1 \leq \alpha_k \leq \beta_k}} (-1)^{|\alpha|} \binom{\beta_1}{\alpha_1} \dots \binom{\beta_k}{\alpha_k - 1} \dots \binom{\beta_d}{\alpha_d} C \mathcal{W}^{\alpha_k} C \mathcal{W}^{\alpha_k} \right. \\
 & \left. + \sum_{0 \leq \alpha'_k \leq \beta'_k} (-1)^{\beta_k + 1 + |\alpha'_k|} \binom{\beta'_k}{\alpha'_k} C \mathcal{W}_k^{\beta_k + 1} \mathcal{W}^{\alpha'_k} C \mathcal{W}^{\alpha'_k} \mathcal{W}_k^{\beta_k + 1} \right) \\
 = & \Phi_\beta(\mathcal{W}, C) - \left(C \mathcal{W}_k C \sum_{\substack{0 \leq \alpha'_k \leq \beta'_k \\ 0 \leq \alpha_k \leq \beta_k - 1}} (-1)^{|\alpha|} \binom{\beta}{\alpha} C \mathcal{W}_k^{\beta_k} \mathcal{W}^{\alpha'_k} C \mathcal{W}^{\alpha'_k} \mathcal{W}_k^{\beta_k} \mathcal{W}_k \right) \\
 & + \mathcal{W}_k C \sum_{0 \leq \alpha'_k \leq \beta'_k} (-1)^{|\alpha'_k + \beta_k|} \binom{\beta'_k}{\alpha'_k} C \mathcal{W}_k^{\beta_k} \mathcal{W}^{\alpha'_k} C \mathcal{W}^{\alpha_k} \mathcal{W}_k^{\beta_k} \mathcal{W}_k \\
 = & \Phi_\beta(\mathcal{W}, C) - C \mathcal{W}_k C \Phi_\beta(\mathcal{W}, C) \mathcal{W}_k.
 \end{aligned}$$

From (2.4), it follows that if $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$, then it is also in $\{\mathbf{T}\text{-}[m+1, \mathcal{C}]\text{-Iso}\}$. Moreover, (2.5) follows directly from (2.4) by summing over k . $\square$

The following presents a result similar to the well-known fact for $(m, \mathcal{C})$ -isometry tuples [22, Theorem 1] and $[m, \mathcal{C}]$ -isometric tuple [2, Theorem 2.10].

THEOREM 2.12. *Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}_c[\mathcal{K}]^d$. Then $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$ for some $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$ is equivalent to*

$$C\mathcal{W}^\beta C\mathcal{W}^\beta = \sum_{k=0}^{m-1} \left(\sum_{\substack{\alpha \in \mathbb{Z}_+^d \\ |\alpha|=k}} \binom{\beta}{\alpha} \Phi_\alpha(\mathcal{W}, \mathcal{C}) \right), \quad \beta \in \mathbb{Z}_+^d. \quad (2.6)$$

Proof. Referring to [7, Remark 2], we have

$$C\mathcal{W}^\beta C\mathcal{W}^\beta = \sum_{k=0}^{\infty} \left(\sum_{\substack{\alpha \in \mathbb{Z}_+^d \\ |\alpha|=k}} \binom{\beta}{\alpha} \Phi_\alpha(\mathcal{W}, \mathcal{C}) \right). \quad (2.7)$$

Assume that $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$. Then $\Phi_\alpha(\mathcal{W}, \mathcal{C}) = 0$ for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| = m$, and, by Proposition 2.11, also for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| \geq m$. Consequently, (2.7) reduces to (2.6).

Conversely, suppose that (2.6) holds. Comparing with (2.7), it follows that $\Phi_\alpha(\mathcal{W}, \mathcal{C}) = 0$ for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| \geq m$. Hence, $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$. $\square$

The following result is analogous to well-known facts for $(m, \mathcal{C})$ -isometric tuples (see [22, Proposition 4]) and for $[m, \mathcal{C}]$ -isometric tuples (see [2, Corollary 2.12]).

COROLLARY 2.13. *Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}_c[\mathcal{K}]^d$. Then $\mathcal{W} \in \{\mathbf{T}\text{-}[2, \mathcal{C}]\text{-Iso}\}$ for some $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$ if and only if*

$$C\mathcal{W}^\beta C\mathcal{W}^\beta = I + \sum_{j=1}^d \beta_j (C\mathcal{W}_j C\mathcal{W}_j - I), \quad \beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d.$$

Proof. The result follows directly from Theorem 2.12. $\square$

THEOREM 2.14. *Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d)$ and $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_d)$ are in $\mathcal{B}_c[\mathcal{K}]^d$ so that $\mathcal{W}_j \mathcal{T}_\ell = \mathcal{T}_\ell \mathcal{W}_j$ for $j, \ell = 1, \dots, d$. If $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$ and $\mathcal{T} \in \{\mathbf{T}\text{-}[2, \mathcal{C}]\text{-Iso}\}$ for some $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$, then the componentwise product $\mathcal{W}\mathcal{T} := (\mathcal{W}_1 \mathcal{T}_1, \dots, \mathcal{W}_d \mathcal{T}_d)$ is in $\{\mathbf{T}\text{-}[m+1, \mathcal{C}]\text{-Iso}\}$.*

Proof. Using the commutation relations $\mathcal{W}_j \mathcal{T}_k = \mathcal{T}_k \mathcal{W}_j$ for all $j, k = 1, \dots, d$ together with Corollary 2.13, we compute

$$\begin{aligned} \Phi_\beta(\mathcal{W}\mathcal{T}, \mathcal{C}) &= \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C}(\mathcal{W}\mathcal{T})^\alpha \mathcal{C}(\mathcal{W}\mathcal{T})^\alpha \\ &= \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C}\mathcal{W}^\alpha \mathcal{C}(\mathcal{C}\mathcal{T}^\alpha \mathcal{C}\mathcal{T}^\alpha) \mathcal{W}^\alpha. \end{aligned}$$

Since $\mathcal{T}$ is toral $[2, \mathcal{C}]$ -isometric, Corollary 2.13 yields

$$\mathcal{C}\mathcal{T}^\alpha \mathcal{C}\mathcal{T}^\alpha = I + \sum_{j=1}^d \alpha_j (\mathcal{C}\mathcal{T}_j \mathcal{C}\mathcal{T}_j - I).$$

Substituting this into the above expression, we obtain

$$\begin{aligned} \Phi_\beta(\mathcal{W}\mathcal{T}, \mathcal{C}) &= \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C}\mathcal{W}^\alpha \mathcal{C}\mathcal{W}^\alpha \\ &\quad + \sum_{j=1}^d (\mathcal{C}\mathcal{T}_j \mathcal{C}) \left(\sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \alpha_j \binom{\beta}{\alpha} \mathcal{C}\mathcal{W}^\alpha \mathcal{C}\mathcal{W}^\alpha \right) \mathcal{T}_j \\ &\quad - \sum_{j=1}^d \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \alpha_j \binom{\beta}{\alpha} \mathcal{C}\mathcal{W}^\alpha \mathcal{C}\mathcal{W}^\alpha. \end{aligned}$$

For $|\beta| = m + 1$, denote

$$K_j := \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \alpha_j \binom{\beta}{\alpha} \mathcal{C}\mathcal{W}^\alpha \mathcal{C}\mathcal{W}^\alpha.$$

A standard combinatorial argument gives

$$K_j = -\beta_j \Phi_{\beta - \varepsilon(j)}(\mathcal{W}, \mathcal{C}).$$

Since $|\beta - \varepsilon(j)| = m$ and $\mathcal{W}$ is toral $[m, \mathcal{C}]$ -isometric, we have

$$\Phi_{\beta - \varepsilon(j)}(\mathcal{W}, \mathcal{C}) = 0,$$

and hence $K_j = 0$ for all $j = 1, \dots, d$.

Therefore,

$$\Phi_\beta(\mathcal{W}\mathcal{T}, \mathcal{C}) = \Phi_\beta(\mathcal{W}, \mathcal{C}) = 0, \quad \text{for all } |\beta| = m + 1,$$

which shows that $\mathcal{W}\mathcal{T} \in \{\mathbf{T}\text{-}[m + 1, \mathcal{C}]\text{-Iso}\}$. $\square$

The next proposition shows that if $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$, then it is possible to construct an operator tuple $\mathbf{T}$ which is an $[m, \mathcal{C}]$ -isometric tuple.

PROPOSITION 2.15. Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}_c[\mathcal{K}]^d$ and define the d -tuple

$$\mathcal{T} = \left(\frac{1}{\sqrt{d}} \mathcal{W}_1, \dots, \frac{1}{\sqrt{d}} \mathcal{W}_d \right).$$

If $\mathcal{W} \in \{\mathbf{T}\text{-}[m+1, \mathcal{C}]\text{-Iso}\}$ for some $\mathcal{C} \in \mathcal{C}[\mathcal{K}]$, then $\mathcal{T}$ is an $[m, \mathcal{C}]$ -isometric tuple.

Proof. We prove the identity

$$\Psi_n(\mathcal{T}, \mathcal{C}) = \frac{(-1)^n}{d^n} \sum_{\substack{\beta \in \mathbb{Z}_+^d \\ |\beta| = n}} \Phi_\beta(\mathcal{W}, \mathcal{C}), \quad (2.8)$$

where

$$\Phi_\beta(\mathcal{W}, \mathcal{C}) = \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \mathcal{C} \mathcal{W}^\alpha \mathcal{C} \mathcal{W}^\alpha, \quad |\beta| = n.$$

Step 1: The case $n = 1$. By (1.7),

$$\Psi_1(\mathcal{T}, \mathcal{C}) = \sum_{k=1}^d \mathcal{C} \mathcal{T}_k \mathcal{C} \mathcal{T}_k - I.$$

For $\mathcal{T} = \frac{1}{\sqrt{d}} \mathcal{W}$, we obtain

$$\begin{aligned} \Psi_1(\mathcal{T}, \mathcal{C}) &= \frac{1}{d} \left(\sum_{k=1}^d \mathcal{C} \mathcal{W}_k \mathcal{C} \mathcal{W}_k - dI \right) \\ &= -\frac{1}{d} \sum_{|\beta|=1} \Phi_\beta(\mathcal{W}, \mathcal{C}), \end{aligned}$$

which proves (2.8) for $n = 1$.

Step 2: Induction step. Assume that (2.8) holds for some $n \geq 1$. Using (1.7), we have

$$\Psi_{n+1}(\mathcal{T}, \mathcal{C}) = \sum_{k=1}^d \mathcal{C} \mathcal{T}_k \mathcal{C} \Psi_n(\mathcal{T}, \mathcal{C}) \mathcal{T}_k - \Psi_n(\mathcal{T}, \mathcal{C}).$$

Substituting the induction hypothesis yields

$$\Psi_{n+1}(\mathcal{T}, \mathcal{C}) = \frac{(-1)^n}{d^n} \sum_{|\beta|=n} \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta}{\alpha} \left[\frac{1}{d} \sum_{k=1}^d \mathcal{C} \mathcal{W}^\alpha \mathcal{W}_k \mathcal{C} \mathcal{W}^\alpha \mathcal{W}_k - \mathcal{C} \mathcal{W}^\alpha \mathcal{C} \mathcal{W}^\alpha \right].$$

Rearranging terms gives

$$\Psi_{n+1}(\mathcal{T}, \mathcal{C}) = \frac{(-1)^{n+1}}{d^{n+1}} \sum_{|\beta|=n} \left[\sum_{k=1}^d \mathcal{C} \mathcal{W}_k \mathcal{C} \Phi_\beta(\mathcal{W}, \mathcal{C}) \mathcal{W}_k - d \Phi_\beta(\mathcal{W}, \mathcal{C}) \right].$$

Using (2.5), we obtain

$$\Psi_{n+1}(\mathcal{T}, \mathcal{C}) = \frac{(-1)^{n+1}}{d^{n+1}} \sum_{|\beta|=n} \sum_{k=1}^d \Phi_{\beta+\varepsilon(k)}(\mathcal{W}, \mathcal{C}) = \frac{(-1)^{n+1}}{d^{n+1}} \sum_{|\beta|=n+1} \Phi_{\beta}(\mathcal{W}, \mathcal{C}).$$

Thus, the identity holds for all $n \in \mathbb{N}$ by induction.

Finally, since $\mathcal{W} \in \{\mathbf{T}\text{-}[m, \mathcal{C}]\text{-Iso}\}$, from Proposition 2.11 we have $\mathcal{W} \in \{\mathbf{T}\text{-}[m+1, \mathcal{C}]\text{-Iso}\}$. Hence, by (2.8), the tuple $\mathcal{T}$ is an $[m, \mathcal{C}]$ -isometric multioperator. $\square$

It should be noted that the converse of Proposition 2.15 does not hold in general, as illustrated by the following example.

EXAMPLE 2.16. Consider the operators

$$\mathcal{W}_1 = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}, \quad \mathcal{W}_2 = \begin{pmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It was shown in [22] that the normalized pair

$$\left(\frac{\mathcal{W}_1}{\sqrt{2}}, \frac{\mathcal{W}_2}{\sqrt{2}} \right)$$

forms a toral $[1, \mathcal{C}]$ -isometric 2-tuple. However, a direct computation yields

$$\mathcal{C}\mathcal{W}_1\mathcal{C}\mathcal{W}_1 \neq I_2,$$

and hence $(\mathcal{W}_1, \mathcal{W}_2) \notin \{\mathbf{T}\text{-}[1, \mathcal{C}]\text{-Iso}\}$.

The following theorem extends and generalizes [28, Proposition 1.3].

THEOREM 2.17. *For $d > 1$, there does not exist a d -tuple of operators that is simultaneously an $[m, \mathcal{C}]$ -isometric tuple and a toral $[m, \mathcal{C}]$ -isometric tuple.*

Proof. Let $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d) \in \mathcal{B}_c[\mathcal{K}]^d$ with $d > 1$. We first observe that

$$\sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} \mathcal{C}\mathcal{W}^\alpha \mathcal{C}\mathcal{W}^\alpha = \sum_{1 \leq j \leq d} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathcal{C}\mathcal{W}_j \mathcal{W}^\alpha \mathcal{C}\mathcal{W}^\alpha \mathcal{W}_j.$$

Suppose that $\mathcal{W}$ is both an $[m, \mathcal{C}]$ -isometric and a toral $[m, \mathcal{C}]$ -isometric d -tuple. Then

$$\Psi_m(\mathcal{W}, \mathcal{C}) = 0, \quad \Phi_\beta(\mathcal{W}, \mathcal{C}) = 0 \quad \text{for all } |\beta| = m.$$

Moreover, by (2.8), we also obtain

$$\Psi_m(\mathcal{T}, \mathcal{C}) = 0, \quad \text{where } \mathcal{T} = \left(\frac{1}{\sqrt{d}} \mathcal{W}_1, \dots, \frac{1}{\sqrt{d}} \mathcal{W}_d \right).$$

Using (1.7), we have

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{W}^\alpha C\mathcal{W}^\alpha = 0,$$

and similarly for $\mathcal{T}$:

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{T}^\alpha C\mathcal{T}^\alpha = 0.$$

Proceeding in the same way for each component $j = 1, \dots, d$, we obtain

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{W}_j \mathcal{W}^\alpha C\mathcal{W}^\alpha \mathcal{W}_j = 0,$$

and the analogous identity for $\mathcal{T}$.

Iterating this argument yields, for all $r \geq 1$,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k+r} \frac{(k+r)!}{\alpha!} C\mathcal{W}^\alpha C\mathcal{W}^\alpha = 0,$$

and similarly for $\mathcal{T}$.

Define

$$a_k := \sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{W}^\alpha C\mathcal{W}^\alpha, \quad b_k := \sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{T}^\alpha C\mathcal{T}^\alpha.$$

Then

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} a_{k+r} = 0, \quad \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} b_{k+r} = 0.$$

As in [6, Theorem 3.1] and [22, Corollary 1], the maps

$$k \mapsto \sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{W}^\alpha C\mathcal{W}^\alpha, \quad k \mapsto \sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{T}^\alpha C\mathcal{T}^\alpha,$$

are polynomials in k of degree at most $m-1$.

Finally, we point out that

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{T}^\alpha C\mathcal{T}^\alpha = \frac{1}{d^k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} C\mathcal{W}^\alpha C\mathcal{W}^\alpha,$$

Which results in a contradiction. Hence, such a simultaneous structure cannot exist. $\square$

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