

A NEW HARDY–HILBERT–TYPE INTEGRAL INEQUALITY INVOLVING ONE EXTENDED DERIVATIVE FUNCTION OF HIGHER-ORDER

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Abstract. By means of the weight functions, the idea of introducing parameters and the techniques of real analysis, a new Hardy-Hilbert-type integral inequality with two internal variables involving one extended derivative functions of m -order ($m \in \mathbf{N}_0$) is obtained. The equivalent statements of the best possible constant factor related to some parameters are considered. The equivalent forms, the operator expressions and the cases of reverses are provided.

1. Introduction

Assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. The following Hardy-Hilbert's inequality with the best possible constant factor $\pi / \sin(\frac{\pi}{p})$ was provided as follows (cf. [9], Theorem 315):

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \quad (1)$$

If $f(x), g(y) \geq 0$, $0 < \int_0^{\infty} f(x) dx < \infty$ and $0 < \int_0^{\infty} g(y) dy < \infty$, then we still have the integral analogue of (1) named in Hardy-Hilbert's integral inequality as follows (cf. [9], Theorem 316):

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y) dy \right)^{\frac{1}{q}}, \quad (2)$$

where the same constant factor $\pi / \sin(\frac{\pi}{p})$ is still the best possible. Inequalities (1) and (2) played an important role in analysis and its applications (cf. [2–13]).

In 2006, by using the Euler-Maclaurin summation formula, Krnić et al. [14] gave an extension of (1) with the kernel as $\frac{1}{(m+n)^{\lambda}}$ ($0 < \lambda \leq 4$). Applying the result of [14], in 2019, Adiyasuren et al. [1] considered an extension of (1) involving two partial sums. Furthermore, in 2020, Mo et al. [17] gave an extension of (2) involving two upper limit

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functions. In 2016–2017, Hong et al. [10, 11] provided several equivalent statements of the extensions of (1) and (2) with multi-parameters. Some similar results were given by [7, 8, 18, 20, 21, 30]. In 2023, [29] gave a Hardy-Hilbert-type integral inequality involving one multiple upper limit function and one derivative function of higher-order. Huang et al. [13] provided a reverse Hardy-Hilbert-type integral inequality involving derivative functions of higher-order. Peng and Yang [19] discussed half-discrete multidimensional Hilbert-type inequality with higher-order derivative function, while this research focuses on a more general kernel related to Hardy-Hilbert-type integral inequality. Although Hong et al. [12] also investigated kernels of integral inequalities involving an upperlimit function similar to ours, our work further extends the study to include higher-order derivative function.

In this paper, following the way of [11] and [29], by means of the weight functions, the idea of introduced parameters and the techniques of real analysis, a new Hardy-Hilbert-type integral inequality with the kernel as $\frac{1}{(U(x)+V(y))^{\lambda+m}}$ ($\lambda > 0$, $m \in \mathbf{N}$) involving one extended derivative function of m -order is provided. The equivalent statements of the best possible constant factor related to several parameters are considered. The equivalent forms, the operator expressions are obtained. The case of reverses are provided, which are extensions of [13].

2. Some lemmas

In what follows, we give the following Assumption A1:

$p > 0$ ($p \neq 1$), $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda_i < \lambda$ ($i = 1, 2$), $\hat{\lambda}_1 := \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$, for $\alpha, \beta \in \mathbf{R} = (-\infty, \infty)$, $U(x)$ ($x > \alpha$) and $V(y)$ ($y > \beta$) are positive functions with $U'(x) > 0$, $V'(y) > 0$, $U(\alpha^+) = V(\beta^+) = 0$, and $U(\infty) = V(\infty) = \infty$. We also assume that $f(x)$ is a nonnegative function in (α, ∞) , and define

$$f_0(x) := f(x); \quad f_k(x) := \left(\frac{f_{k-1}(x)}{U'(x)} \right)' \quad (x \in (\alpha, \infty))$$

$$(k = 1, \dots, m; m \in \mathbf{N}),$$

satisfying for any $k \in \{1, \dots, m\}$,

$$f_{k-1}(x) = o(e^{tU(x)}) \quad (t > 0; x \rightarrow \infty), \quad f_{k-1}(\alpha^+) = 0,$$

and $f_m(x)$ (resp. $g(y)$) ($m \in \mathbf{N}_0$) is a nonnegative continuous function unless at finite points in $x \in (\alpha, \infty)$ (resp. $y \in (\beta, \infty)$), such that

$$0 < \int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx < \infty \quad \text{and}$$

$$0 < \int_{\beta}^{\infty} \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy < \infty.$$

We indicate the gamma function as follows:

$$\Gamma(\alpha) := \int_0^{\infty} e^{-t} t^{\alpha-1} dt \quad (\alpha > 0) \quad (3)$$

satisfying $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ ($\alpha > 0$). Define the following beta function (cf. [22]):

$$B(u, v) := \int_0^\infty \frac{t^{u-1} dt}{(1+t)^{u+v}} = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \quad (u, v > 0). \quad (4)$$

By using (6), for $\lambda > 0$, $x > \alpha$, $y > \beta$, we still have the following formula related to the gamma function:

$$\frac{1}{(U(x) + V(y))^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(U(x)+V(y))t} t^{\lambda-1} dt. \quad (5)$$

LEMMA 1. *With regards to Assumption A1, for $t > 0$, $m \in \mathbf{N}_0$, we have the following expressions:*

$$\int_\alpha^\infty e^{-tU(x)} f(x) dx = t^{-m} \int_\alpha^\infty e^{-tU(x)} f_m(x) dx. \quad (6)$$

Proof. For $m = 0$, $f_0(x) = f(x)$, (6) is naturally valid; for $m \in \mathbf{N}$, $k \in \{1, \dots, m\}$, by the assumption A1, we find $e^{-tU(x)} \frac{f_{k-1}(x)}{U'(x)} \Big|_\alpha^\infty = 0$, and then by integration by parts, it follows that

$$\begin{aligned} \int_\alpha^\infty e^{-tU(x)} f_k(x) dx &= \int_\alpha^\infty e^{-tU(x)} d \left(\frac{f_{k-1}(x)}{U'(x)} \right) \\ &= \left[e^{-tU(x)} \frac{f_{k-1}(x)}{U'(x)} \right]_\alpha^\infty - \int_\alpha^\infty \frac{f_{k-1}(x)}{U'(x)} d e^{-tU(x)} \\ &= t \int_\alpha^\infty e^{-tU(x)} f_{k-1}(x) dx. \end{aligned} \quad (7)$$

Substitution of $k = 1, \dots, m$ in (7), by simplification, for $f_0(x) = f(x)$, we obtain (6).

The lemma is proved. $\square$

LEMMA 2. *With regards to Assumption A1, define the following weight functions:*

$$\varpi(\lambda_2, x) := (U(x))^{\lambda-\lambda_2} \int_\beta^\infty \frac{(V(t))^{\lambda_2-1} V'(t) dt}{(U(x) + V(t))^\lambda}, \quad x \in (\alpha, \infty), \quad (8)$$

$$\omega(\lambda_1, y) := (V(y))^{\lambda-\lambda_1} \int_\alpha^\infty \frac{(U(t))^{\lambda_1-1} U'(t) dt}{(U(t) + V(y))^\lambda}, \quad y \in (\beta, \infty). \quad (9)$$

We have the following expressions:

$$\varpi(\lambda_2, x) = B(\lambda_2, \lambda - \lambda_2) \quad (x \in (\alpha, \infty)), \quad (10)$$

$$\omega(\lambda_1, y) = B(\lambda_1, \lambda - \lambda_1) \quad (y \in (\beta, \infty)). \quad (11)$$

Proof. For a fixed x and $x > \alpha$, setting $u = \frac{V(t)}{U(x)}$, we find

$$\begin{aligned} \varpi(\lambda_2, x) &= (U(x))^{\lambda-\lambda_2} \int_0^\infty \frac{(uU(x))^{\lambda_2-1} U(x)}{(U(x) + uU(x))^\lambda} du \\ &= \int_0^\infty \frac{u^{\lambda_2-1} du}{(1+u)^\lambda} = B(\lambda_2, \lambda - \lambda_2), \end{aligned}$$

namely, (10) follows. In the same way, we obtain (11).

The lemma is proved. $\square$

LEMMA 3. *With regards to Assumption A1, (i) for $p > 1$, we have the following extended Hardy-Hilbert's integral inequality:*

$$I_0 := \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{f_m(x)g(y)dxdy}{(U(x)+V(y))^{\lambda}} < B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ \times \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}; \quad (12)$$

(ii) for $0 < p < 1$, we have the reverse of (12).

Proof. (i) By using the Hölder's inequality (cf. [15]), we obtain

$$I_0 = \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{1}{(U(x)+V(y))^{\lambda}} \left[\frac{(V(y))^{(\lambda_2-1)/p} (V'(y))^{1/p}}{(U(x))^{(\lambda_1-1)/q} (U'(x))^{1/q}} f_m(x) \right] \\ \times \left[\frac{(U(x))^{(\lambda_1-1)/q} (U'(x))^{1/q}}{(V(y))^{(\lambda_2-1)/p} (V'(y))^{1/p}} g(y) \right] dxdy \\ \leq \left\{ \int_{\alpha}^{\infty} \left[\int_{\beta}^{\infty} \frac{(U'(x))^{1-p} V'(y)}{(U(x)+V(y))^{\lambda}} \frac{(V(y))^{\lambda_2-1} dy}{(U(x))^{(\lambda_1-1)(p-1)}} f_m^p(x) dx \right]^{\frac{1}{p}} \right. \\ \times \left. \left\{ \int_{\beta}^{\infty} \left[\int_{\alpha}^{\infty} \frac{(V'(y))^{1-q} U'(x)}{(U(x)+V(y))^{\lambda}} \frac{(U(x))^{\lambda_1-1} dx}{(V(y))^{(\lambda_2-1)(q-1)}} g^q(y) dy \right]^{\frac{1}{q}} \right\} \right. \\ = \left[\int_{\alpha}^{\infty} \varpi(\lambda_2, x) \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}} \\ \times \left[\int_{\beta}^{\infty} \omega(\lambda_1, y) \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}. \quad (13)$$

If (13) keeps the form of equality, then there exist constants A and B , such that they are not both zero, satisfying

$$\frac{A(V(y))^{\lambda_2-1} V'(y) f_m^p(x)}{(U(x))^{(\lambda_1-1)(p-1)} (U'(x))^{p-1}} = \frac{B(U(x))^{\lambda_1-1} U'(x) g^q(y)}{(V(y))^{(\lambda_2-1)(q-1)} (V'(y))^{q-1}} \\ a.e. \text{ in } (\alpha, \infty) \times (\beta, \infty).$$

Assume that $A \neq 0$. For fixed $a, e, y \in (\beta, \infty)$, we have

$$\frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) \\ = \frac{B g^q(y)}{A(V(y))^{(\lambda_2-1)q} (V'(y))^q} (U(x))^{-1-b} U'(x) \text{ a.e. in } (\alpha, \infty),$$

where $b := \lambda - \lambda_1 - \lambda_2 \in \mathbf{R}$. Since $\int_{\alpha}^{\infty} (U(x))^{-1-b} U'(x) dx = \infty$, the above expression contradicts the fact that

$$0 < \int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx < \infty.$$

Therefore, by using (10), (11) and (13), we have (12).

(ii) By applying the reverse Hölder's inequality and in the same way, we obtain the reverse of (13).

The lemma is proved. $\square$

3. Main results

THEOREM 1. *With regards to Assumption A1, (i) for $p > 1$, we have the following Hardy-Hilbert-type integral inequality involving one extended derivative function of m -order ($m \in \mathbf{N}_0$):*

$$\begin{aligned} I &:= \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(U(x) + V(y))^{\lambda+m}} dx dy \\ &< \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ &\quad \times \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}; \quad (14) \end{aligned}$$

(ii) for $0 < p < 1$, we have the reverse of (14).

In particular, for $p > 1$, $\lambda_1 + \lambda_2 = \lambda$, we reduce (14) to the following:

$$\begin{aligned} &\int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y) dx dy}{(U(x) + V(y))^{\lambda+m}} < \frac{\Gamma(\lambda) B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)} \\ &\quad \times \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\lambda_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\lambda_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \quad (15) \end{aligned}$$

where the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B(\lambda_1, \lambda_2)$ is the best possible.

Proof. (i) By using (6) and the Fubini theorem (cf. [16]), we find

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+m)} \int_{\beta}^{\infty} \int_{\alpha}^{\infty} f(x)g(y) \left[\int_0^{\infty} t^{\lambda+m-1} e^{-(U(x)+V(y))t} dt \right] dx dy \\ &= \frac{1}{\Gamma(\lambda+m)} \int_0^{\infty} t^{\lambda+m-1} \left(\int_{\alpha}^{\infty} e^{-U(x)t} f(x) dx \right) \left(\int_{\beta}^{\infty} e^{-V(y)t} g(y) dy \right) dt \\ &= \frac{1}{\Gamma(\lambda+m)} \int_0^{\infty} t^{\lambda-1} \left(\int_{\alpha}^{\infty} e^{-U(x)t} f_n(x) dx \right) \left(\int_{\beta}^{\infty} e^{-V(y)t} g(y) dy \right) dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\lambda+m)} \int_{\beta}^{\infty} \int_{\alpha}^{\infty} f_m(x)g(y) \left[\int_0^{\infty} t^{\lambda-1} e^{-(U(x)+V(y))t} dt \right] dx dy \\
&= \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{f_m(x)g(y)dx dy}{(U(x)+V(y))^{\lambda}} = \frac{\Gamma(\lambda)I_0}{\Gamma(\lambda+m)}.
\end{aligned}$$

Then by using (12), we have (14).

(ii) In the same way, by using the reverse of (12), we have the reverse of (14).

For $\lambda_1 + \lambda_2 = \lambda$ in (14), by simplification, we have (15). For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we set

$$\tilde{f}(x) := \begin{cases} 0, & \alpha < x < \alpha + 1, \\ [(U(x))^{\lambda_1 - \frac{\varepsilon}{p} + m - 1} - P_{m-1}(U(x))]U'(x), & x \geq \alpha + 1, \end{cases}$$

and for $k = 0, 1, \dots, m$ ($m \in \mathbf{N}_0$),

$$\widehat{f}_k(x) := \begin{cases} 0, & \alpha < x < \alpha + 1, \\ \prod_{i=0}^{k-1} (\lambda_1 - \frac{\varepsilon}{p} + i) [(U(x))^{\lambda_1 - \frac{\varepsilon}{p} + m - k - 1} - P_{m-k-1}(U(x))]U'(x), & x \geq \alpha + 1, \end{cases}$$

where we define $\prod_{i=0}^{-1} (\lambda_1 - \frac{\varepsilon}{p} + i) := 1$, $P_{-1}(U(x)) := 0$; for $k = 0, \dots, m-1$ ($m \in \mathbf{N}$), $P_{m-k-1}(t)$ is a positive polynomial of $(m-k-1)$ -order, satisfying $P_{m-k-1}(U(\alpha+1)) = (U(\alpha+1))^{\lambda_1 - \frac{\varepsilon}{p} + m - k - 1}$,

$$\begin{aligned}
\widehat{f}_0(x) &:= \tilde{f}(x); \widehat{f}_k(x) = \left(\frac{\widehat{f}_{k-1}(x)}{U'(x)} \right)' \quad (x \in (\alpha, \infty)), \\
\widehat{f}_k(x) &= o(e^{tU(x)}) \quad (t > 0; x \rightarrow \infty), \quad \widehat{f}_k(\alpha^+) = 0 \\
&\quad (k = 1, \dots, m; m \in \mathbf{N}),
\end{aligned}$$

and

$$\widehat{f}_m(x) = \begin{cases} 0, & \alpha < x < \alpha + 1, \\ \prod_{i=0}^{m-1} (\lambda_1 - \frac{\varepsilon}{p} + i)(U(x))^{\lambda_1 - \frac{\varepsilon}{p} - 1} U'(x), & x \geq \alpha + 1. \end{cases}$$

We also set the following function:

$$\widetilde{g}(y) := \begin{cases} 0, & \beta < y < \beta + 1, \\ (V(y))^{\lambda_2 - \frac{\varepsilon}{q} - 1} V'(y), & y \geq \beta + 1. \end{cases}$$

If there exists a positive constant $M \leq \frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)}$, such that (15) is valid if we

replace $\frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)}$ by M , then in particular, setting $b_\varepsilon := \prod_{i=0}^{m-1} (\lambda_1 - \frac{\varepsilon}{p} + i) (> 0)$, since

$$\begin{aligned} \tilde{J} &:= \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\lambda_1)-1}}{(U'(x))^{p-1}} \widehat{f}_m^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\lambda_2)-1}}{(V'(y))^{q-1}} \widetilde{g}^q(y) dy \right]^{\frac{1}{q}} \\ &= b_\varepsilon \left[\int_{\alpha+1}^{\infty} U(x)^{-\varepsilon-1} dU(x) \right]^{\frac{1}{p}} \left[\int_{\beta+1}^{\infty} (V(y))^{-\varepsilon-1} dV(y) \right]^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} b_\varepsilon (U(\alpha+1))^{-\frac{\varepsilon}{p}} (V(\beta+1))^{-\frac{\varepsilon}{q}}, \end{aligned}$$

we have

$$\begin{aligned} \tilde{I} &:= \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{\widetilde{f}(x) \widetilde{g}(y) dx dy}{(U(x) + V(y))^{\lambda+m}} < M \tilde{J} \\ &= \frac{M}{\varepsilon} b_\varepsilon (U(\alpha+1))^{-\frac{\varepsilon}{p}} (V(\beta+1))^{-\frac{\varepsilon}{q}}. \end{aligned} \quad (16)$$

In view of the Fubini Theorem (cf. [16]), setting $v = V(y)/U(x)$, it follows that

$$\begin{aligned} \tilde{I} &= \int_{\alpha+1}^{\infty} \left[\int_{\beta+1}^{\infty} \frac{(V(y))^{\lambda_2 - \frac{\varepsilon}{q} - 1} V'(y)}{(U(x) + V(y))^{\lambda+m}} dy \right] \\ &\quad \times [(U(x))^{\lambda_1 - \frac{\varepsilon}{p} + m - 1} - P_{m-1}(U(x))] U'(x) dx = I_0 - I_1, \end{aligned}$$

where we indicate

$$\begin{aligned} I_0 &:= \int_{\alpha+1}^{\infty} \left[\int_{\beta+1}^{\infty} \frac{(V(y))^{\lambda_2 - \frac{\varepsilon}{q} - 1} V'(y)}{(U(x) + V(y))^{\lambda+m}} dy \right] (U(x))^{\lambda_1 - \frac{\varepsilon}{p} + m - 1} U'(x) dx, \\ I_1 &:= \int_{\alpha+1}^{\infty} \left[\int_{\beta+1}^{\infty} \frac{(V(y))^{\lambda_2 - \frac{\varepsilon}{q} - 1} V'(y)}{(U(x) + V(y))^{\lambda+m}} dy \right] P_{m-1}(U(x)) U'(x) dx. \end{aligned}$$

For a fixed x and $x > \alpha$, setting $v = V(y)/U(x)$, we find

$$\begin{aligned} I_0 &= \int_{\alpha+1}^{\infty} (U(x))^{-\varepsilon-1} U'(x) \left[\int_{\frac{V(\beta+1)}{U(x)}}^{\infty} \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+v)^{\lambda+m}} dv \right] dx \\ &= \int_{\alpha+1}^{\infty} (U(x))^{-\varepsilon-1} U'(x) \left[\int_{\frac{V(\beta+1)}{U(x)}}^{\frac{V(\beta+1)}{U(\alpha+1)}} \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+v)^{\lambda+m}} dv \right] dx \\ &\quad + \int_{\alpha+1}^{\infty} (U(x))^{-\varepsilon-1} U'(x) \left[\int_{\frac{V(\beta+1)}{U(\alpha+1)}}^{\infty} \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+v)^{\lambda+m}} dv \right] dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{V(\beta+1)}{U(\alpha+1)}} \left[\int_{U^{-1}(\frac{V(\beta+1)}{v})}^{\infty} (U(x))^{-\varepsilon-1} U'(x) dx \right] \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1} dv}{(1+v)^{\lambda+m}} \\
&\quad + \frac{1}{\varepsilon} (U(\alpha+1))^{-\varepsilon} \int_{\frac{V(\beta+1)}{U(\alpha+1)}}^{\infty} \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+v)^{\lambda+m}} dv \\
&= \frac{1}{\varepsilon} \left[(V(\beta+1))^{-\varepsilon} \int_0^{\frac{V(\beta+1)}{U(\alpha+1)}} \frac{v^{\lambda_2 + \frac{\varepsilon}{p} - 1}}{(1+v)^{\lambda+m}} dv + (U(\alpha+1))^{-\varepsilon} \int_{\frac{V(\beta+1)}{U(\alpha+1)}}^{\infty} \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+v)^{\lambda+m}} dv \right].
\end{aligned}$$

We also obtain

$$\begin{aligned}
0 < I_1 &= \int_{\alpha+1}^{\infty} \frac{P_{m-1}(U(x))U'(x)}{(U(x)+V(y))^{\frac{\lambda_1}{2}+m}} \left[\int_{\beta+1}^{\infty} \frac{(V(y))^{\lambda_2 - \frac{\varepsilon}{q} - 1} V'(y) dy}{(U(x)+V(y))^{\lambda_2 + \frac{\lambda_1}{2}}} \right] dx \\
&\leq \int_{\alpha+1}^{\infty} \frac{U'(x) dx}{O((U(x))^{\frac{\lambda_1}{2}+1})} \int_{\beta+1}^{\infty} \frac{V'(y) dy}{(V(y))^{\frac{\varepsilon}{q} + \frac{\lambda_1}{2} + 1}} \leq M_1 < \infty.
\end{aligned}$$

So by using (16) and the above results, it follows that

$$\begin{aligned}
&(V(\beta+1))^{-\varepsilon} \int_0^{\frac{V(\beta+1)}{U(\alpha+1)}} \frac{v^{\lambda_2 + \frac{\varepsilon}{p} - 1}}{(1+v)^{\lambda+m}} dv + (U(\alpha+1))^{-\varepsilon} \int_{\frac{V(\beta+1)}{U(\alpha+1)}}^{\infty} \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+v)^{\lambda+m}} dv - \varepsilon I_1 \\
&= \varepsilon \tilde{I} < M b_{\varepsilon} (U(\alpha+1))^{-\frac{\varepsilon}{p}} (V(\beta+1))^{-\frac{\varepsilon}{q}}.
\end{aligned}$$

As $\varepsilon \rightarrow 0^+$ in the above inequality, in view of the continuity of the beta function, we find

$$\frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)} b_0 = B(\lambda_1+m, \lambda_2) \leq M b_0,$$

namely, $M \geq \frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)}$.

Hence, $M = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B(\lambda_1, \lambda_2)$ is the best possible constant factor in (15).

The theorem is proved. $\square$

REMARK 1. For $p > 1$, $\hat{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$, it follows that $\hat{\lambda}_1 + \hat{\lambda}_2 = \lambda$. We find $0 < \hat{\lambda}_1 < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda$ and then $0 < \hat{\lambda}_2 = \lambda - \hat{\lambda}_1 < \lambda$. By using the Hölder's inequality (cf. [15]), we obtain

$$\begin{aligned}
0 < B(\hat{\lambda}_1, \hat{\lambda}_2) &= \int_0^{\infty} \frac{u^{\hat{\lambda}_1-1} du}{(1+u)^{\lambda}} = \int_0^{\infty} \frac{u^{\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} - 1}}{(1+u)^{\lambda}} du \\
&= \int_0^{\infty} \frac{1}{(1+u)^{\lambda}} \left(u^{\frac{\lambda-\lambda_2-1}{p}} \right) \left(u^{\frac{\lambda_1-1}{q}} \right) du \\
&\leq \left[\int_0^{\infty} \frac{u^{(\lambda-\lambda_2)-1}}{(1+u)^{\lambda}} du \right]^{\frac{1}{p}} \left[\int_0^{\infty} \frac{u^{\lambda_1-1}}{(1+u)^{\lambda}} du \right]^{\frac{1}{q}} \\
&= B^{\frac{1}{p}}(\lambda - \lambda_2, \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1).
\end{aligned} \tag{17}$$

THEOREM 2. With regards to Assumption A1, for $p > 1$, if the constant factor

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (14) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. By using (15) (for $\lambda_i = \widehat{\lambda}_i$ ($i = 1, 2$)), since

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

is the best possible constant factor in (14), we have

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \leq \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B(\widehat{\lambda}_1, \widehat{\lambda}_2) (\in \mathbf{R}_+),$$

namely,

$$B(\widehat{\lambda}_1, \widehat{\lambda}_2) \geq B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1).$$

It follows that (17) keeps the form of equality.

We observe that (17) keeps the form of equality if and only if there exist constants A and B , such that they are not both zero and $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ a.e. in $\mathbf{R}_+$ (cf. [15]). Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_2-\lambda_1} = B/A$ a.e. in $\mathbf{R}_+$, namely, $\lambda - \lambda_2 - \lambda_1 = 0$, and then $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. $\square$

For $U(x) = x$ ($x \in (0, \infty)$) in Theorem 1 and Theorem 2, we have $f_m(x) = f^{(m)}(x)$ and

COROLLARY 1. For $p > 1$, we have the following Hardy-Hilbert-type integral inequality involving one derivative functions of m -order:

$$\begin{aligned} & \int_0^\infty \int_\beta^\infty \frac{f(x)g(y)}{(x+U(y))^{\lambda+m}} dx dy \\ & < \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ & \times \left[\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_\beta^\infty \frac{(V(y))^{q(1-\widehat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^p(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (18)$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we reduce (18) to the following:

$$\begin{aligned} & \int_0^\infty \int_\beta^\infty \frac{f(x)g(y)}{(x+V(y))^{\lambda+m}} dx dy < \frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)} \\ & \times \left[\int_0^\infty x^{p(1-\lambda_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_\beta^\infty \frac{(V(y))^{q(1-\lambda_2)-1}}{(V'(y))^{q-1}} g^p(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (19)$$

where, the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B(\lambda_1, \lambda_2)$ is the best possible. On the other hand, if the constant factor in (18) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.

4. The case of reverses

For $0 < p < 1$ in Theorem 1, we have the following reverse of (14):

$$\begin{aligned} & \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(U(x) + V(y))^{\lambda+m}} dx dy \\ & > \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ & \times \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \quad (20) \end{aligned}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we reduce (20) as follows:

$$\begin{aligned} & \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y) dx dy}{(U(x) + V(y))^{\lambda+m}} > \frac{\Gamma(\lambda) B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)} \\ & \times \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\lambda_1)-1}}{(U'(x))^{p-1}} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\lambda_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}. \quad (21) \end{aligned}$$

THEOREM 3. *With regards to Assumption A1, for $0 < p < 1$, the constant factor $\frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)}$ in (21) is the best possible.*

Proof. For any $0 < \varepsilon < \lambda_1 \min\{p, |q|\}$, we set functions $\tilde{f}(x)$, $\tilde{g}(y)$ and $\tilde{f}_n(x)$ as in Theorem 1. If there exists a positive constant $M \geq \frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)}$, such that (21) is valid when we replace $\frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)}$ by M , then in particular, we have

$$\tilde{I} = \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{\tilde{f}(x)\tilde{g}(y)}{(U(x) + V(y))^{\lambda+m}} dx dy > M\tilde{J}, \quad (22)$$

where, in view of the proof of Theorem 1, we have

$$\begin{aligned} \tilde{J} &= \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\lambda_1)-1}}{(U'(x))^{p-1}} \tilde{f}_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\lambda_2)-1}}{(V'(y))^{q-1}} \tilde{g}^q(y) dy \right]^{\frac{1}{2}} \\ &= \frac{b_{\varepsilon}}{\varepsilon} (U(\alpha+1))^{-\frac{\varepsilon}{p}} (V(\beta+1))^{-\frac{\varepsilon}{q}}. \end{aligned}$$

By applying the Fubini Theorem (cf. [16]) and the proof of Theorem 2, setting $v = V(y)/U(x)$, it follows that

$$\begin{aligned} \tilde{I} &< \int_{\alpha+1}^{\infty} \left[\int_{\beta+1}^{\infty} \frac{(V(y))^{\lambda_2 - \frac{\varepsilon}{q} - 1} V'(y) dy}{(U(x) + V(y))^{\lambda+m}} \right] (U(x))^{\lambda_1 - \frac{\varepsilon}{p} + m - 1} U'(x) dx \\ &= \frac{1}{\varepsilon} \left[(V(\beta+1))^{-\varepsilon} \int_0^{\frac{V(\beta+1)}{U(\alpha+1)}} \frac{v^{\lambda_2 + \frac{\varepsilon}{p} - 1}}{(1+v)^{\lambda+m}} dv + (U(\alpha+1))^{-\varepsilon} \int_{\frac{V(\beta+1)}{U(\alpha+1)}}^{\infty} \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+v)^{\lambda+m}} dv \right] \end{aligned}$$

By using (22) and the above results, we have

$$\left[(V(\beta+1))^{-\varepsilon} \int_0^{\frac{V(\beta+1)}{U(\alpha+1)}} \frac{v^{\lambda_2 + \frac{\varepsilon}{p} - 1}}{(1+v)^{\lambda+m}} dv + (U(\alpha+1))^{-\varepsilon} \int_{\frac{V(\beta+1)}{U(\alpha+1)}}^{\infty} \frac{v^{\lambda_2 - \frac{\varepsilon}{q} - 1}}{(1+v)^{\lambda+m}} dv \right] \\ > \varepsilon \tilde{I} > Mb_{\varepsilon} (U(\alpha+1))^{-\frac{\varepsilon}{p}} (V(\beta+1))^{-\frac{\varepsilon}{q}}.$$

As $\varepsilon \rightarrow 0^+$ in the above inequality, in view of the continuity of the beta function, we find

$$\frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)b_0}{\Gamma(\lambda+m)} = \frac{\Gamma(\lambda)B(\lambda_1+m, \lambda_2)}{\Gamma(\lambda+m)} \geq Mb_0.$$

Hence, $M = \frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)}$ is the best possible constant factor in (21).

The theorem is proved. $\square$

REMARK 2. For $0 < p < 1$, $\widehat{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} = \frac{\lambda - \lambda_1 - \lambda_2}{p} + \lambda_1$, $\widehat{\lambda}_2 = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$, it follows that $\widehat{\lambda}_1 + \widehat{\lambda}_2 = \lambda$. For $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, we find $0 < \widehat{\lambda}_1 < \lambda$, and then $0 < \widehat{\lambda}_2 = \lambda - \widehat{\lambda}_1 < \lambda$. By using the reverse Hölder's inequality (cf. [15]), we obtain

$$\begin{aligned} \infty > B(\widehat{\lambda}_1, \widehat{\lambda}_2) &= \int_0^{\infty} \frac{u^{\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} - 1}}{(1+u)^{\lambda}} du \\ &= \int_0^{\infty} \frac{1}{(1+u)^{\lambda}} \left(u^{\frac{\lambda - \lambda_2 - 1}{p}} \right) \left(u^{\frac{\lambda_1 - 1}{q}} \right) du \\ &\geq \left[\int_0^{\infty} \frac{u^{\lambda - \lambda_2 - 1}}{(1+u)^{\lambda}} du \right]^{\frac{1}{p}} \left[\int_0^{\infty} \frac{u^{\lambda_1 - 1}}{(1+u)^{\lambda}} du \right]^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) > 0. \end{aligned} \quad (23)$$

THEOREM 4. With regards to Assumption A1, for $0 < p < 1$, if the constant factor

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (20) is the best possible, then for $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. By using (21) (for $\lambda_i = \widehat{\lambda}_i$ ($i = 1, 2$)), since

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

is the best possible constant factor in (20), we have

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \geq \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B(\widehat{\lambda}_1, \widehat{\lambda}_2) (\in \mathbf{R}_+),$$

namely,

$$B(\widehat{\lambda}_1, \widehat{\lambda}_2) \leq B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1).$$

It follows that (23) keeps the form of equality.

We observe that (23) keeps the form of equality if and only if there exist constants A and B , such that they are not both zero and $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ a.e. in R_+ (cf. [15]). Assume that $A \neq 0$. We have $u^{\lambda-\lambda_2-\lambda_1} = B/A$ a.e. in R_+ , namely, $\lambda - \lambda_2 - \lambda_1 = 0$, and then $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. $\square$

For $U(x) = x$ ($x \in (0, \infty)$) in Theorem 3 and Theorem 4, we have $f_m(x) = f^{(m)}(x)$ and

COROLLARY 2. For $0 < p < 1$, we have the following reverse Hardy-Hilbert-type integral inequality involving one derivative function of m -order:

$$\begin{aligned} \int_{\beta}^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{(x+V(y))^{\lambda+m}} dx dy &> \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ &\times \left[\int_0^{\infty} x^{p(1-\widehat{\lambda}_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\widehat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (24)$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we reduce (24) as follows:

$$\begin{aligned} \int_{\beta}^{\infty} \int_0^{\infty} \frac{f(x)g(y)dx dy}{(x+V(y))^{\lambda+m}} &> \frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)} \\ &\times \left[\int_0^{\infty} x^{p(1-\lambda_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\lambda_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (25)$$

where the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B(\lambda_1, \lambda_2)$ is the best possible. On the other hand, if the constant factor in (24) is the best possible, then for $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, we have $\lambda_1 + \lambda_2 = \lambda$.

5. Equivalent forms and the operator expressions

THEOREM 5. With regards to Assumption A1, (i) for $p > 1$, we have the following Hardy-Hilbert-type integral inequality which is equivalent to (14):

$$\begin{aligned} J &:= \left\{ \int_{\beta}^{\infty} \left[(V(y))^{p\widehat{\lambda}_2-1} V'(y) \int_{\alpha}^{\infty} \frac{f(x)dx}{(U(x)+V(y))^{\lambda+m}} \right]^p dy \right\}^{\frac{1}{p}} \\ &< \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ &\times \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\widehat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \quad (26)$$

(ii) for $0 < p < 1$, we have the reverse of (26) which is equivalent to (20).

In particular, for $p > 1$, $\lambda_1 + \lambda_2 = \lambda$, we reduce (26) as follows:

$$\left\{ \int_{\beta}^{\infty} \left[(V(y))^{\hat{\lambda}_2-1} V'(y) \int_{\alpha}^{\infty} \frac{f(x)dx}{(U(x)+V(y))^{\lambda+m}} \right]^p dy \right\}^{\frac{1}{p}} \\ < \frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)} \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\lambda_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}}; \quad (27)$$

for $0 < p < 1$, we have the reverse of (27), where the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}B(\lambda_1, \lambda_2)$ in (27) as well as its reverse is the best possible.

Proof. (i) By applying the Hölder's inequality, we have

$$I = \int_{\beta}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)dxdy}{(U(x)+V(y))^{\lambda+m}} \\ = \int_{\beta}^{\infty} \left[\frac{(V(y))^{\frac{1}{p}-\hat{\lambda}_2}}{(V'(y))^{-1/p}} \int_{\alpha}^{\infty} \frac{f(x)dx}{(U(x)+V(y))^{\lambda+m}} \right] \left[\frac{(V(y))^{\frac{1}{p}-\hat{\lambda}_2}g(y)}{(V'(y))^{1/p}} \right] dy \\ \leq J \left[\int_{\beta}^{\infty} \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}. \quad (28)$$

Then by using (26), we have (14).

On the other hand, assume that (14) is valid. We set

$$g(y) := \left[(V(y))^{\hat{\lambda}_2-1} V'(y) \int_{\alpha}^{\infty} \frac{f(x)dx}{(U(x)+V(y))^{\lambda+m}} \right]^{p-1}, \quad y > \beta.$$

If $J = 0$, then (26) is naturally valid; if $J = \infty$, then it is impossible that makes (26) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. We have

$$0 < \int_{\beta}^{\infty} \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy = J^p = I \\ < \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}} J^{p-1}, \\ 0 < \left\{ \int_{\beta}^{\infty} \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}} g^q(y) dy \right\}^{\frac{1}{p}} = J \\ < \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left[\int_{\alpha}^{\infty} \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}} f_m^p(x) dx \right]^{\frac{1}{p}}.$$

This proves (26), which is equivalent to (14).

(ii) In the same way, for $0 < p < 1$, we have the reverse of (26) which is equivalent (20).

For $\lambda_1 + \lambda_2 = \lambda$ in (26), by simplification, we have (27).

For $p > 1$, the constant factor in (27) is the best possible. Otherwise, by (28), we would reach a contradiction that the same constant factor in (15) is not the best possible. In the same way, for $0 < p < 1$, we can show that the same constant factor in the reverse of (27) is the best possible.

The theorem is proved. $\square$

THEOREM 6. *With regards to Assumption AI, (i) for $p > 1$, if the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (26) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$; (ii) for $0 < p < 1$, if the constant factor

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in the reverse of (26) is the best possible, then for $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. (i) Since

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

is the best possible constant factor in (26), by the equivalency of (26) and (14), we can show that the same constant factor in (14) is the best possible. Then by using Theorem 2, we have $\lambda_1 + \lambda_2 = \lambda$.

(ii) Since

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

is the best possible constant factor in the reverse of (26), by the equivalency of the reverse of (26) and the reverse of (14), we can show that the same constant factor in the reverse of (14) is the best possible. Then by using Theorem 4, for $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, we have $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. $\square$

For $U(x) = x$ ($x \in (0, \infty)$) in Theorem 5 and Theorem 6, we have $f_m(x) = f^{(m)}(x)$ and

COROLLARY 3. *(i) For $p > 1$, we have the following Hardy-Hilbert-type integral inequality which is equivalent to (18) involving one derivative functions of m -order:*

$$\left\{ \int_{\beta}^{\infty} \left[(V(y))^{p\hat{\lambda}_2-1} V'(y) \int_0^{\infty} \frac{f(x)dx}{(x+V(y))^{\lambda+m}} \right]^p dy \right\}^{\frac{1}{p}} \\ < \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left[\int_0^{\infty} x^{p(1-\hat{\lambda}_1)-1} (f^{\{m\}}(x))^p dx \right]^{\frac{1}{p}}. \quad (29)$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we reduce (29) to the equivalent form of (19) as follows:

$$\left\{ \int_{\beta}^{\infty} \left[(V(y))^{p\lambda_2-1} V'(y) \int_0^{\infty} \frac{f(x)dx}{(x+V(y))^{\lambda+m}} \right]^p dy \right\}^{\frac{1}{p}} < \frac{\Gamma(\lambda)B(\lambda_1, \lambda_2)}{\Gamma(\lambda+m)} \left[\int_0^{\infty} x^{p(1-\hat{\lambda}_1)-1} (f^{[m]}(x))^p dx \right]^{\frac{1}{p}}, \quad (30)$$

where the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}B(\lambda_1, \lambda_2)$ is the best possible. On the other hand, if the constant factor in (29) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.

(ii) For $0 < p < 1$, we have the reverse of (20) which is equivalent to (29). In particular, for $\lambda_1 + \lambda_2 = \lambda$, we reduce the reverse of (29) which is the equivalent form of (21), where the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}B(\lambda_1, \lambda_2)$ is the best possible. On the other hand, if the constant factor in the reverse of (29) is the best possible, then for $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, we have $\lambda_1 + \lambda_2 = \lambda$.

In the following, we consider the operator expressions.

For $p > 1$, we set functions $\varphi(x) := \frac{(U(x))^{p(1-\hat{\lambda}_1)-1}}{(U'(x))^{p-1}}$, $\psi(y) := \frac{(V(y))^{q(1-\hat{\lambda}_2)-1}}{(V'(y))^{q-1}}$, then, we have

$$\Psi^{1-p}(y) := (V(y))^{p\hat{\lambda}_2-1} V'(y) \quad (x \in (\alpha, \infty), y \in (\beta, \infty)).$$

Define some real normed linear spaces as follows:

$$\begin{aligned} L_{p,\varphi}((\alpha, \infty)) &:= \left\{ f = f(x) : \|f\|_{p,\varphi} = \left(\int_{\alpha}^{\infty} \varphi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\psi}((\beta, \infty)) &:= \left\{ g = g(y) : \|g\|_{q,\psi} = \left(\int_{\beta}^{\infty} \psi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \\ L_{p,\Psi^{1-p}}((\beta, \infty)) &:= \left\{ h = h(y) : \|h\|_{p,\Psi^{1-p}} = \left(\int_{\beta}^{\infty} \Psi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

For $f \in \tilde{L}_{p,\varphi}((\alpha, \infty)) := \{f = f(x); f_0(x) = f(x); f_k(x) = \left(\frac{f_{k-1}(x)}{U'(x)}\right)' f_{k-1}(x) = o(e^{tU(x)}) \ (t > 0; x \rightarrow \infty), f_{k-1}(\alpha^+) = 0 \ (k = 1, \dots, m; m \in \mathbf{N}), \text{ and } f_m(x) \text{ is a non-negative continuous function unless at finite points in } x \in (\alpha, \infty), f_m \in L_{p,\varphi}((\alpha, \infty)) \ (m \in \mathbf{N}_0)\}$, setting

$$h(y) := \int_{\alpha}^{\infty} \frac{f(x)dx}{(U(x) + V(y))^{\lambda+m}} \quad (y \in (\beta, \infty)).$$

By using (26), we have $h \in L_{p,\Psi^{1-p}}((\beta, \infty))$, satisfying

$$\begin{aligned} \|h\|_{p,\Psi^{1-p}} &= \left[\int_{\beta}^{\infty} \Psi^{1-p}(y) |h(y)|^p dy \right]^{\frac{1}{p}} \\ &< \frac{\Gamma(\lambda)B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)}{\Gamma(\lambda+m)} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \|f_m\|_{p,\varphi}. \end{aligned} \quad (31)$$

DEFINITION 1. Define a Hardy-Hilbert-type integral operator $T: \widetilde{L}_{p,\varphi}((\alpha, \infty)) \rightarrow L_{p,\psi^{1-p}}((\beta, \infty))$ as follows: for any $f \in \widetilde{L}_{p,\varphi}((\alpha, \infty))$, there exists a unique representation $h = Tf \in L_{p,\psi^{1-p}}((\beta, \infty))$, satisfying for any $y \in (\beta, \infty)$, $Tf(y) = h(y)$.

Define the formal inner product of Tf and $g \in L_{q,\psi}((\beta, \infty))$, and the norm of T as follows:

$$(Tf, g) := \int_{\beta}^{\infty} \left(\int_{\alpha}^{\infty} \frac{f(x)dx}{(U(x) + V(y))^{\lambda+m}} \right) g(y) dy = I,$$

$$\|T\| = \sup_{f_m(\neq 0) \in L_{p,\varphi}((\alpha, \infty))} \frac{\|Tf\|_{p,\psi^{1-p}}}{\|f_m\|_{p,\varphi}}.$$

In view of (26), it follows that

$$\|Tf\|_{p,\psi^{1-p}} = \|h\|_{p,\psi^{1-p}} \leq \frac{\Gamma(\lambda)B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)}{\Gamma(\lambda + m)} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \|f_m\|_{p,\varphi}$$

and then the operator T is bounded satisfying

$$\|T\| \leq \frac{\Gamma(\lambda)B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)}{\Gamma(\lambda + m)} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1).$$

By using Theorem 5 and Theorem 6, we have

THEOREM 7. If $f(> 0) \in \widetilde{L}_{p,\varphi}((\alpha, \infty))$, $g(> 0) \in L_{q,\psi}((\beta, \infty))$, then we have the following equivalent inequalities:

$$(Tf, g) < \frac{\Gamma(\lambda)B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)}{\Gamma(\lambda + m)} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \|f_m\|_{p,\varphi} \|g\|_{q,\psi}, \quad (32)$$

$$\|Tf\|_{p,\psi^{1-p}} < \frac{\Gamma(\lambda)B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)}{\Gamma(\lambda + m)} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \|f_m\|_{p,\varphi}. \quad (33)$$

Moreover, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor

$$\frac{\Gamma(\lambda)B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)}{\Gamma(\lambda + m)} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (32) and (33) is the best possible, namely, $\|T\| = \frac{\Gamma(\lambda)B^{\frac{1}{p}}(\lambda_1, \lambda_2)}{\Gamma(\lambda + m)}$.

6. Conclusions

In this paper, by means of the weight functions, the idea of introducing parameters and the techniques of real analysis, a new Hardy-Hilbert-type integral inequality with the kernel as $\frac{1}{(U(x)+V(y))^{\lambda+m}}$ ($\lambda > 0$) involving one extended derivative function of m -order and the equivalent statements of the best possible constant factor related to several

parameters are considered in Theorem 1 and Theorem 2. The equivalent forms and the operator expressions are obtained in Theorems 5–7. The cases of the reverses are also provided in Theorem 3 and Theorem 4.

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Data availability statement. We declare that the data and material in this paper can be used publicly.

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