

GENERALIZED UNCERTAINTY PRINCIPLE IN THE CONTEXT OF DIFFERENTIAL FORMS

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(Communicated by T. Burić)

Abstract. This paper extends the uncertainty principle from function spaces to the more general space of differential forms. A brief introduction to differential forms provides a natural foundation for this generalization. By applying a general extension lemma, the Fourier transform is extended to differential forms, establishing the corresponding inversion theorem and Parseval's identity. Utilizing the method of phase derivatives and Cauchy's inequality, a new inequality is derived, leading to a generalized uncertainty principle within the space of differential forms.

1. Introduction

The uncertainty principle is a fundamental concept in quantum mechanics, first proposed by Heisenberg in 1927 as a hypothesis that became a natural law, serving as a critical starting point for quantum theory against classical concepts [7]. This principle states that the canonical coordinates of a microscopic system and their conjugate momenta cannot be simultaneously measured with arbitrary precision; theoretically, there exists a lower bound on the accuracy. This principle is the source of the wave-particle duality observed in microscopic phenomena and serves as the physical foundation for the statistical interpretation of the wave function.

Although Heisenberg possessed profound physical insights, the mathematical rigor of his formulation was not stringent; the precise mathematical expression was later provided by Kennard [10],

$$\Delta x \Delta p \geq \hbar/2 \quad (1.1)$$

where Δx is the standard deviation of position, Δp is the standard deviation of momentum, and $\hbar$ is the reduced Planck constant.

A more general inequality regarding non-commuting operators was established by Robertson [11]. When operators A and B satisfy $[A, B] = -i\hbar C$, it follows that

$$\Delta A \cdot \Delta B \geq \hbar|C_0|/2 \quad (1.2)$$

From a mathematical perspective, the uncertainty principle is expressed as an inequality, stating that a non-zero function and its Fourier transform cannot be simultaneously tightly localized in both the frequency and time domains. The precise quantitative formulation is as follows [6]:

Mathematics subject classification (2020): 42B10, 58A10.

Keywords and phrases: Differential forms, Fourier transformation, uncertainty principle, inequality.

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For $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, we have

$$\int (x-a)^2 |f(x)|^2 dx \int (\xi-b)^2 |\widehat{f}(\xi)|^2 d\xi \geq \frac{\|f\|_2^4}{16\pi^2} \quad (1.3)$$

where $\widehat{f}(\xi)$ is the Fourier transform of f , defined as

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx. \quad (1.4)$$

In recent years, research on the uncertainty principle has progressively deepened, and more generalized inequalities have been proposed and proven [2, 4–6, 8, 12, 14–17]. Folland and Sitaram [13] provide a comprehensive overview of the development of the uncertainty principle up to the 21st century, covering topics such as uncertainty inequalities in higher-dimensional spaces and logarithmic uncertainty inequalities. Phase space decomposition, a relatively recent method, has been adopted in several modern studies [4, 5, 12, 14–17]. The phase derivative approach has been extensively used to generalize the uncertainty principle in various settings [4, 5, 15–17], including studies conducted within the framework of Clifford algebra [5, 15]. The Hardy space decomposition is another frequently employed method, and power spectrum estimation first appeared in [2]. However, to date, no research has been found on establishing the uncertainty principle for differential forms, which forms the basis of our work.

In Section 2, we provide a brief review of the basic concepts and definitions of differential forms. Since differential forms are a natural extension of functions, this facilitates the natural extension of the uncertainty principle. In Section 3, using a general extension lemma, we extend the definition of the Fourier transform from the space $\mathcal{S}$ to the space $L^2(\mathbb{R}^n, \wedge^l)$, and prove the corresponding inversion theorem and Parseval's identity. In Section 4, we establish a generalized uncertainty principle in the space $L^2(\mathbb{R}^n, \wedge^l)$ by deriving a new inequality through the phase derivative method and the use of Cauchy's inequality.

Differential forms, as a crucial tool in differential geometry, offer a clear and concise representation of Riemannian manifolds and antisymmetric covariant tensor fields. General relativity is constructed using the language of differential geometry. Moreover, differential forms, as a natural extension of functions, can establish corresponding uncertainty principles. Given that the uncertainty principle is central to quantum mechanics, differential forms may play a significant role in future efforts to unify general relativity and quantum mechanics into a cohesive theory, thus bringing limitless vitality to the development of mathematics and physics.

2. Introduction to differential forms

The concept of differential forms and their formation of exterior algebra through wedge products and exterior derivatives originated with Cartan [3] and has extensive applications in fields such as partial differential equations, algebraic topology, differential geometry, and mathematical physics. Differential forms can be interpreted from both geometric and algebraic perspectives, but this paper primarily adopts the algebraic viewpoint to establish the corresponding definitions. We begin with an introduction to

exterior algebra.

2.1. Exterior algebra

DEFINITION 2.1. [9] Let $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)$ denote a standard orthogonal basis for the n -dimensional Euclidean space $\mathbb{R}^n$. An l -form ω on $\mathbb{R}^n$ is defined as an l -th order antisymmetric linear function on the n -fold Cartesian product $\mathbb{R}^n \times \dots \times \mathbb{R}^n$, given by

$$\omega(\xi_1, \dots, \xi_l) = \text{sgn}(i_1, \dots, i_l) \omega(\xi_{i_1}, \dots, \xi_{i_l}) \quad (2.1)$$

where $\xi_1, \dots, \xi_l \in \mathbb{R}^n$, $(i_1, \dots, i_l)$ is a permutation of $(1, \dots, l)$, and $\text{sgn}(i_1, \dots, i_l)$ denotes the sign of the permutation, with $l = 1, 2, \dots, n$.

DEFINITION 2.2. [9] The space $\bigwedge^l(\mathbb{R}^n)$ denotes the collection of all l -forms on $\mathbb{R}^n$.

In particular, $\bigwedge^1(\mathbb{R}^n)$ represents the dual space of $\mathbb{R}^n$, with the dual basis of $\{e_1, e_2, \dots, e_n\}$ denoted by $\{e^1, e^2, \dots, e^n\}$, forming the standard basis for $\bigwedge^1(\mathbb{R}^n)$.

DEFINITION 2.3. [9] The space $\bigwedge(\mathbb{R}^n)$ is defined as the direct sum of all $\bigwedge^l(\mathbb{R}^n)$:

$$\bigwedge(\mathbb{R}^n) = \bigoplus_{l=0}^n \bigwedge^l(\mathbb{R}^n) \quad (2.2)$$

which constitutes a graded algebra concerning the exterior (wedge) product, where $\bigwedge^0(\mathbb{R}^n)$ corresponds to the space of functions.

The wedge product satisfies the following operations [9]: for $\alpha, \alpha_1, \alpha_2 \in \bigwedge^l(\mathbb{R}^n)$, $\beta, \beta_1, \beta_2 \in \bigwedge^k(\mathbb{R}^n)$, and $\gamma \in \bigwedge^l(\mathbb{R}^n)$, we have:

(1) Antisymmetry:

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

(2) Associativity:

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$$

(3) Distributivity:

$$(\alpha_1 + \alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \alpha_2 \wedge \beta$$

$$\alpha \wedge (\beta_1 + \beta_2) = \alpha \wedge \beta_1 + \alpha \wedge \beta_2.$$

It is noteworthy that the antisymmetry property implies that if $\alpha \in \bigwedge^1(\mathbb{R}^n)$, then $\alpha \wedge \alpha = 0$.

In summary, the elements $1, e^1, e^2, \dots, e^n$ form a generating set for $\bigwedge(\mathbb{R}^n)$. Therefore, the set $\{1, e^1, e^2, \dots, e^n, e^1 \wedge e^2, \dots, e^{n-1} \wedge e^n, \dots, e^1 \wedge e^2 \wedge \dots \wedge e^n\}$ constitutes an ordered standard basis for $\bigwedge(\mathbb{R}^n)$. The basis of $\bigwedge^l(\mathbb{R}^n)$ consists of elements of the form $e^\tau = e^{\tau_1} \wedge e^{\tau_2} \wedge \dots \wedge e^{\tau_l}$, where the ordered l -tuple $\tau = (\tau_1, \tau_2, \dots, \tau_l)$ satisfies $1 \leq \tau_1 < \dots < \tau_l \leq n$. Thus, we have $\dim \bigwedge^l(\mathbb{R}^n) = \binom{n}{l}$, and from equation 2.2, it follows that $\dim \bigwedge(\mathbb{R}^n) = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$.

For any given 1-forms $a^1, a^2, \dots, a^k$, it holds that [9]

$$a^1 \wedge \dots \wedge a^k = \sum_{1 \leq \tau_1 < \dots < \tau_k \leq n} \det \begin{pmatrix} a_{\tau_1}^1 & \dots & a_{\tau_k}^1 \\ \vdots & \ddots & \vdots \\ a_{\tau_1}^k & \dots & a_{\tau_k}^k \end{pmatrix} e^{\tau_1} \wedge \dots \wedge e^{\tau_k}. \quad (2.3)$$

2.2. Differential forms

Let $\Omega \subset \mathbb{R}^n$ be an open set with a smooth boundary. An l -form on Ω is a locally integrable function from Ω to $\wedge^l(\mathbb{R}^n)$. In particular, as previously stated, 0-forms are generalized functions. Thus, the coordinate functions $x_1, x_2, \dots, x_n$ defined on Ω can be viewed as 0-forms.

DEFINITION 2.4. [9] If $\omega(x) \in \wedge^l(\mathbb{R}^n)$ is defined at $x \in \Omega$, then the value of $\omega(x)$ at the vectors $\xi_1, \dots, \xi_l \in \mathbb{R}^n$ is defined as

$$\omega(x)(\xi_1, \dots, \xi_l) = \omega(x; \xi_1, \dots, \xi_l). \quad (2.4)$$

Let $\mathcal{D}'(\Omega, \wedge^k)$ denote the set of all smooth k -forms defined on Ω . $\mathcal{D}'(\Omega)$ is the space of all smooth functions defined on Ω . A vector-valued differential form can be defined similarly to the gradient of a function.

DEFINITION 2.5. [9] For $\omega \in \mathcal{D}'(\Omega, \wedge^l)$,

$$\nabla \omega = \left(\frac{\partial \omega}{\partial x_1}, \dots, \frac{\partial \omega}{\partial x_n} \right). \quad (2.5)$$

Similarly, the directional gradient of a differential form can be defined as follows [1]:

$$\nabla_{\xi_{l+1}} \omega(x; \xi_1, \dots, \xi_l) = \nabla \omega(x; \xi_1, \dots, \xi_l) \cdot \xi_{l+1} \quad (2.6)$$

where $\cdot$ denotes the usual inner product.

DEFINITION 2.6. [1] If $\omega(x; \xi_1, \dots, \xi_l)$ is differentiable at $x \in \Omega$, the exterior derivative $d\omega(x; \xi_1, \dots, \xi_l)$ is an $(l+1)$ -form given by

$$d\omega(x; \xi_1, \dots, \xi_l, \xi_{l+1}) = \sum_{i=1}^{l+1} (-1)^{i+1} \nabla_{\xi_i} \omega(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{l+1}). \quad (2.7)$$

The 1-forms $dx_1, dx_2, \dots, dx_n$ are constant functions from Ω to $\wedge^1(\mathbb{R}^n)$, with the constant value of dx_i simply being the constant value of e^i , for $i = 1, 2, \dots, n$. Therefore, each l -form $\omega : \Omega \rightarrow \wedge^l(\mathbb{R}^n)$ can be uniquely represented as

$$\omega(x) = \sum_{\tau} \omega_{\tau}(x) dx_{\tau} = \sum_{1 \leq \tau_1 < \dots < \tau_l \leq n} \omega_{\tau_1 \tau_2 \dots \tau_l}(x) dx_{\tau_1} \wedge dx_{\tau_2} \wedge \dots \wedge dx_{\tau_l} \quad (2.8)$$

where each $\omega_{\tau}(x)$ is a generalized function in $\mathcal{D}'(\Omega)$.

Let $p \in \Omega$, and define the tangent space at that point as $T_p \mathbb{R}^n = \{q - p \mid q \in \mathbb{R}^n\}$. Let $\{(e_1)_p, \dots, (e_n)_p\}$ be the standard orthogonal basis of $T_p \mathbb{R}^n$. Then $\{(dx_1)(p), \dots, (dx_n)(p)\}$ is the dual basis of $\{(e_1)_p, \dots, (e_n)_p\}$, yielding

$$(dx_i)(p)(e_j)_p = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Thus, $dx_1, dx_2, \dots, dx_n$ are 1-forms from $\Omega \rightarrow \wedge^1(\mathbb{R}^n)$. In fact, $\wedge^1(\mathbb{R}^n) = (\mathbb{R}^n)^*$.

Similar to function theory, an integrable function space can also be defined on differential forms. Therefore, we define differential forms with L^p integrable coefficients as follows:

$$L^p(\mathbb{R}^n, \wedge^l) = \left\{ \omega = \sum_{\tau} \omega_{\tau} dx_{\tau} \mid \omega_{\tau} \in L^p(\mathbb{R}^n) \right\}. \quad (2.9)$$

We also define the norms

$$\|\omega\|_{p, \Omega} = \left(\int_{\Omega} |\omega(x)|^p dx \right)^{1/p}, \quad \|\omega\|_{\infty, \Omega} = \text{ess sup}_{x \in \Omega} |\omega(x)| \quad (2.10)$$

where $|\omega(x)| = (\sum_{\tau} (\omega_{\tau}(x))^2)^{1/2}$.

Correspondingly, we define the inner product on $L^2(\mathbb{R}^n, \wedge^l)$ as

$$\langle \omega, \eta \rangle = \int_{\mathbb{R}^n} \sum_{\tau} \omega_{\tau}(x) \eta_{\tau}(x) dx,$$

for $\omega = \sum_{\tau} \omega_{\tau} dx_{\tau}$ and $\eta = \sum_{\tau} \eta_{\tau} dx_{\tau}$ in $L^2(\mathbb{R}^n, \wedge^l)$. This definition is consistent with the norm $\|\omega\|_{2, \mathbb{R}^n} = \langle \omega, \omega \rangle^{1/2}$.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all infinitely differentiable functions ω_{τ} that satisfy the following inequality:

$$\sup_{x \in \mathbb{R}^n} \left| x^{\alpha} \left(\frac{\partial}{\partial x} \right)^{\beta} \omega_{\tau}(x) \right| < \infty,$$

for all multi-indices α and β , meaning ω_{τ} and all its derivatives are rapidly decreasing functions. Hence, we obtain the following definition:

$$\mathcal{S}(\mathbb{R}^n, \wedge^l) = \left\{ \omega = \sum_{\tau} \omega_{\tau} dx_{\tau} \mid \omega_{\tau} \in \mathcal{S}(\mathbb{R}^n) \right\}. \quad (2.11)$$

3. Fourier transform on differential forms

The Fourier transform of a function ω_{τ} on $\mathbb{R}^n$ is defined as

$$\widehat{\omega}_{\tau}(\xi) = \int_{\mathbb{R}^n} \omega_{\tau}(x) e^{-2\pi i x \xi^T} dx. \quad (3.1)$$

The corresponding inverse transform is given by

$$\omega_{\tau}(x) = \int_{\mathbb{R}^n} \widehat{\omega}_{\tau}(\xi) e^{2\pi i x \xi^T} d\xi. \quad (3.2)$$

We know that if $\omega_\tau \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform is a bijection on this space, i.e., the inversion formula 3.2 holds. Moreover, we have the Plancherel identity

$$\int_{\mathbb{R}^n} |\widehat{\omega}_\tau(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\omega_\tau(x)|^2 dx. \quad (3.3)$$

Now, we move to the more general case of differential forms. We aim to reconstruct the symmetry between ω_τ and $\widehat{\omega}_\tau$ in a more general setting, which holds for the $\mathcal{S}$ class.

This will be done in two steps. The first step is to extend the space $\mathcal{S}$ to $\mathcal{S}(\mathbb{R}^n, \wedge^l)$, which naturally leads to the following definition:

DEFINITION 3.1. For $\omega \in \mathcal{S}(\mathbb{R}^n, \wedge^l)$, its Fourier transform $\widehat{\omega}(\xi)$ is defined as

$$\widehat{\omega}(\xi) = \sum_{\tau} \int_{\mathbb{R}^n} \omega_\tau(x) e^{-2\pi i \xi x^T} dx d\tau \quad (3.4)$$

This means that the Fourier transform is applied to each component of the differential form separately, and then the differential form is reassembled. This definition automatically preserves the Fourier inversion, i.e.,

$$\omega(x) = \sum_{\tau} \int_{\mathbb{R}^n} \widehat{\omega}_\tau(\xi) e^{2\pi i \xi x^T} d\xi d\tau \quad (3.5)$$

as well as the following Plancherel identity:

$$\begin{aligned} \|\widehat{\omega}(\xi)\|_{L^2} &= \left(\int_{\mathbb{R}^n} |\widehat{\omega}(\xi)|^2 d\xi \right)^{1/2} = \left(\int_{\mathbb{R}^n} \sum_{\tau} |\widehat{\omega}_\tau(\xi)|^2 d\xi \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^n} \sum_{\tau} |\omega_\tau(x)|^2 dx \right)^{1/2} = \left(\int_{\mathbb{R}^n} |\omega(x)|^2 dx \right)^{1/2} = \|\omega(x)\|_{L^2}. \end{aligned} \quad (3.6)$$

The second step is to generalize this to L^2 space because the most elegant and useful formulation of Fourier inversion is through L^2 theory, and the most widely applied context in quantum mechanics is Hilbert space.

The Fourier transform on $L^2(\mathbb{R}^n, \wedge^l)$ is defined as its extension from $\mathcal{S}(\mathbb{R}^n, \wedge^l)$. To do this, we temporarily denote by F_0 and F the Fourier transform on $\mathcal{S}(\mathbb{R}^n, \wedge^l)$ and its extension to $L^2(\mathbb{R}^n, \wedge^l)$, respectively.

The main result of the proof is given in the following theorem:

THEOREM 3.2. The Fourier transform F_0 , initially defined on $\mathcal{S}(\mathbb{R}^n, \wedge^l)$, can be uniquely extended as a unitary map F from $L^2(\mathbb{R}^n, \wedge^l)$ to itself, specifically:

$$\|F(\omega)\|_{L^2} = \|\omega\|_{L^2} \quad (3.7)$$

for all $\omega \in L^2(\mathbb{R}^n, \wedge^l)$.

Proof. The following limiting process gives the extensions of F : If $\{\omega_d\}$ is a sequence of differential forms in $\mathcal{S}(\mathbb{R}^n, \wedge^l)$ converging to $\omega \in L^2(\mathbb{R}^n, \wedge^l)$, then $F_0(\omega_d)$ converges to an element in $L^2(\mathbb{R}^n, \wedge^l)$, and we define this element to be the Fourier

transform of ω . To apply this method, we need to show that every differential form in $L^2(\mathbb{R}^n, \wedge^l)$ can be approximated by elements in $\mathcal{S}(\mathbb{R}^n, \wedge^l)$. $\square$

LEMMA 3.3. *The space $\mathcal{S}(\mathbb{R}^n, \wedge^l)$ is dense in $L^2(\mathbb{R}^n, \wedge^l)$, i.e., for any $\omega \in L^2(\mathbb{R}^n, \wedge^l)$, there exists a sequence of differential forms $\omega_d \in \mathcal{S}(\mathbb{R}^n, \wedge^l)$ such that $\|\omega - \omega_d\|_{L^2} \rightarrow 0$ as $d \rightarrow \infty$.*

Proof. To prove this lemma, we use a known result: the space $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ [13]. This ensures that for any given $\omega_\tau \in L^2(\mathbb{R}^n)$ and any $\varepsilon > 0$, there exists a sequence of functions $\{\omega_{\tau_d}\} \subset \mathcal{S}(\mathbb{R}^n)$ and d_τ such that, when $d > d_\tau$, we have

$$\begin{aligned} \|\omega_\tau - \omega_{\tau_d}\|_{L^2} &= \left(\int_{\mathbb{R}^n} |\omega_\tau - \omega_{\tau_d}|^2 dx \right)^{1/2} \\ &\leq \frac{\varepsilon}{\sqrt{C_n^l}}. \end{aligned} \quad (3.8)$$

Thus, for $\sum_\tau \omega_\tau dx_\tau = \omega \in L^2(\mathbb{R}^n, \wedge^l)$, for each ω_τ , we obtain the corresponding sequence $\{\omega_{\tau_d}\}$ and construct a sequence of differential forms $\{\omega_d\} = \sum_\tau \omega_{\tau_d} dx_\tau$. By construction, we know that $\{\omega_{\tau_d}\} \subset \mathcal{S}(\mathbb{R}^n, \wedge^l)$. Taking $D = \max_\tau d_\tau$, for $d > D$, we have

$$\|\omega - \omega_d\|_{L^2} = \left(\int_{\mathbb{R}^n} \sum_\tau |\omega_\tau - \omega_{\tau_d}|^2 dx \right)^{1/2} \leq \left(\sum_\tau \left(\frac{\varepsilon}{\sqrt{C_n^l}} \right)^2 \right)^{1/2} = \left(C_n^l \frac{\varepsilon^2}{C_n^l} \right)^{1/2} = \varepsilon.$$

Consequently, the proof of the lemma is established. $\square$

The definition of the Fourier transform on $L^2(\mathbb{R}^n, \wedge^l)$ combines the denseness of $\mathcal{S}(\mathbb{R}^n, \wedge^l)$ and a general “extension principle”.

LEMMA 3.4. [13] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Suppose S is a dense subspace of $\mathcal{H}_1$, and $T_0 : S \rightarrow \mathcal{H}_2$ is a linear operator such that $\|T_0(f)\|_2 \leq c\|f\|_1$ for all $f \in S$, where c is a constant. Then, T_0 can be uniquely extended to a linear operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, satisfying $\|T(f)\|_2 \leq c\|f\|_1$ for all $f \in \mathcal{H}_1$.*

In the current case of the Fourier transform, we apply the lemma to $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R}^n, \wedge^l)$ (endowed with the L^2 -norm), $\mathcal{S} = \mathcal{S}(\mathbb{R}^n, \wedge^l)$, and $T_0 = F_0$, where F_0 is the Fourier transform defined on $\mathcal{S}(\mathbb{R}^n, \wedge^l)$. By definition, the Fourier transform on $L^2(\mathbb{R}^n, \wedge^l)$ is the unique (bounded) extension of F_0 to $L^2(\mathbb{R}^n, \wedge^l)$, guaranteed by Lemma 3.4. Thus, if $\omega \in L^2(\mathbb{R}^n, \wedge^l)$ and $\{\omega_d\}$ is any sequence in $\mathcal{S}(\mathbb{R}^n, \wedge^l)$ converging to ω (i.e., $\|\omega - \omega_d\|_{L^2} \rightarrow 0$ as $d \rightarrow \infty$), then the Fourier transform of ω is defined as

$$F(\omega) = \lim_{d \rightarrow \infty} F_0(\omega_d),$$

where the limit is taken in the L^2 -sense. Clearly, the argument in the proof of the lemma shows that in this extended case, F still satisfies the identity:

$$\|F(\omega)\|_{L^2} = \|\omega\|_{L^2}, \quad \text{for all } \omega \in L^2(\mathbb{R}^n, \wedge^l).$$

Since F is invertible on L^2 (hence F is a unitary mapping), this fact also follows from similar properties of F_0 on $\mathcal{S}(\mathbb{R}^n, \wedge^l)$. Recall that on $\mathcal{S}(\mathbb{R}^n, \wedge^l)$, F_0^{-1} is given by the formula (Equation 3.5):

$$F_0^{-1}(g)(x) = \sum_{\tau} \int_{\mathbb{R}^n} g_{\tau}(\xi) e^{2\pi i \xi \mathbf{x}^T} d\xi dx_{\tau},$$

and satisfies the identity $\|F_0^{-1}(g)\|_{L^2} = \|g\|_{L^2}$. Therefore, similar to the earlier discussion, we can extend F_0^{-1} to $L^2(\mathbb{R}^n, \wedge^l)$ via a limiting process. Given $\omega \in L^2(\mathbb{R}^n, \wedge^l)$, we choose a sequence $\{\omega_d\}$ in $\mathcal{S}(\mathbb{R}^n, \wedge^l)$ such that $\|\omega - \omega_d\|_{L^2} \rightarrow 0$, leading to

$$\omega_d = F_0^{-1}(F_0(\omega_d)) = F_0(F_0^{-1}(\omega_d)),$$

and taking the limit as $d \rightarrow \infty$, we get

$$\omega = F^{-1}F(\omega) = FF^{-1}(\omega),$$

proving that F is invertible. This concludes the proof of Theorem 3.2.

Theorem 3.2 provides a rather abstract definition of the Fourier transform on $L^2(\mathbb{R}^n, \wedge^l)$. In light of this, we can also define the Fourier transform more concretely as follows. If $\omega \in L^2(\mathbb{R}^n, \wedge^l)$, then

$$\widehat{\omega}(\xi) = \lim_{R \rightarrow \infty} \sum_{\tau} \int_{|x| \leq R} \omega_{\tau}(x) e^{-2\pi i \xi \mathbf{x}^T} dx dx_{\tau},$$

where the limit is taken in the L^2 -norm. Notice that if χ_R denotes the characteristic function of the ball $\{x \in \mathbb{R}^d : |x| \leq R\}$, then for each R , the function $\omega_{\tau}\chi_R$ belongs to both L^1 and L^2 , and $\omega_{\tau}\chi_R \rightarrow \omega_{\tau}$ in the L^2 -norm.

In fact, it can be shown that

$$F(\omega)(\xi) = \widehat{\omega}(\xi) = \sum_{\tau} \int_{\mathbb{R}^n} \omega_{\tau}(x) e^{-2\pi i \xi \mathbf{x}^T} dx dx_{\tau} \quad (3.9)$$

by applying Lemma 3.4. It remains only to prove that this is a bounded linear operator. Linearity is obvious, so we proceed to prove the boundedness.

Proof.

$$\begin{aligned} \|F(\omega)(\xi)\|_{\infty} &= \text{ess sup} \left| \sum_{\tau} \int_{\mathbb{R}^n} \omega_{\tau}(x) e^{-2\pi i \xi \mathbf{x}^T} dx dx_{\tau} \right| \\ &\leq \text{ess sup} \left(\sum_{\tau} \left(\int_{\mathbb{R}^n} (\omega_{\tau}(x) e^{-2\pi i \xi \mathbf{x}^T})^2 dx \right)^{\frac{1}{2}} \right) \\ &\leq \text{ess sup} \left(\sum_{\tau} \left(\int_{\mathbb{R}^n} |\omega_{\tau}(x)| |e^{-2\pi i \xi \mathbf{x}^T}|^2 dx \right)^{\frac{1}{2}} \right) \\ &= \left(\sum_{\tau} \left(\int_{\mathbb{R}^n} |\omega_{\tau}(x)|^2 dx \right)^{\frac{1}{2}} \right) \end{aligned}$$

$$\begin{aligned}
&\leqslant (C_n^l)^{\frac{1}{2}} \left(\sum_{\tau} \left(\int_{\mathbb{R}^n} |\omega_{\tau}(x)| dx \right)^2 \right)^{\frac{1}{2}} \\
&\leqslant (C_n^l)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \sum_{\tau} |\omega_{\tau}(x)| dx \right) \\
&= (C_n^l)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left(\sum_{\tau} |\omega_{\tau}(x)| \right)^2 dx \right)^{\frac{1}{2}} \\
&\leqslant (C_n^l)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (C_n^l)^{\frac{1}{2}} \left(\sum_{\tau} |\omega_{\tau}(x)|^2 \right)^{\frac{1}{2}} dx \right) \\
&= (C_n^l)^{\frac{1}{2}} \int_{\mathbb{R}^n} \left(\sum_{\tau} |\omega_{\tau}(x)|^2 \right)^{\frac{1}{2}} dx = C_n^l \|\omega\|_1. \quad \square
\end{aligned}$$

The third-to-last inequality derives from the power mean inequality. For non-negative real numbers $x_1, \dots, x_n$, we have:

$$\frac{1}{n} \cdot \sum_{i=1}^n x_i \leqslant \left(\frac{1}{n} \cdot \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$$

Equality holds if and only if $x_1 = x_2 = \dots = x_n$.

In this context, let n be C_n^l and let x_i be $|\omega_{\tau}(y)|$. This substitution yields the desired inequality.

4. Generalized uncertainty principle

4.1. Main lemma and related inequalities

Subsequent symbols and function properties are as follows. For

$$\omega_{\tau}(\mathbf{x}) = \lambda_{\tau}(\mathbf{x}) e^{2\pi i \varphi_{\tau}(\mathbf{x})} \in L^2(\mathbb{R}^n)$$

and $\mathbf{a} = (a_1, a_2, \dots, a_n)$, $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$, assume that for $1 \leqslant k \leqslant n$, the partial derivatives $\frac{\partial \lambda_{\tau}}{\partial x_k}$, $\frac{\partial \varphi_{\tau}}{\partial x_k}$, $\frac{\partial \omega_{\tau}}{\partial x_k}$ exist at any point $\mathbf{x} \in \mathbb{R}^n$, and that $\mathbf{x} \omega(\mathbf{x}), \xi \widehat{\omega}(\xi) \in L^2(\mathbb{R}^n)$. Let $\nabla_{\mathbf{x}} \varphi_{\tau} = \left(\frac{\partial \varphi_{\tau}}{\partial x_1}, \frac{\partial \varphi_{\tau}}{\partial x_2}, \dots, \frac{\partial \varphi_{\tau}}{\partial x_n} \right)$, and unless otherwise specified, the integrals are taken over $\mathbb{R}^n$.

LEMMA 4.1.

$$\begin{aligned}
&\int \sum_{\tau} (\xi_k - b)^2 |\widehat{\omega}_{\tau}(\xi)|^2 d\xi \\
&= \frac{1}{4\pi^2} \int \sum_{\tau} \left(\frac{\partial \lambda_{\tau}}{\partial x_k} \right)^2 d\mathbf{x} + \int \sum_{\tau} \left(\frac{\partial \varphi_{\tau}}{\partial x_k} - b \right)^2 \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x}.
\end{aligned} \tag{4.1}$$

Proof. Since $\frac{1}{2\pi i} \frac{\partial \omega_\tau}{\partial x_k}$ and $\xi_k \widehat{\omega}_\tau(\xi)$ form a Fourier transform pair, by the Parseval's theorem, we get

$$\begin{aligned} & \int \sum_{\tau} (\xi_k - b)^2 |\widehat{\omega}_\tau(\xi)|^2 d\xi \\ &= \frac{1}{4\pi^2} \int \sum_{\tau} \left[\left(\frac{\partial \lambda_\tau}{\partial x_k} \right)^2 + 4\pi^2 \left(\frac{\partial \varphi_\tau}{\partial x_k} \right)^2 \lambda_\tau^2(\mathbf{x}) \right] d\mathbf{x} \\ & \quad + b^2 \int \sum_{\tau} \lambda_\tau^2(\mathbf{x}) d\mathbf{x} - 2b \int \sum_{\tau} \frac{\partial \varphi_\tau}{\partial x_k} \lambda_\tau^2(\mathbf{x}) d\mathbf{x} \\ & \quad + \frac{bi}{\pi} \int \sum_{\tau} \frac{\partial \lambda_\tau}{\partial x_k} \lambda_\tau(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Noting that $\int \xi_k |\widehat{\omega}_\tau(\xi)|^2 d\xi$ is real, we get $\int \sum_{\tau} \frac{\partial \lambda_\tau}{\partial x_k} \lambda_\tau(\mathbf{x}) d\mathbf{x} = 0$, and the lemma is proven after simplification. $\square$

LEMMA 4.2. For $a_i(x), b_i(x) \in L^2(\mathbb{R}^n)$ with corresponding indices, we have

$$\left(\int \sum a_i(x) b_i(x) dx \right)^2 \leq \left(\int \sum a_i^2(x) dx \right) \left(\int \sum b_i^2(x) dx \right). \quad (4.2)$$

Proof. For $a_I(x) = (a_1(x), \dots, a_i(x), \dots, a_\tau(x))$ and $b_I(x) = (b_1(x), \dots, b_i(x), \dots, b_\tau(x))$, we define the inner product as follows:

DEFINITION 4.3.

$$\langle a_I(x), b_I(x) \rangle = \int \sum a_i(x) b_i(x) dx. \quad (4.3)$$

It is easy to verify that this definition satisfies the basic properties of inner products. According to $|\langle u, v \rangle| \leq \|u\| \|v\|$, substituting $a_I(x)$ and $b_I(x)$ for u and v , the lemma is proven. $\square$

4.2. Main Theorem

For $\omega(x) \in L^2(\mathbb{R}^n, \Lambda^l)$, we aim to prove the following:

THEOREM 4.4.

$$\int |\mathbf{x} - \mathbf{a}|^2 |\omega(\mathbf{x})|^2 d\mathbf{x} \int |\xi - \mathbf{b}|^2 |\widehat{\omega}(\xi)|^2 d\xi \geq \frac{n^2}{16\pi^2} \|\omega\|_2^4. \quad (4.4)$$

Proof. From the properties of differential forms and the definition of their corresponding Fourier transforms, we have

$$|\omega(\mathbf{x})|^2 = \sum_{\tau} |\omega_\tau(\mathbf{x})|^2, \quad |\widehat{\omega}(\xi)|^2 = \sum_{\tau} |\widehat{\omega}_\tau(\xi)|^2. \quad (4.5)$$

Thus, it suffices to prove

$$\int \sum_{\tau} |\mathbf{x} - \mathbf{a}|^2 |\omega_{\tau}(\mathbf{x})|^2 d\mathbf{x} \int \sum_{\tau} |\xi - \mathbf{b}|^2 |\widehat{\omega}_{\tau}(\xi)|^2 d\xi \geq \frac{n^2}{16\pi^2} \|\omega\|_2^4. \quad \square \quad (4.6)$$

We will prove a stronger result to derive this inequality. Specifically, we will prove the following:

THEOREM 4.5.

$$\begin{aligned} & \int \sum_{\tau} |\mathbf{x} - \mathbf{a}|^2 |\omega_{\tau}(\mathbf{x})|^2 d\mathbf{x} \int \sum_{\tau} |\xi - \mathbf{b}|^2 |\widehat{\omega}_{\tau}(\xi)|^2 d\xi \\ & \geq \frac{n^2}{16\pi^2} \|\omega\|_2^4 + \left[\int \sum_{\tau} |\mathbf{x} - \mathbf{a}| |\nabla_{\mathbf{x}} \varphi_{\tau} - \mathbf{b}|^T \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \right]^2. \end{aligned} \quad (4.7)$$

Proof. From Lemma 4.1, we have

$$\begin{aligned} & \int \sum_{\tau} (x_k - a)^2 |\omega_{\tau}(\mathbf{x})|^2 d\mathbf{x} \int \sum_{\tau} (\xi_k - b)^2 |\widehat{\omega}_{\tau}(\xi)|^2 d\xi \\ & = \frac{1}{4\pi^2} I_1 + I_2, \end{aligned}$$

where

- $I_1 = \int \sum_{\tau} (x_k - a)^2 \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \int \sum_{\tau} \left(\frac{\partial \lambda_{\tau}}{\partial x_k} \right)^2 d\mathbf{x},$
- $I_2 = \int \sum_{\tau} (x_k - a)^2 \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \int \sum_{\tau} \left(\frac{\partial \varphi_{\tau}}{\partial x_k} - b \right)^2 \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x}.$

From Lemma 4.2, we have

$$\begin{aligned} I_1 & \geq \left[\int \sum_{\tau} (x_k - a) \frac{\partial \lambda_{\tau}}{\partial x_k} \lambda_{\tau}(\mathbf{x}) d\mathbf{x} \right]^2 \\ I_2 & \geq \left[\int \sum_{\tau} \left| (x_k - a) \left(\frac{\partial \varphi_{\tau}}{\partial x_k} - b \right) \right| \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \right]^2. \end{aligned}$$

Since $\int \sum_{\tau} \frac{\partial \lambda_{\tau}}{\partial x_k} \lambda_{\tau}(\mathbf{x}) d\mathbf{x} = 0$, as shown in the proof of Lemma 4.1, we get

$$I_1 \geq \left[\int \sum_{\tau} x_k \frac{\partial \lambda_{\tau}}{\partial x_k} \lambda_{\tau}(\mathbf{x}) d\mathbf{x} \right]^2$$

Due to the rapid decay of $x_k \lambda_{\tau}^2(\mathbf{x})$, we have

$$I_1 \geq \left[\frac{1}{2} \int \sum_{\tau} \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \right]^2 = \frac{\|\omega\|_2^4}{4}.$$

Thus,

$$\begin{aligned} & \int \sum_{\tau} (x_k - a)^2 |\omega_{\tau}(\mathbf{x})|^2 d\mathbf{x} \int \sum_{\tau} (\xi_k - b)^2 |\widehat{\omega}(\xi)|^2 d\xi \\ & \geq \frac{\|\omega\|_2^4}{16\pi^2} + \left[\int \sum_{\tau} \left| (x_k - a) \left(\frac{\partial \varphi_{\tau}}{\partial x_k} - b \right) \right| \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \right]^2. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \int \sum_{\tau} |\mathbf{x} - \mathbf{a}|^2 |\omega_{\tau}(\mathbf{x})|^2 d\mathbf{x} \int \sum_{\tau} |\xi - \mathbf{b}|^2 |\widehat{\omega}_{\tau}(\xi)|^2 d\xi \\ & = \sum_{k=1}^n \int \sum_{\tau} (x_k - a_k)^2 |\omega_{\tau}(\mathbf{x})|^2 d\mathbf{x} \times \sum_{k=1}^n \int \sum_{\tau} (\xi_k - b_k)^2 |\widehat{\omega}_{\tau}(\xi)|^2 d\xi \\ & \geq \left[\sum_{k=1}^n \left(\int \sum_{\tau} (x_k - a_k)^2 |\omega_{\tau}(\mathbf{x})|^2 d\mathbf{x} \times \int \sum_{\tau} (\xi_k - b_k)^2 |\widehat{\omega}_{\tau}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \right]^2 \\ & \geq \left[\sum_{k=1}^n \left(\frac{\|\omega\|_2^4}{16\pi^2} + \left(\int \sum_{\tau} \left| (x_k - a_k) \left(\frac{\partial \varphi_{\tau}}{\partial x_k} - b_k \right) \right| \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \right)^2 \right)^{\frac{1}{2}} \right]^2 \\ & \geq \frac{\|\omega\|_2^4}{16\pi^2} \left[n^2 + \left(\sum_{k=1}^n \frac{4\pi}{\|\omega\|_2^2} \int \sum_{\tau} \left| (x_k - a_k) \left(\frac{\partial \varphi_{\tau}}{\partial x_k} - b_k \right) \right| \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \right)^2 \right] \\ & = \frac{n^2}{16\pi^2} \|\omega\|_2^4 + \left[\int \sum_{\tau} |\mathbf{x} - \mathbf{a}| |\nabla_{\mathbf{x}} \varphi_{\tau} - \mathbf{b}|^T \lambda_{\tau}^2(\mathbf{x}) d\mathbf{x} \right]^2. \end{aligned}$$

This completes the proof. $\square$

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(Received October 31, 2024)

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