

WEIGHTED HÖLDER WITH NUMERICAL RADIUS INEQUALITIES

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(Communicated by M. Krnić)

Abstract. The main aim of the current work is a discussion of the matrix Hölder inequality for unitarily invariant norms, with some applications to norm and numerical radius bounds.

1. Introduction

The algebra of all $n \times n$ complex matrices, denoted as $\mathbb{M}_n$, has attracted numerous researchers' attention due to its applications in different fields.

Finding possible relations among distinct matrix quantities has been one of the most active research areas in this field, where tighter relations are sought.

As a basic idea in such relations, extending known scalar relations to possible matrix versions comes first. For example, it is a simple observation that if a, b are positive real numbers, then $2ab \leq a^2 + b^2$. This is known as the arithmetic-geometric mean inequality. Though simple, this inequality, with its weighted form, has been a keen tool in mathematical inequalities.

The matrix version of the arithmetic-geometric mean inequality states that [23]

$$2\|AXB^*\| \leq \|A^*AX + XB^*B\|, \quad (1.1)$$

where $A, B, X \in \mathbb{M}_n$ and $\|\cdot\|$ is an arbitrary unitarily invariant norm on $\mathbb{M}_n$.

Here, we recall some notations and terminologies. First, upper case letters, through this paper, will denote elements of $\mathbb{M}_n$. When $A \in \mathbb{M}_n$, its conjugate transpose is denoted by A^* . On the other hand, a matrix norm (that is, a sub-multiplicative norm on $\mathbb{M}_n$) is termed as unitarily invariant if, for all $A \in \mathbb{M}_n$ and all unitary elements $U, V \in \mathbb{M}_n$, one has $\|UAV\| = \|A\|$.

Among the most useful examples of unitarily invariant norms is the usual operator norm $\|\cdot\|$, defined as $\|A\| = \sup_{\|x\|=1} \|Ax\|$, where the supremum is taken over all such unit vectors in $\mathbb{C}^n$.

When $X = I$, the identity matrix, (1.1) was shown earlier in [24] in the form

$$2\|AB^*\| \leq \|A^*A + B^*B\|. \quad (1.2)$$

Mathematics subject classification (2020): Primary 47A63; Secondary 15A42, 15A60, 15A18.

Keywords and phrases: Hölder inequality, operator norm, numerical radius, Aluthge transform.

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This was an extension of the corresponding singular value inequality, proved in the same reference.

Another celebrated matrix inequality is the matrix Schwarz inequality, which asserts that if $A, B \in \mathbb{M}_n$, and $r > 0$, then [29]

$$\| \| \| AB^* |^r \| \| ^2 \leq \| \| \| (A^* A)^r \| \| \| \| \| (B^* B)^r \| \| \| . \quad (1.3)$$

This was extended in [30] to the form

$$\| \| \| AB^* |^r \| \| \leq \| \| \| (A^* A)^{\frac{rp}{2}} \| \| \|^{1/p} \| \| \| (B^* B)^{\frac{rq}{2}} \| \| \|^{1/q},$$

where, in addition to the earlier assumptions, p, q are conjugate exponents. That is, $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$.

Recently, a unified form that interpolates between (1.2) and (1.3) was shown in [22] in the form

$$\| \| \| AB^* |^r \| \| \leq \| \| \| (Z(\alpha))^{\frac{rp}{2}} \| \| \|^{1/p} \| \| \| (Z(1-\alpha))^{\frac{rq}{2}} \| \| \|^{1/q}, \quad (1.4)$$

where

$$Z(\alpha) = \alpha A^* A + (1-\alpha) B^* B.$$

In our current work, we employ (1.4) to obtain more extended forms that generalize the results mentioned above. For example, our first main result will be focused on finding an interpolated upper bound of Schwarz type for $\left\| \left\| \sum_{i=1}^N A_i X_i^{1/2} Y_i^{1/2} B_i^* \right\|^r \right\|$, where X is a positive (semi-definite) element of $\mathbb{M}_n$, and $A_i, B_i \in \mathbb{M}_n$ are arbitrary, as the reader can see in Theorem 3.1 below.

Once we show this, many special and simpler cases will be treated. This treatment will lead to several known results as special cases.

After establishing these Hölder-type inequalities, we present some applications that involve new bounds for the usual operator norm of certain matrix quantities. The presented forms will be shown to be sharper than known bounds. In this line, for example, we show that for $A, B \in \mathbb{M}_n$,

$$\begin{aligned} \|A + B\| &\leq \sqrt{\frac{1}{2} \left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2} \right)} \\ &\quad \times \sqrt{\frac{1}{2} \left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2} \right)}. \end{aligned}$$

This is sharper than some recent progress, as the reader can find in Remark 4.2 below.

Although the main discussion revolves around norm inequalities, insightful applications involving the numerical radius will be presented and compared with the literature through numerical calculations and rigorous approaches. We recall here that given $A \in \mathbb{M}_n$, its numerical radius is defined by $\omega(A) = \sup_{\|x\|=1} |\langle Ax, x \rangle|$. This quantity

has received renowned attention in the literature, as one can find in [4–5], [13–21], [2, 6, 7, 26, 32, 33, 35, 37, 38]. Research about the numerical radius involves, among other things, finding possible relations between the numerical radius and the operator norm. A basic relation in this regard is [27, Theorem 1.3-1]

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|,$$

which asserts the equivalence of the numerical radius and the operator norm.

Block matrices have played a major role in investigating numerical radius bounds, as one can see in [31, 33]. In Corollary 4.2 below, we prove that

$$\omega\left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix}\right) \leq \frac{1}{4}\sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|^2}} \\ \times \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2}}.$$

This bound will be shown to be sharper than celebrated bounds from the literature, as one can see in Remark 4.3.

Many other results will be displayed, and our findings will be compared with the literature whenever possible.

2. Preliminaries

It is proved in [12] that if $A_i, B_i, X_i, Y_i \in \mathbb{M}_n$ such that X_i and Y_i are positive, $i = 1, 2, \dots, N$, then

$$2s_j\left(\sum_{i=1}^N A_i X_i^{1/2} Y_i^{1/2} B_i^*\right) \leq s_j^2(W), \quad j = 1, \dots, N, \quad (2.1)$$

where $W = \begin{bmatrix} A_1 X_1^{1/2} & A_2 X_2^{1/2} & \dots & A_N X_N^{1/2} \\ B_1 Y_1^{1/2} & B_2 Y_2^{1/2} & \dots & B_N Y_N^{1/2} \end{bmatrix}$. The notation $s_j(\cdot)$ denotes the j -th largest singular value. The norm version of (2.1) is

$$2\left\| \sum_{i=1}^N A_i X_i^{1/2} Y_i^{1/2} B_i^* \right\| \leq \|W\|^2. \quad (2.2)$$

In the same reference, it is shown that if $A, B, X \in \mathbb{M}_n$ are positive, then

$$2s_j(M + R) \leq s_j((H + |K^*|) \oplus (L + |K|)), \quad j = 1, 2, \dots, N, \quad (2.3)$$

where $M = A^{1/2} X^{1/2} Y^{1/2} A^{1/2}$, $R = B^{1/2} X^{1/2} Y^{1/2} B^{1/2}$, $H = X^{1/2} A X^{1/2} + Y^{1/2} A Y^{1/2}$, $K = X^{1/2} A^{1/2} B^{1/2} X^{1/2} + Y^{1/2} A^{1/2} B^{1/2} Y^{1/2}$ and $L = X^{1/2} B X^{1/2} + Y^{1/2} B Y^{1/2}$. Here, $\oplus$ denotes the direct sum.

In particular, letting $Y = X$, we have

$$s_j \left(A^{1/2} X A^{1/2} + B^{1/2} X B^{1/2} \right) \leq s_j \left((P + |Q^*|) \oplus (R + |Q|) \right), \quad j = 1, 2, \dots, N$$

where $P = X^{1/2} A X^{1/2}$, $Q = X^{1/2} A^{1/2} B^{1/2} X^{1/2}$, and $R = X^{1/2} B X^{1/2}$.

Moreover, in the same reference, it is established that if $A, B, X_1, X_2, Y_1, Y_2 \in \mathbb{M}_n$ are such that X_1, X_2, Y_1, Y_2 are positive, then

$$2s_j(E - F) \leq s_j((J + |S^*|) \oplus (T + |S|)), \quad j = 1, 2, \dots, N, \quad (2.4)$$

where $E = A X_1^{1/2} Y_1^{1/2} A^*$, $F = B X_2^{1/2} Y_2^{1/2} B^*$, $J = X_1^{1/2} A^* A X_1^{1/2} + Y_1^{1/2} A^* A Y_1^{1/2}$, $S = X_1^{1/2} A^* B X_2^{1/2} - Y_1^{1/2} A^* B Y_2^{1/2}$, and $T = X_2^{1/2} B^* B X_2^{1/2} + Y_2^{1/2} B^* B Y_2^{1/2}$.

Inequalities (2.3) and (2.4) have the following norm versions, respectively:

$$2\|M + R\| \leq \|((H + |K^*|) \oplus (L + |K|))\|, \quad (2.5)$$

and

$$2\|E - F\| \leq \|((J + |S^*|) \oplus (T + |S|))\|. \quad (2.6)$$

In the subsequent section, we give generalizations and Höldert-type inequalities that extend (2.2), (2.5) and (2.6).

The following basic lemmas will be needed to accomplish our aim.

LEMMA 2.1. *Let $A \in \mathbb{M}_n$ be Hermitian. Then*

$$\pm A \leq |A|.$$

That is, $|A| - (\pm A)$ is positive.

LEMMA 2.2. *Let $A, B, X, Y \in \mathbb{M}_n$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then*

$$\| |A X Y^* B^*|^r \| \leq \left\| (F(\alpha))^{\frac{rp}{2}} \right\|^{1/p} \left\| (F(1-\alpha))^{\frac{rq}{2}} \right\|^{1/q}, \quad (2.7)$$

where

$$F(\alpha) = \alpha X^* A^* A X + (1-\alpha) Y^* B^* B Y.$$

Proof. Letting $A = AX$ and $B = BY$ in (1.4) leads to (2.7). $\square$

LEMMA 2.3. *Let $A, B, X, Y \in \mathbb{M}_n$ be such that X and Y are positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then*

$$\left\| A X^{1/2} Y^{1/2} B^* \right\|^r \leq \left\| (H(\alpha))^{\frac{rp}{2}} \right\|^{1/p} \left\| (H(1-\alpha))^{\frac{rq}{2}} \right\|^{1/q}, \quad (2.8)$$

where

$$H(\alpha) = \alpha X^{1/2} A^* A X^{1/2} + (1-\alpha) Y^{1/2} B^* B Y^{1/2}.$$

Proof. Substituting X by $X^{1/2}$ and Y by $Y^{1/2}$ in (2.7) implies (2.8). $\square$

LEMMA 2.4. Let $A, B, X \in \mathbb{M}_n$ be such that X and Y are positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then

$$\left\| |AXB^*|^r \right\| \leq \left\| (R(\alpha))^{\frac{rp}{2}} \right\|^{1/p} \left\| (R(1-\alpha))^{\frac{rq}{2}} \right\|^{1/q},$$

where

$$R(\alpha) = \alpha X^{1/2} A^* A X^{1/2} + (1-\alpha) X^{1/2} B^* B X^{1/2}.$$

Proof. The result is observed by letting $Y = X$ in (2.8). $\square$

LEMMA 2.5. Let $A, B, X, Y \in \mathbb{M}_n$ be positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then

$$\left\| |A^{1/2} X^{1/2} Y^{1/2} B^{1/2}|^r \right\| \leq \left\| (G(\alpha))^{\frac{rp}{2}} \right\|^{1/p} \left\| (G(1-\alpha))^{\frac{rq}{2}} \right\|^{1/q},$$

where

$$G(\alpha) = \alpha X^{1/2} A X^{1/2} + (1-\alpha) Y^{1/2} B Y^{1/2}.$$

Proof. The result is deduced from inequality (2.8) by letting $A = A^{1/2}$ and $B = B^{1/2}$. $\square$

Other lemmas that we will need when studying the numerical radius are stated next.

LEMMA 2.6. [33] Let $A, B \in \mathbb{M}_n$. Then

$$\omega \left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right) = \frac{1}{2} \sup_{\theta \in \mathbb{R}} \|A + e^{i\theta} B^*\|.$$

In particular,

$$\omega(A) = \sup_{\theta \in \mathbb{R}} \left\| \Re \left(e^{i\theta} A \right) \right\|,$$

where $\Re$ denotes the real part.

An immediate consequence of Lemma 2.6 is that, for $A, B \in \mathbb{M}_n$,

$$\frac{1}{2} \|A + B^*\| \leq \omega \left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right) \leq \frac{\|A\| + \|B\|}{2}. \quad (2.9)$$

In Corollary 4.2 below, we will be able to present a new upper bound for $\omega \left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right)$. Interestingly, this bound will be sharper than (2.9), as we justify in Remark 4.3.

While the above lemmas are stated to use them in our proofs, we now list some results for comparison reasons. It has been shown, very recently, in [10, Corollary 2.6] that, when $T_1, T_2 \in \mathbb{M}_n$,

$$\|T_1 T_2^* + T_2^* T_1\| \leq \|T_1\| \|T_2\| + \frac{1}{2} \|T_2 T_1 + T_1 T_2\|. \quad (2.10)$$

In Theorem 4.1 below, we will improve this bound.

Another norm bound, shown in [9, Corollary 2.4], states that

$$\|XY - YX\| \leq \|X\| \|Y\| + \frac{1}{2} \|X^* Y - Y X^*\|. \quad (2.11)$$

We will improve this bound in Corollary 4.1.

In [10, Corollary 2.8] it is shown that if $X, Y, Z \in \mathbb{M}_n$, then

$$\|XZ + ZY\| \leq \max(\|X\|, \|Y\|) \|Z\| + \frac{1}{2} \|X^* Z + Z Y^*\|. \quad (2.12)$$

In Remark 4.1 below, we show how Theorem 4.1 improves this important bound.

It has been shown in [39, Theorem 2.14] that

$$\|A + B\| \leq \sqrt{\max\{\|A\|, \|B\|\} + \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|} \sqrt{\max\{\|A\|, \|B\|\} + \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|}. \quad (2.13)$$

Theorem 4.2 below will present a refinement of this bound, as explained in Remark 4.2.

3. Hölder-type inequalities

At this stage of our discussion, we present the main result of this section, which is a generalization of inequality (2.2). We should mention that this result has been recently shown for the case $r = 1, p = 2$ in [8].

THEOREM 3.1. *Let $A_i, B_i, X_i, Y_i \in \mathbb{M}_n$ be such that X_i and Y_i are positive, $i = 1, 2, \dots, N$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then*

$$\left\| \left\| \sum_{i=1}^N A_i X_i^{1/2} Y_i^{1/2} B_i^* \right\|^r \right\| \leq \left\| \left\| (H(\alpha))^{\frac{rp}{2}} \right\|^{1/p} \right\| \left\| (H(1-\alpha))^{\frac{rq}{2}} \right\|^{1/q}, \quad (3.1)$$

where

$$H(\alpha) = \left\| \begin{bmatrix} \sqrt{\alpha} A_1 X_1^{1/2} & \sqrt{\alpha} A_2 X_2^{1/2} & \dots & \sqrt{\alpha} A_N X_N^{1/2} \\ \sqrt{1-\alpha} B_1 Y_1^{1/2} & \sqrt{1-\alpha} B_2 Y_2^{1/2} & \dots & \sqrt{1-\alpha} B_N Y_N^{1/2} \end{bmatrix} \right\|^2.$$

Proof. On $\oplus_{j=1}^N \mathbb{M}_n$, define

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 & \dots & B_N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_N \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_N \end{bmatrix}.$$

Let

$$V_{ij} = \alpha X_i^{1/2} A_i^* A_j X_j^{1/2} + (1 - \alpha) Y_i^{1/2} B_i^* B_j Y_j^{1/2},$$

and

$$H(\alpha) = \alpha X^{1/2} A^* A X^{1/2} + (1 - \alpha) Y^{1/2} B^* B Y^{1/2}.$$

Then

$$A X^{1/2} Y^{1/2} B^* = \begin{bmatrix} \sum_{i=1}^N A_i X_i^{1/2} Y_i^{1/2} B_i^* & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (3.2)$$

and

$$\begin{aligned} H(\alpha) &= \begin{bmatrix} V_{11} & \cdots & V_{1n} \\ V_{21} & \cdots & V_{2n} \\ \vdots & \ddots & \vdots \\ V_{n1} & \cdots & V_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{\alpha} X_1^{1/2} A_1^* \sqrt{1 - \alpha} Y_1^{1/2} B_1^* \\ \sqrt{\alpha} X_2^{1/2} A_2^* \sqrt{1 - \alpha} Y_2^{1/2} B_2^* \\ \cdots & \cdots \\ \sqrt{\alpha} X_n^{1/2} A_n^* \sqrt{1 - \alpha} Y_n^{1/2} B_n^* \end{bmatrix} \\ &\quad \times \begin{bmatrix} \sqrt{\alpha} A_1 X_1^{1/2} & \sqrt{\alpha} A_2 X_2^{1/2} & \cdots & \sqrt{\alpha} A_n X_n^{1/2} \\ \sqrt{1 - \alpha} B_1 Y_1^{1/2} & \sqrt{1 - \alpha} B_2 Y_2^{1/2} & \cdots & \sqrt{1 - \alpha} B_n Y_n^{1/2} \end{bmatrix} \\ &= \left| \begin{bmatrix} \sqrt{\alpha} A_1 X_1^{1/2} & \sqrt{\alpha} A_2 X_2^{1/2} & \cdots & \sqrt{\alpha} A_n X_n^{1/2} \\ \sqrt{1 - \alpha} B_1 Y_1^{1/2} & \sqrt{1 - \alpha} B_2 Y_2^{1/2} & \cdots & \sqrt{1 - \alpha} B_n Y_n^{1/2} \end{bmatrix} \right|^2. \end{aligned} \quad (3.3)$$

Similarly,

$$H(1 - \alpha) = \left| \begin{bmatrix} \sqrt{1 - \alpha} A_1 X_1^{1/2} & \sqrt{1 - \alpha} A_2 X_2^{1/2} & \cdots & \sqrt{1 - \alpha} A_n X_n^{1/2} \\ \sqrt{\alpha} B_1 Y_1^{1/2} & \sqrt{\alpha} B_2 Y_2^{1/2} & \cdots & \sqrt{\alpha} B_n Y_n^{1/2} \end{bmatrix} \right|^2. \quad (3.4)$$

Substituting (3.2), (3.3) and (3.4) in inequality (2.8), we get (3.1). $\square$

REMARK 3.1. Letting $r = 1$, $p = q = 2$, $\alpha = \frac{1}{2}$ in (3.1), we get (2.2).

A considerable special case of (3.1) is stated next.

COROLLARY 3.1. Let $A_i, B_i, X_i, Y_i \in \mathbb{M}_n$ be such that X_i and Y_i are positive, $i = 1, 2$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then

$$\| \| |D|^r \| \leq \| \| (E(\alpha))^{\frac{rp}{2}} \| \|^{1/p} \| \| (E(1-\alpha))^{\frac{rq}{2}} \| \|^{1/q}, \quad (3.5)$$

where

$$D = A_1 X_1^{1/2} Y_1^{1/2} B_1^* + A_2 X_2^{1/2} Y_2^{1/2} B_2^*,$$

and

$$E(\alpha) = \left\| \left[\begin{array}{cc} \sqrt{\alpha} A_1 X_1^{1/2} & \sqrt{\alpha} A_2 X_2^{1/2} \\ \sqrt{1-\alpha} B_1 Y_1^{1/2} & \sqrt{1-\alpha} B_2 Y_2^{1/2} \end{array} \right] \right\|^2.$$

Proof. Inequality (3.5) follows from (3.1) by letting $N = 2$. $\square$

COROLLARY 3.2. Let $A_i, B_i \in \mathbb{M}_n$, $i = 1, 2$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then

$$\| \| |J|^r \| \leq \| \| (K(\alpha))^{\frac{rp}{2}} \| \|^{1/p} \| \| (K(1-\alpha))^{\frac{rq}{2}} \| \|^{1/q}, \quad (3.6)$$

where

$$J = A_1 B_1^* + A_2 B_2^*$$

and

$$K(\alpha) = \left\| \left[\begin{array}{cc} \sqrt{\alpha} A_1 & \sqrt{\alpha} A_2 \\ \sqrt{1-\alpha} B_1 & \sqrt{1-\alpha} B_2 \end{array} \right] \right\|^2.$$

Proof. Letting $X_i = Y_i = I$, $i = 1, 2$ in (3.5), we get (3.6). $\square$

COROLLARY 3.3. Let $A, B \in \mathbb{M}_n$, and let $\alpha \in [0, 1]$. Then

$$\| \| AB^* \| \leq \| \| Z(\alpha) \| \|^{1/2} \| \| Z(1-\alpha) \| \|^{1/2}, \quad (3.7)$$

where

$$Z(\alpha) = \alpha A^* A + (1-\alpha) B^* B.$$

Proof. Letting $A_2 = B_2 = 0$, $r = 1$, $p = q = 2$ in (3.6) implies

$$\begin{aligned} \| \| AB^* \| &\leq \left\| \left\| \left[\begin{array}{cc} \sqrt{\alpha} A & 0 \\ \sqrt{1-\alpha} B & 0 \end{array} \right] \right\|^2 \right\|^{1/2} \left\| \left\| \left[\begin{array}{cc} \sqrt{1-\alpha} A & 0 \\ \sqrt{\alpha} B & 0 \end{array} \right] \right\|^2 \right\|^{1/2} \\ &= \| \| \alpha A^* A + (1-\alpha) B^* B \| \|^{1/2} \| \| (1-\alpha) A^* A + \alpha B^* B \| \|^{1/2}. \quad \square \end{aligned}$$

REMARK 3.2. Letting $\alpha = \frac{1}{2}$ in (3.7), we get the arithmetic-geometric mean inequality (1.2).

Another special case of inequality (3.1) is stated in the following result.

COROLLARY 3.4. Let $A, B, X, Y \in \mathbb{M}_n$ be such that X and Y are positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then

$$\| \| |G|^r \| \leq \| \| (H(\alpha))^{\frac{rp}{2}} \| \|^{1/p} \| \| (H(1-\alpha))^{\frac{rq}{2}} \| \|^{1/q}, \quad (3.8)$$

where

$$G = AX^{1/2}Y^{1/2}B^* + BX^{1/2}Y^{1/2}A^*,$$

and

$$H(\alpha) = \left\| \begin{bmatrix} \sqrt{\alpha}AX^{1/2} & \sqrt{\alpha}BX^{1/2} \\ \sqrt{1-\alpha}BY^{1/2} & \sqrt{1-\alpha}AY^{1/2} \end{bmatrix} \right\|^2.$$

Proof. Substituting $N = 2$, $A_1 = B_2 = A$, $A_2 = B_1 = B$, $X_1 = X_2 = X$ and $Y_1 = Y_2 = Y$ in (3.1), we get (3.8). $\square$

COROLLARY 3.5. Let $A, B, X \in \mathbb{M}_n$ be such that X is positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then

$$\begin{aligned} & \| \| |AXB^* + BXA^*|^r \| \| \\ & \leq \| \| (J(\alpha))^{\frac{rp}{2}} \| \|^{1/p} \| \| (J(1-\alpha))^{\frac{rq}{2}} \| \|^{1/q}, \end{aligned} \quad (3.9)$$

where

$$J(\alpha) = \left\| \begin{bmatrix} \sqrt{\alpha}AX^{1/2} & \sqrt{\alpha}BX^{1/2} \\ \sqrt{1-\alpha}BX^{1/2} & \sqrt{1-\alpha}AX^{1/2} \end{bmatrix} \right\|^2.$$

Proof. Letting $Y = X$ in (3.8) implies (3.9). $\square$

COROLLARY 3.6. Let $A, B, X \in \mathbb{M}_n$ and let $\alpha \in [0, 1]$. Then

$$\| \| AB^* + BA^* \| \| \leq \| \| (M(\alpha)) \| \|^{1/2} \| \| (M(1-\alpha)) \| \|^{1/2}, \quad (3.10)$$

where

$$M(\alpha) = \begin{bmatrix} \alpha A^*A + (1-\alpha)B^*B & \alpha A^*B + (1-\alpha)B^*A \\ \alpha B^*A + (1-\alpha)A^*B & (1-\alpha)A^*A + \alpha B^*B \end{bmatrix}.$$

Proof. Letting $X = I$, $r = 1$, $p = q = 2$ in (3.9), yields (3.10). $\square$

REMARK 3.3. Letting $\alpha = \frac{1}{2}$ in (3.10), we obtain

$$2\| \| AB^* + BA^* \| \| \leq \| \| \begin{bmatrix} A^*A + B^*B & A^*B + B^*A \\ A^*B + B^*A & A^*A + B^*B \end{bmatrix} \| \|.$$

According to Lemma 2.1 and (3.1), we have the following general inequality, which is a generalization of (2.5).

COROLLARY 3.7. Let $A, B, X, Y \in \mathbb{M}_n$ be positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then

$$\| |M + R|^r \| \leq \| |Z_1| \|^{1/p} \| |Z_2| \|^{1/q}, \quad (3.11)$$

where $M = A^{1/2}X^{1/2}Y^{1/2}A^{1/2}$, $R = B^{1/2}X^{1/2}Y^{1/2}B^{1/2}$,

$$Z_1 = ((H(\alpha) + |K^*(\alpha)|) \oplus (L(\alpha) + |K(\alpha)|))^{\frac{rp}{2}},$$

$$Z_2 = ((H(1-\alpha) + |K^*(1-\alpha)|) \oplus (L(1-\alpha) + |K(1-\alpha)|))^{\frac{rq}{2}},$$

$$H(\alpha) = \alpha X^{1/2}AX^{1/2} + (1-\alpha)Y^{1/2}AY^{1/2},$$

$$K(\alpha) = \alpha X^{1/2}A^{1/2}B^{1/2}X^{1/2} + (1-\alpha)Y^{1/2}A^{1/2}B^{1/2}Y^{1/2},$$

and $L(\alpha) = \alpha X^{1/2}BX^{1/2} + (1-\alpha)Y^{1/2}BY^{1/2}$.

In particular, letting $Y = X$, we have

$$\| (A^{1/2}XA^{1/2} + B^{1/2}XB^{1/2})^r \| \leq \| |W|^{\frac{rp}{2}} \|^{1/p} \| |W|^{\frac{rq}{2}} \|^{1/q},$$

where $W = ((P + |Q^*|) \oplus (R + |Q|))$, $P = X^{1/2}AX^{1/2}$, $Q = X^{1/2}A^{1/2}B^{1/2}X^{1/2}$, and $R = X^{1/2}BX^{1/2}$.

In addition, letting $X = I$, we have

$$\| (A + B)^r \| \leq \| |V|^{\frac{rp}{2}} \|^{1/p} \| |V|^{\frac{rq}{2}} \|^{1/q},$$

where $V = ((A + |B^{1/2}A^{1/2}|) \oplus (B + |A^{1/2}B^{1/2}|))$.

Moreover, letting $r = 1$, $p = q = 2$, leads to the well-known result [34]

$$\| A + B \| \leq \| (A + |B^{1/2}A^{1/2}|) \oplus (B + |A^{1/2}B^{1/2}|) \|.$$

Proof. Letting $N = 2$, $A_1 = B_1 = A^{1/2}$, $A_2 = B_2 = B^{1/2}$, $X_1 = X_2 = X$, $Y_1 = Y_2 = Y$ in (3.1), leads to

$$\begin{aligned} & \| |A^{1/2}X^{1/2}Y^{1/2}A^{1/2} + B^{1/2}X^{1/2}Y^{1/2}B^{1/2}|^r \| \\ & \leq \left\| \left(\left| \begin{bmatrix} \sqrt{\alpha}A^{1/2}X^{1/2} & \sqrt{\alpha}B^{1/2}X^{1/2} \\ \sqrt{1-\alpha}A^{1/2}Y^{1/2} & \sqrt{1-\alpha}B^{1/2}Y^{1/2} \end{bmatrix} \right|^2 \right)^{\frac{rp}{2}} \right\|^{1/p} \\ & \quad \times \left\| \left(\left| \begin{bmatrix} \sqrt{1-\alpha}A^{1/2}X^{1/2} & \sqrt{1-\alpha}B^{1/2}X^{1/2} \\ \sqrt{\alpha}A^{1/2}Y^{1/2} & \sqrt{\alpha}B^{1/2}Y^{1/2} \end{bmatrix} \right|^2 \right)^{\frac{rq}{2}} \right\|^{1/q} \\ & = \left\| \left(\begin{bmatrix} H(\alpha) & K(\alpha) \\ K^*(\alpha) & L(\alpha) \end{bmatrix} \right)^{\frac{rp}{2}} \right\|^{1/p} \left\| \left(\begin{bmatrix} H(1-\alpha) & K(1-\alpha) \\ K^*(1-\alpha) & L(1-\alpha) \end{bmatrix} \right)^{\frac{rq}{2}} \right\|^{1/q}. \end{aligned}$$

Using Lemma 2.1, we deduce

$$\begin{aligned}
 & \left\| \left(\begin{bmatrix} H(\alpha) & K(\alpha) \\ K^*(\alpha) & L(\alpha) \end{bmatrix} \right)^{\frac{rp}{2}} \right\| \\
 &= \left\| \left(\begin{bmatrix} H(\alpha) & 0 \\ 0 & L(\alpha) \end{bmatrix} + \begin{bmatrix} 0 & K(\alpha) \\ K^*(\alpha) & 0 \end{bmatrix} \right)^{\frac{rp}{2}} \right\| \\
 &\leq \left\| \left(\begin{bmatrix} H(\alpha) & 0 \\ 0 & L(\alpha) \end{bmatrix} + \left| \begin{bmatrix} 0 & K(\alpha) \\ K^*(\alpha) & 0 \end{bmatrix} \right| \right)^{\frac{rp}{2}} \right\| \\
 &= \left\| \left(\begin{bmatrix} H(\alpha) & 0 \\ 0 & L(\alpha) \end{bmatrix} + \begin{bmatrix} |K^*(\alpha)| & 0 \\ 0 & |K(\alpha)| \end{bmatrix} \right)^{\frac{rp}{2}} \right\| \\
 &= \left\| \left(\begin{bmatrix} H(\alpha) + |K^*(\alpha)| & 0 \\ 0 & L(\alpha) + |K(\alpha)| \end{bmatrix} \right)^{\frac{rp}{2}} \right\|
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & \left\| \left(\begin{bmatrix} H(1-\alpha) & K(1-\alpha) \\ K^*(1-\alpha) & L(1-\alpha) \end{bmatrix} \right)^{\frac{rq}{2}} \right\| \\
 &\leq \left\| \left(\begin{bmatrix} H(1-\alpha) + |K^*(1-\alpha)| & 0 \\ 0 & L(1-\alpha) + |K(1-\alpha)| \end{bmatrix} \right)^{\frac{rq}{2}} \right\|.
 \end{aligned} \tag{3.12}$$

Thus, inequality (3.11) follows from inequalities (3.12) and (3.12). $\square$

REMARK 3.4. Letting $\alpha = \frac{1}{2}$, $r = 1$, $p = q = 2$ in (3.11), we obtain (2.5). In that sense, (3.11) is a generalization of (2.5).

Using Lemma 2.1 and (3.1) implies the following general inequality, which is a generalization of (2.6).

COROLLARY 3.8. Let $A, B, X_1, X_2, Y_1, Y_2 \in \mathbb{M}_n$ be such that X_1, X_2, Y_1, Y_2 are positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$, and $r \geq 0$. Then

$$\begin{aligned}
 & 2 \left\| |E - F|^r \right\| \\
 &\leq \left\| \left((H(\alpha) + |K^*(\alpha)|) \oplus (L(\alpha) + |K(\alpha)|) \right)^{\frac{rp}{2}} \right\|^{1/p} \\
 &\quad \times \left\| \left((H(1-\alpha) + |K^*(1-\alpha)|) \oplus (L(1-\alpha) + |K(1-\alpha)|) \right)^{\frac{rq}{2}} \right\|^{1/q},
 \end{aligned}$$

where $E = AX_1^{1/2}Y_1^{1/2}A^*$, $F = BX_2^{1/2}Y_2^{1/2}B^*$,

$$\begin{aligned}
 H(\alpha) &= \alpha X_1^{1/2}A^*AX_1^{1/2} + (1-\alpha)Y_1^{1/2}A^*AY_1^{1/2}, \\
 K(\alpha) &= \alpha X_1^{1/2}A^*BX_2^{1/2} - (1-\alpha)Y_1^{1/2}A^*BY_2^{1/2},
 \end{aligned}$$

and

$$L(\alpha) = \alpha X_2^{1/2} B^* B X_2^{1/2} + (1 - \alpha) Y_2^{1/2} B^* B Y_2^{1/2}.$$

Proof. Letting $N = 2$, $A_1 = B_1 = A$, and $A_2 = -B_2 = B$ in (3.1) yields

$$\begin{aligned} & \left\| \left| A X_1^{1/2} Y_1^{1/2} A^* - B X_2^{1/2} Y_2^{1/2} B^* \right|^r \right\| \\ & \leq \left\| \left(\left| \begin{bmatrix} \sqrt{\alpha} A X_1^{1/2} & \sqrt{\alpha} B X_2^{1/2} \\ \sqrt{1-\alpha} A Y_1^{1/2} & -\sqrt{1-\alpha} B Y_2^{1/2} \end{bmatrix} \right|^2 \right)^{\frac{rp}{2}} \right\|^{1/p} \\ & \quad \times \left\| \left(\left| \begin{bmatrix} \sqrt{1-\alpha} A X_1^{1/2} & \sqrt{1-\alpha} B X_2^{1/2} \\ \sqrt{\alpha} A Y_1^{1/2} & -\sqrt{\alpha} B Y_2^{1/2} \end{bmatrix} \right|^2 \right)^{\frac{rq}{2}} \right\|^{1/q} \\ & = \left\| \left(\begin{bmatrix} H(\alpha) & K(\alpha) \\ K^*(\alpha) & L(\alpha) \end{bmatrix} \right)^{\frac{rp}{2}} \right\|^{1/p} \\ & \quad \times \left\| \left(\begin{bmatrix} H(1-\alpha) & K(1-\alpha) \\ K^*(1-\alpha) & L(1-\alpha) \end{bmatrix} \right)^{\frac{rq}{2}} \right\|^{1/q}. \end{aligned}$$

We give the desired result by applying the same steps used in the proof of Corollary 3.7. $\square$

REMARK 3.5. Letting $\alpha = \frac{1}{2}$, $r = 1$, $p = q = 2$ in (3.13), we obtain (2.6). In that sense, (3.13) is a generalization of (2.6).

COROLLARY 3.9. Let $A, B \in \mathbb{M}_n$ be positive and let $\alpha \in [0, 1]$. Then

$$\|A - B\| \leq \|(H_1(\alpha) + |K_1^*(\alpha)|) \oplus (L_1(\alpha) + |K_1(\alpha)|)\|, \quad (3.13)$$

where $H_1(\alpha) = \alpha A + (1 - \alpha)A$, $K_1(\alpha) = \alpha A^{1/2} B^{1/2} - (1 - \alpha) A^{1/2} B^{1/2}$ and $L_1(\alpha) = \alpha B + (1 - \alpha)B$. In particular, if $\alpha = \frac{1}{2}$, then

$$\|A - B\| \leq \|A \oplus B\|.$$

Proof. If $r = 1$, $p = q = 2$, $X = Y = I$ in (3.13), then (3.13) follows. $\square$

4. Norm and numerical radius bounds

This section implements the above results in obtaining significant bounds for the usual operator norm of certain quantities and the numerical radius. These bounds will be shown to be sharper than existing celebrated bounds.

The first result is the following improvement of (2.10).

THEOREM 4.1. Let $T_1, T_2 \in \mathbb{M}_n$. Then

$$\|T_1 T_2^* + T_2^* T_1\| \leq \frac{1}{2} (\|T_1\| \|T_2\| + \max \{\|T_1 T_2^*\|, \|T_2^* T_1\|\} + \|T_2 T_1 + T_1 T_2\|).$$

Proof. It follows from Corollary 3.2 that

$$\begin{aligned} & \|T_1 T_2^* + T_2^* T_1\| \\ & \leq \frac{1}{2} \left\| \begin{bmatrix} T_1^* T_1 + T_2^* T_2 & T_1^* T_2^* + T_2^* T_1^* \\ T_2 T_1 + T_1 T_2 & T_2 T_2^* + T_1 T_1^* \end{bmatrix} \right\| \\ & = \frac{1}{2} \left\| \begin{bmatrix} T_1^* T_1 + T_2^* T_2 & O \\ O & T_2 T_2^* + T_1 T_1^* \end{bmatrix} + \begin{bmatrix} O & T_1^* T_2^* + T_2^* T_1^* \\ T_2 T_1 + T_1 T_2 & O \end{bmatrix} \right\| \\ & \leq \frac{1}{2} \left(\left\| \begin{bmatrix} T_1^* T_1 + T_2^* T_2 & O \\ O & T_2 T_2^* + T_1 T_1^* \end{bmatrix} \right\| + \left\| \begin{bmatrix} O & T_1^* T_2^* + T_2^* T_1^* \\ T_2 T_1 + T_1 T_2 & O \end{bmatrix} \right\| \right) \\ & = \frac{1}{2} (\max \{\|T_1^* T_1 + T_2^* T_2\|, \|T_2 T_2^* + T_1 T_1^*\|\} + \|T_2 T_1 + T_1 T_2\|). \end{aligned}$$

That is,

$$\|T_1 T_2^* + T_2^* T_1\| \leq \frac{1}{2} (\max \{\|T_1^* T_1 + T_2^* T_2\|, \|T_2 T_2^* + T_1 T_1^*\|\} + \|T_2 T_1 + T_1 T_2\|).$$

Now, by letting $T_1 = \sqrt{\frac{\|T_2\|}{\|T_1\|}} T_1$ and $T_2 = \sqrt{\frac{\|T_1\|}{\|T_2\|}} T_2$, we conclude that

$$\begin{aligned} & \|T_1 T_2^* + T_2^* T_1\| \\ & \leq \frac{1}{2} \left(\max \left\{ \left\| \frac{\|T_2\|}{\|T_1\|} T_1^* T_1 + \frac{\|T_1\|}{\|T_2\|} T_2^* T_2 \right\|, \left\| \frac{\|T_1\|}{\|T_2\|} T_2 T_2^* + \frac{\|T_2\|}{\|T_1\|} T_1 T_1^* \right\| \right\} + \|T_2 T_1 + T_1 T_2\| \right) \\ & \leq \frac{1}{2} (\max \{\|T_1\| \|T_2\| + \| |T_1| |T_2| \|, \|T_1\| \|T_2\| + \| |T_2^*| |T_1^*| \| \} + \|T_2 T_1 + T_1 T_2\|) \\ & = \frac{1}{2} (\|T_1\| \|T_2\| + \max \{\|T_1 T_2^*\|, \|T_2^* T_1\|\} + \|T_2 T_1 + T_1 T_2\|) \end{aligned}$$

where in the second inequality, we employed the following fact for positive matrices [36]

$$\|X + Y\| \leq \max \{\|X\|, \|Y\|\} + \left\| X^{\frac{1}{2}} Y^{\frac{1}{2}} \right\|.$$

This completes the proof. $\square$

In the following remark, we show how Theorem 4.1 improves (2.12).

REMARK 4.1. Letting $T_1 = \begin{bmatrix} Y & O \\ O & X \end{bmatrix}$ and $T_2 = \begin{bmatrix} O & Z^* \\ O & O \end{bmatrix}$, in Theorem 4.1, we obtain

$$\|XZ + ZY\| \leq \frac{1}{2} (\max \{\|X\|, \|Y\|\} \|Z\| + \max \{\|XZ\|, \|ZY\|\} + \|X^* Z + ZY^*\|). \quad (4.1)$$

Now we prove the following improvement of (2.11).

COROLLARY 4.1. Let $X, Y \in \mathbb{M}_n$. Then

$$\|XY - YX\| \leq \frac{1}{2} (\|X\| \|Y\| + \max\{\|XY\|, \|YX\|\} + \|X^*Y - YX^*\|).$$

Proof. If we replace Y by $-Y$, in (4.1), we get

$$\|XZ - ZY\| \leq \frac{1}{2} (\max\{\|X\|, \|Y\|\} \|Z\| + \max\{\|XZ\|, \|ZY\|\} + \|X^*Z - ZY^*\|).$$

We reach the desired result by substituting Z and Y with Y and X , respectively. $\square$

THEOREM 4.2. Let $A, B \in \mathbb{M}_n$. Then for any $v \in [0, 1]$,

$$\begin{aligned} & \|A + B\| \\ & \leq \sqrt{\frac{1}{2} \left(\|A\|^{2v} + \|B\|^{2(1-v)} + \sqrt{\left(\|A\|^{2v} - \|B\|^{2(1-v)} \right)^2 + 4 \left\| |A^*|^v |B^*|^{1-v} \right\|^2} \right)} \\ & \quad \times \sqrt{\frac{1}{2} \left(\|A\|^{2(1-v)} + \|B\|^{2v} + \sqrt{\left(\|A\|^{2(1-v)} - \|B\|^{2v} \right)^2 + 4 \left\| |A|^{1-v} |B|^v \right\|^2} \right)}. \end{aligned}$$

In particular,

$$\begin{aligned} \|A + B\| & \leq \sqrt{\frac{1}{2} \left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|^2} \right)} \\ & \quad \times \sqrt{\frac{1}{2} \left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2} \right)}. \end{aligned}$$

Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B , respectively. Let $t > 0$. Setting $A_1 = \sqrt{t}U|A|^v$, $B_1 = \frac{1}{\sqrt{t}}|A|^{1-v}$, $A_2 = \sqrt{t}V|B|^{1-v}$, and $B_2 = \frac{1}{\sqrt{t}}|B|^v$, in Corollary 3.2, we have

$$\begin{aligned} & \|A + B\| \\ & \leq \frac{1}{2} \left\| \begin{bmatrix} t|A|^{2v} + \frac{1}{t}|A|^{2(1-v)} & t|A|^v U^* V |B|^{1-v} + \frac{1}{t}|A|^{1-v} |B|^v \\ t|B|^{1-v} V^* U |A|^v + \frac{1}{t}|B|^v |A|^{1-v} & t|B|^{2(1-v)} + \frac{1}{t}|B|^{2v} \end{bmatrix} \right\| \\ & = \frac{1}{2} \omega \left(\begin{bmatrix} t|A|^{2v} + \frac{1}{t}|A|^{2(1-v)} & t|A|^v U^* V |B|^{1-v} + \frac{1}{t}|A|^{1-v} |B|^v \\ t|B|^{1-v} V^* U |A|^v + \frac{1}{t}|B|^v |A|^{1-v} & t|B|^{2(1-v)} + \frac{1}{t}|B|^{2v} \end{bmatrix} \right) \\ & = \frac{1}{2} \omega \left(t \begin{bmatrix} |A|^{2v} & |A|^v U^* V |B|^{1-v} \\ |B|^{1-v} V^* U |A|^v & |B|^{2(1-v)} \end{bmatrix} + \frac{1}{t} \begin{bmatrix} |A|^{2(1-v)} & |A|^{1-v} |B|^v \\ |B|^v |A|^{1-v} & |B|^{2v} \end{bmatrix} \right) \\ & \leq \frac{1}{2} \left(t \omega \left(\begin{bmatrix} |A|^{2v} & |A|^v U^* V |B|^{1-v} \\ |B|^{1-v} V^* U |A|^v & |B|^{2(1-v)} \end{bmatrix} \right) + \frac{1}{t} \omega \left(\begin{bmatrix} |A|^{2(1-v)} & |A|^{1-v} |B|^v \\ |B|^v |A|^{1-v} & |B|^{2v} \end{bmatrix} \right) \right). \end{aligned}$$

That is,

$$\|A + B\| \leq \frac{1}{2} \left(t \omega \left(\begin{bmatrix} |A|^{2\nu} & |A|^\nu U^* V |B|^{1-\nu} \\ |B|^{1-\nu} V^* U |A|^\nu & |B|^{2(1-\nu)} \end{bmatrix} \right) + \frac{1}{t} \omega \left(\begin{bmatrix} |A|^{2(1-\nu)} & |A|^{1-\nu} |B|^\nu \\ |B|^\nu |A|^{1-\nu} & |B|^{2\nu} \end{bmatrix} \right) \right),$$

for any $t > 0$. Taking minimum over $t > 0$, we get

$$\|A + B\| \leq \sqrt{\omega \left(\begin{bmatrix} |A|^{2\nu} & |A|^\nu U^* V |B|^{1-\nu} \\ |B|^{1-\nu} V^* U |A|^\nu & |B|^{2(1-\nu)} \end{bmatrix} \right) \omega \left(\begin{bmatrix} |A|^{2(1-\nu)} & |A|^{1-\nu} |B|^\nu \\ |B|^\nu |A|^{1-\nu} & |B|^{2\nu} \end{bmatrix} \right)}.$$

On the other hand,

$$\begin{aligned} & \sqrt{\omega \left(\begin{bmatrix} |A|^{2\nu} & |A|^\nu U^* V |B|^{1-\nu} \\ |B|^{1-\nu} V^* U |A|^\nu & |B|^{2(1-\nu)} \end{bmatrix} \right) \omega \left(\begin{bmatrix} |A|^{2(1-\nu)} & |A|^{1-\nu} |B|^\nu \\ |B|^\nu |A|^{1-\nu} & |B|^{2\nu} \end{bmatrix} \right)} \\ & \leq \sqrt{\omega \left(\begin{bmatrix} \|A\|^{2\nu} & \| |A|^\nu U^* V |B|^{1-\nu} \| \\ \| |B|^{1-\nu} V^* U |A|^\nu \| & \|B\|^{2(1-\nu)} \end{bmatrix} \right) \omega \left(\begin{bmatrix} \|A\|^{2(1-\nu)} & \| |A|^{1-\nu} |B|^\nu \| \\ \| |B|^\nu |A|^{1-\nu} \| & \|B\|^{2\nu} \end{bmatrix} \right)} \\ & \leq \sqrt{\frac{1}{2} \left(\|A\|^{2\nu} + \|B\|^{2(1-\nu)} + \sqrt{\left(\|A\|^{2\nu} - \|B\|^{2(1-\nu)} \right)^2 + 4 \| |A|^\nu U^* V |B|^{1-\nu} \|^2} \right)} \\ & \quad \times \sqrt{\frac{1}{2} \left(\|A\|^{2(1-\nu)} + \|B\|^{2\nu} + \sqrt{\left(\|A\|^{2(1-\nu)} - \|B\|^{2\nu} \right)^2 + 4 \| |A|^{1-\nu} |B|^\nu \|^2} \right)}, \end{aligned}$$

where we have used [3, Corollary 2] to obtain the last inequality. So,

$$\begin{aligned} & \|A + B\| \\ & \leq \sqrt{\frac{1}{2} \left(\|A\|^{2\nu} + \|B\|^{2(1-\nu)} + \sqrt{\left(\|A\|^{2\nu} - \|B\|^{2(1-\nu)} \right)^2 + 4 \| |A|^\nu U^* V |B|^{1-\nu} \|^2} \right)} \\ & \quad \times \sqrt{\frac{1}{2} \left(\|A\|^{2(1-\nu)} + \|B\|^{2\nu} + \sqrt{\left(\|A\|^{2(1-\nu)} - \|B\|^{2\nu} \right)^2 + 4 \| |A|^{1-\nu} |B|^\nu \|^2} \right)}. \end{aligned}$$

Since

$$\| |A|^\nu U^* V |B|^{1-\nu} \| = \| U \left(|A|^\nu U^* V |B|^{1-\nu} \right) V^* \| = \| |A^*|^\nu |B^*|^{1-\nu} \|,$$

we deduce the desired result. $\square$

The following remark shows that Theorem 4.2 is a considerable improvement of (2.13).

REMARK 4.2. Theorem 4.2 improves (2.13). Indeed,

$$\begin{aligned}
 \|A + B\| &\leq \sqrt{\frac{1}{2} \left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|^2} \right)} \\
 &\quad \times \sqrt{\frac{1}{2} \left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2} \right)} \\
 &\leq \sqrt{\frac{1}{2} \left(\|A\| + \|B\| + \left| \|A\| - \|B\| \right| + 2 \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\| \right)} \\
 &\quad \times \sqrt{\frac{1}{2} \left(\|A\| + \|B\| + \left| \|A\| - \|B\| \right| + 2 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\| \right)} \\
 &= \sqrt{\max\{\|A\|, \|B\|\} + \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|} \sqrt{\max\{\|A\|, \|B\|\} + \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|}.
 \end{aligned}$$

COROLLARY 4.2. Let $A, B \in \mathbb{M}_n$. Then for any $\nu \in [0, 1]$,

$$\begin{aligned}
 &\omega \left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix} \right) \\
 &\leq \frac{1}{4} \sqrt{\|A\|^{2\nu} + \|B\|^{2(1-\nu)} + \sqrt{\left(\|A\|^{2\nu} - \|B\|^{2(1-\nu)} \right)^2 + 4 \left\| |A^*|^\nu |B^*|^{1-\nu} \right\|^2}} \\
 &\quad \times \sqrt{\|A\|^{2(1-\nu)} + \|B\|^{2\nu} + \sqrt{\left(\|A\|^{2(1-\nu)} - \|B\|^{2\nu} \right)^2 + 4 \left\| |A|^{1-\nu} |B|^\nu \right\|^2}}.
 \end{aligned}$$

In particular,

$$\begin{aligned}
 \omega \left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix} \right) &\leq \frac{1}{4} \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|^2}} \\
 &\quad \times \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2}}.
 \end{aligned}$$

Proof. By replacing B by $e^{i\theta}B$, in Theorem 4.2, we obtain

$$\begin{aligned}
 &\|A + e^{i\theta}B\| \\
 &\leq \sqrt{\frac{1}{2} \left(\|A\|^{2\nu} + \|B\|^{2(1-\nu)} + \sqrt{\left(\|A\|^{2\nu} - \|B\|^{2(1-\nu)} \right)^2 + 4 \left\| |A^*|^\nu |B^*|^{1-\nu} \right\|^2} \right)} \\
 &\quad \times \sqrt{\frac{1}{2} \left(\|A\|^{2(1-\nu)} + \|B\|^{2\nu} + \sqrt{\left(\|A\|^{2(1-\nu)} - \|B\|^{2\nu} \right)^2 + 4 \left\| |A|^{1-\nu} |B|^\nu \right\|^2} \right)}.
 \end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$ gives the desired result upon using Lemma 2.6. $\square$

REMARK 4.3. We know, from (2.9), that

$$\omega \left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix} \right) \leq \frac{\|A\| + \|B\|}{2}. \quad (4.2)$$

In Corollary 4.2, we have shown that

$$\begin{aligned} \omega \left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix} \right) &\leq \frac{1}{4} \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|^2}} \\ &\quad \times \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2}}. \end{aligned} \quad (4.3)$$

In this remark, we show that our bound is sharper than the previous bound. Notice that

$$\left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|^2 \leq \|A\| \|B\|.$$

Therefore,

$$\begin{aligned} &\sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|^2}} \\ &\leq \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \|A\| \|B\|}} \\ &= \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| + \|B\|)^2}} \\ &= \sqrt{2 (\|A\| + \|B\|)}. \end{aligned}$$

Similarly,

$$\sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2}} \leq \sqrt{2 (\|A\| + \|B\|)}.$$

Consequently, Corollary 4.2 implies

$$\begin{aligned} \omega \left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix} \right) &\leq \frac{1}{4} \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|^2}} \\ &\quad \times \sqrt{\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^2}} \\ &\leq \frac{\|A\| + \|B\|}{2}. \end{aligned}$$

Thus, we have shown that (4.3) is a refinement of (4.2).

As a final application of our discussion, we present the following upper bound of the norm of the real part of a matrix. This result is stated in terms of the weighted Aluthge transform $\widetilde{A}_v$, which is defined as $\widetilde{A}_v = |A|^v U |A|^{1-v}$, where $A = U |A|$ is the polar decomposition of A . This weighted transform was defined in [25] as a generalization of the original Aluthge transform defined in [11]. When $v = \frac{1}{2}$, we simply write $\widetilde{A}$ instead of $\widetilde{A}_{1/2}$. The Aluthge transform has appeared in significant work treating the numerical radius inequalities, as one can find in [1, 31, 40].

PROPOSITION 4.1. Let $A \in \mathbb{M}_n$. Then for any $v \in [0, 1]$,

$$\|\Re A\| \leq \frac{1}{2} \omega \left(\begin{bmatrix} |A|^{2(1-v)} & (\widetilde{A}_v)^* \\ \widetilde{A}_v & |A|^{2v} \end{bmatrix} \right).$$

In particular,

$$\|\Re A\| \leq \frac{1}{2} \omega \left(\begin{bmatrix} |A| & (\widetilde{A})^* \\ \widetilde{A} & |A| \end{bmatrix} \right).$$

Proof. If we set $T_1 = \sqrt{t} U |A|^{1-v}$, $T_2 = \frac{1}{\sqrt{t}} |A|^v$, $T_3 = \sqrt{t} |A|^v$, and $T_4 = \frac{1}{\sqrt{t}} U |A|^{1-v}$, in Corollary 3.2, we obtain

$$\begin{aligned} \|A + A^*\| &\leq \frac{1}{2} \left\| \begin{bmatrix} t|A|^{2(1-v)} + \frac{1}{t}|A|^{2v} & t(\widetilde{A}_v)^* + \frac{1}{t}\widetilde{A}_v \\ t\widetilde{A}_v + \frac{1}{t}(\widetilde{A}_v)^* & t|A|^{2v} + \frac{1}{t}|A|^{2(1-v)} \end{bmatrix} \right\| \\ &= \frac{1}{2} \omega \left(\begin{bmatrix} t|A|^{2(1-v)} + \frac{1}{t}|A|^{2v} & t(\widetilde{A}_v)^* + \frac{1}{t}\widetilde{T}_v \\ t\widetilde{A}_v + \frac{1}{t}(\widetilde{A}_v)^* & t|A|^{2v} + \frac{1}{t}|A|^{2(1-v)} \end{bmatrix} \right) \\ &= \frac{1}{2} \omega \left(t \begin{bmatrix} |A|^{2(1-v)} & (\widetilde{A}_v)^* \\ \widetilde{A}_v & |A|^{2v} \end{bmatrix} + \frac{1}{t} \begin{bmatrix} |A|^{2v} & \widetilde{A}_v \\ (\widetilde{A}_v)^* & |A|^{2(1-v)} \end{bmatrix} \right) \\ &\leq \frac{1}{2} \left(t \omega \left(\begin{bmatrix} |A|^{2(1-v)} & (\widetilde{A}_v)^* \\ \widetilde{A}_v & |A|^{2v} \end{bmatrix} \right) + \frac{1}{t} \omega \left(\begin{bmatrix} |A|^{2v} & \widetilde{A}_v \\ (\widetilde{A}_v)^* & |A|^{2(1-v)} \end{bmatrix} \right) \right). \end{aligned}$$

Thus, we have shown that

$$\|A + A^*\| \leq \frac{1}{2} \left(t \omega \left(\begin{bmatrix} |A|^{2(1-v)} & (\widetilde{A}_v)^* \\ \widetilde{A}_v & |A|^{2v} \end{bmatrix} \right) + \frac{1}{t} \omega \left(\begin{bmatrix} |A|^{2v} & \widetilde{A}_v \\ (\widetilde{A}_v)^* & |A|^{2(1-v)} \end{bmatrix} \right) \right),$$

for any $t > 0$. Since this is true for all $t > 0$, we may take the minimum over such values of t to deduce that

$$\|\Re A\| \leq \frac{1}{2} \sqrt{\omega \left(\begin{bmatrix} |A|^{2(1-v)} & (\widetilde{A}_v)^* \\ \widetilde{A}_v & |A|^{2v} \end{bmatrix} \right) \omega \left(\begin{bmatrix} |A|^{2v} & \widetilde{A}_v \\ (\widetilde{A}_v)^* & |A|^{2(1-v)} \end{bmatrix} \right)}. \quad (4.4)$$

But, for any $A, B, C, D \in \mathbb{M}_n$, we have

$$\omega \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \omega \left(\begin{bmatrix} D & C \\ B & A \end{bmatrix} \right).$$

This, together with (4.4), implies the desired result. $\square$

REMARK 4.4. We know that $\|\Re A\| \leq \|A\|$, for any $A \in \mathbb{M}_n$. Proposition 4.1 provides a refinement of this basic inequality, as follows. Let $\nu = \frac{1}{2}$ in Proposition 4.1 and notice that

$$\begin{aligned} \|\Re A\| &\leq \frac{1}{2} \omega \left(\begin{bmatrix} |A| & (\tilde{A})^* \\ \tilde{A} & |A| \end{bmatrix} \right) \\ &= \frac{1}{2} \omega \left(\begin{bmatrix} |A| & O \\ O & |A| \end{bmatrix} + \begin{bmatrix} O & (\tilde{A})^* \\ \tilde{A} & O \end{bmatrix} \right) \\ &\leq \frac{1}{2} \left(\omega(|A|) + \omega \left(\begin{bmatrix} O & (\tilde{A})^* \\ \tilde{A} & O \end{bmatrix} \right) \right) \\ &= \frac{1}{2} (\|A\| + \|\tilde{A}\|) \\ &\leq \|A\|. \end{aligned}$$

REMARK 4.5. The argument in Remark 4.4 may lure us to think of the inequality

$$\|\Re A\| \leq \frac{1}{2} (\omega(A) + \|\tilde{A}\|).$$

In fact, this inequality is not always true, as one can check with the matrix $A = \begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix}$.

In this case, we have

$$\begin{aligned} \Re A &= \begin{bmatrix} 3 & 3.5 \\ 3.5 & 2 \end{bmatrix}, \quad \tilde{A} \approx \begin{bmatrix} 3.61315 & 3.00182 \\ 3.00182 & 1.38685 \end{bmatrix}, \quad \|\Re A\| \approx 6.03553, \\ \frac{1}{2} (\omega(A) + \|\tilde{A}\|) &\approx 5.86855. \end{aligned}$$

Declarations

Availability of data and materials. Not applicable.

Conflict of interest. The authors declare that they have no conflict of interest.

Funding. Not applicable.

Authors' contributions. Authors declare that they have contributed equally to this paper. All authors have read and approved this version.

Acknowledgements. Not applicable.

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(Received April 24, 2024)

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