

NEW INEQUALITIES INVOLVING $(n!)^{1/n}$

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Abstract. Let $G_n = (n!)^{1/n}$ denote the geometric mean of the first n positive integers. In this paper, we prove that for $n \geq 2$,

$$1 < \frac{G_n^2}{G_{n-1}G_{n+1}} < 1 + \frac{1}{n^2}.$$

Both constants are asymptotically sharp: $n^2(G_n^2/(G_{n-1}G_{n+1}) - 1) \rightarrow 1$ as $n \rightarrow \infty$.

1. Introduction

Let $G_n = (n!)^{1/n}$ denote the geometric mean of the first n positive integers. When investigating a conjecture on an upper bound for permanents of $(0, 1)$ -matrices, Minc and Sathre [12] discovered several noteworthy inequalities involving G_n . One of them is the following:

$$\frac{n}{n+1} < \frac{G_n}{G_{n+1}} < 1, \quad n \in \mathbb{N} := \{1, 2, \dots\}. \quad (1.1)$$

It is easy to see that

$$\frac{(n+1)^n}{n!} = \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right)^2 \cdots \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n}\right)^{n^2},$$

which yields

$$\frac{n+1}{\sqrt[n]{n!}} \leq \left(1 + \frac{1}{n}\right)^n < e, \quad n \in \mathbb{N}. \quad (1.2)$$

By using (1.2), Hardy [10] presented a proof of Carleman's inequality

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

where $a_n \geq 0$ for $n \in \mathbb{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$.

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REMARK 1.1. We see from (1.2) that

$$\frac{n}{\sqrt[n]{n!}} < \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N}. \quad (1.3)$$

We find that, for $n \in \mathbb{N}$,

$$\begin{aligned} \frac{n}{n+1} < \frac{\sqrt[n]{n!}}{n+1\sqrt[n+1]{(n+1)!}} &\iff \left(\frac{n}{n+1}\right)^{n(n+1)} < \frac{(n!)^{n+1}}{[(n+1)!]^n} = \frac{n!}{(n+1)^n} \\ &\iff \left(\frac{n}{n+1}\right)^{n+1} < \frac{\sqrt[n]{n!}}{n+1} \iff \frac{n}{\sqrt[n]{n!}} < \left(1 + \frac{1}{n}\right)^n. \end{aligned}$$

This shows that the left-hand side of (1.1) is equivalent to the inequality (1.3).

In [6], it was proved that for $x > 0$,

$$\frac{x}{[\Gamma(x+1)]^{1/x}} < \left(1 + \frac{1}{x}\right)^x. \quad (1.4)$$

In [15], it was shown that the function $\frac{[\Gamma(x+1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x$ is strictly logarithmically completely monotonic on $(0, \infty)$. Since

$$\lim_{x \rightarrow \infty} \frac{[\Gamma(x+1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x = 1,$$

the inequality (1.4) is an immediate consequence of the fact that $\frac{[\Gamma(x+1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x$ is strictly decreasing on $(0, \infty)$.

Let $s > 0$ be a real number. Chen and Qi [7] proved that the function $f_s(x) = \frac{[\Gamma(x+s+1)]^{1/(x+s)}}{[\Gamma(x+1)]^{1/x}}$ is strictly logarithmically completely monotonic on $(0, \infty)$. Since

$$\lim_{x \rightarrow \infty} \frac{[\Gamma(x+2)]^{1/(x+1)}}{[\Gamma(x+1)]^{1/x}} = 1,$$

the second inequality in (1.1) is an immediate consequence of the fact that $\frac{[\Gamma(x+2)]^{1/(x+1)}}{[\Gamma(x+1)]^{1/x}}$ is strictly decreasing on $(0, \infty)$.

Alzer [1] provided a refinement of inequality (1.1) as follows:

$$\frac{n+1}{n+2} < \frac{G_n}{G_{n+1}} \leq \frac{n+1}{n+2\sqrt{2}-1}, \quad n \in \mathbb{N}. \quad (1.5)$$

Chen [4] proved that for $n \in \mathbb{N}$,

$$(\sqrt{2\pi n})^{\frac{1}{n(n+1)}} \left(1 - \frac{1}{n+a}\right) < \frac{G_n}{G_{n+1}} \leq (\sqrt{2\pi n})^{\frac{1}{n(n+1)}} \left(1 - \frac{1}{n+b}\right),$$

with the best possible constants

$$a = \frac{1}{2} \quad \text{and} \quad b = \frac{1}{2^{3/4}\pi^{1/4} - 1} = 0.807\dots$$

Chen and Mortici [5] presented diverse asymptotic expansions for $\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}}$. For example, these authors established the following asymptotic expansion:

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{2x-1}{2x+1} \left[\sqrt{2\pi x} \exp\left(\frac{2}{3x} - \frac{1}{4x^2} + \frac{187}{720x^3} - \frac{3}{16x^4} + \dots\right) \right]^{\frac{1}{x(x+1)}} \quad (1.6)$$

as $x \rightarrow \infty$. Motivated by (1.6), Chen and Mortici [5] proved that for $x \geq 1$,

$$\begin{aligned} \frac{2x-1}{2x+1} \left[\sqrt{2\pi x} \exp\left(\frac{2}{3x} - \frac{1}{4x^2}\right) \right]^{\frac{1}{x(x+1)}} &< \frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \\ &< \frac{2x-1}{2x+1} \left[\sqrt{2\pi x} \exp\left(\frac{2}{3x}\right) \right]^{\frac{1}{x(x+1)}}. \end{aligned}$$

For more information on inequalities involving G_n , please refer to [2, 9, 13].

All of the inequalities recalled above bound the consecutive ratio G_n/G_{n+1} (equivalently, $\Gamma(x+2)^{1/(x+1)}/\Gamma(x+1)^{1/x}$). In this paper, we instead consider the second-order ratio

$$\frac{G_n^2}{G_{n-1}G_{n+1}}.$$

We prove sharp two-sided bounds for this ratio (Theorem 1); the lower bound is a quantitative form of the strict log-concavity of $\{G_n\}$, while the upper bound $1 + 1/n^2$ has the leading constant 1, which is asymptotically sharp (Remark 3.1). Two conjectures of Turán type, derived from the same asymptotic, are also proposed.

2. Lemmas

The classical Euler gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

the psi (or digamma) function $\psi(x) = \Gamma'(x)/\Gamma(x)$, and the polygamma functions $\psi^{(n)}(x)$ for $n \in \mathbb{N}$ and $x > 0$ are important special functions and have extensive applications.

LEMMA 1. (see [11, p. 32]) *The logarithm of the gamma function has the following asymptotic expansion:*

$$\begin{aligned}\ln \Gamma(x+1) &\sim \left(x + \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{n+1}}{n(n+1)x^n} \\ &= \left(x + \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \dots\end{aligned}\quad (2.1)$$

as $x \rightarrow \infty$, where B_n are the Bernoulli numbers defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

LEMMA 2. ([14, Corollary 1]) *For $x > 0$, the following inequality holds:*

$$\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \ln x < \frac{1}{2x}. \quad (2.2)$$

LEMMA 3. (Alzer [3, Theorem 9]) *For $x > 0$, the following inequality holds:*

$$\frac{1}{x} - \frac{1}{2x^2} < \psi'(x+1) < \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3}. \quad (2.3)$$

The right-hand side of (2.3) can also be found in [8, Theorem 4].

3. Main result

The main result of this paper is the following theorem.

THEOREM 1. *For $n \geq 2$, we have*

$$1 < \frac{G_n^2}{G_{n-1}G_{n+1}} < 1 + \frac{1}{n^2}. \quad (3.1)$$

Proof. Let us prove the left-hand side inequality first. Define the function $f(x)$, for $x > 1$, by

$$f(x) = \frac{\Gamma(x+1)^{2/x}}{\Gamma(x)^{1/(x-1)}\Gamma(x+2)^{1/(x+1)}}. \quad (3.2)$$

Taking the logarithm of both sides of (3.2), we have

$$\begin{aligned}\ln f(x) &= \frac{2}{x} \ln \Gamma(x+1) - \frac{1}{x-1} [\ln \Gamma(x+1) - \ln x] \\ &\quad - \frac{1}{x+1} \left[\ln \Gamma(x+1) + \ln x + \ln \left(1 + \frac{1}{x}\right) \right] \\ &= -\frac{2}{x(x^2-1)} \ln \Gamma(x+1) + \frac{2}{x^2-1} \ln x - \frac{1}{x+1} \ln \left(1 + \frac{1}{x}\right).\end{aligned}$$

Using the asymptotic expansion (2.1), we find that

$$\ln f(x) = \frac{1}{x^2} + \frac{\frac{3}{2} - \ln(2\pi x)}{x^3} + O\left(\frac{\ln x}{x^5}\right), \quad x \rightarrow \infty, \quad (3.3)$$

which yields

$$f(\infty) = \lim_{x \rightarrow \infty} f(x) = 1.$$

Direct calculation shows that

$$f(2) = \left(\frac{4}{3}\right)^{1/3} < 1 + \frac{1}{2^2}.$$

To prove the left-hand side of (3.1), it suffices to show that the function $f(x)$ is strictly decreasing on $(1, \infty)$. By differentiating $\ln f(x)$, we get

$$\begin{aligned} -\frac{x^2(x^2-1)^2}{2(3x^2-1)} \frac{f'(x)}{f(x)} &= -\ln \Gamma(x+1) + \frac{x(x^2-1)}{3x^2-1} \psi(x+1) + \frac{2x^3}{3x^2-1} \ln x \\ &\quad - \frac{x^2(x-1)^2}{2(3x^2-1)} \ln\left(1 + \frac{1}{x}\right) - \frac{x(x-1)(3x+1)}{2(3x^2-1)} =: g(x). \end{aligned}$$

By differentiating $g(x)$ and using the inequalities (2.2) and (2.3) as well as

$$\ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}, \quad x > 0,$$

we have

$$\begin{aligned} g'(x) &= -\frac{6x^2(x^2-1)}{(3x^2-1)^2} [\psi(x+1) - \ln x] + \frac{x(x^2-1)}{3x^2-1} \psi'(x+1) \\ &\quad - \frac{x(x-1)(3x^3-2x+1)}{(3x^2-1)^2} \ln\left(1 + \frac{1}{x}\right) + \frac{(x-1)(6x^4+3x^3+7x^2+7x+1)}{2(x+1)(3x^2-1)^2} \\ &> -\frac{6x^2(x^2-1)}{(3x^2-1)^2} \frac{1}{2x} + \frac{x(x^2-1)}{3x^2-1} \left(\frac{1}{x} - \frac{1}{2x^2}\right) \\ &\quad - \frac{x(x-1)(3x^3-2x+1)}{(3x^2-1)^2} \frac{1}{x} + \frac{(x-1)(6x^4+3x^3+7x^2+7x+1)}{2(x+1)(3x^2-1)^2} \\ &= \frac{(x-1)^2(6x^4+6x^3+3x^2-1)}{2x(x+1)(3x^2-1)^2} > 0, \quad x > 1, \end{aligned}$$

where the positivity of $6x^4+6x^3+3x^2-1$ on $(1, \infty)$ follows from the expansion

$$6x^4+6x^3+3x^2-1 = 14+48y+57y^2+30y^3+6y^4 \quad (x=1+y, y>0).$$

Hence, the function $g(x)$ is strictly increasing on $(1, \infty)$, and we have

$$g(x) > g(1) = 0 \implies f'(x) < 0, \quad x > 1.$$

Therefore $f(x)$ is strictly decreasing on $(1, \infty)$, and since $f(\infty) = 1$ it follows that $f(x) > 1$ for all $x > 1$. In particular, $f(n) > 1$ for $n \geq 2$, which is the left-hand side of (3.1).

For the proof of the right-hand side inequality, we shall consider the function $F(x)$ defined, for $x > 1$, by

$$\begin{aligned} F(x) &= \ln f(x) - \ln \left(1 + \frac{1}{x^2} \right) \\ &= -\frac{2}{x(x^2-1)} \ln \Gamma(x+1) + \frac{2}{x^2-1} \ln x - \frac{1}{x+1} \ln \left(1 + \frac{1}{x} \right) - \ln \left(1 + \frac{1}{x^2} \right), \end{aligned}$$

where $f(x)$ is given in (3.2). Using the asymptotic expansion (2.1), we find that

$$F(x) = \frac{\frac{3}{2} - \ln(2\pi x)}{x^3} + \frac{1}{2x^4} + \frac{\frac{25}{12} - \ln(2\pi x)}{x^5} + O\left(\frac{\ln x}{x^6}\right), \quad x \rightarrow \infty,$$

which yields

$$F(\infty) = \lim_{x \rightarrow \infty} F(x) = 0.$$

To prove the right-hand side of (3.1), it suffices to show that the function $F(x)$ is strictly increasing on $(1, \infty)$. By differentiating $F(x)$, we get

$$\begin{aligned} \frac{x^2(x^2-1)^2}{2(3x^2-1)} F'(x) &= \ln \Gamma(x+1) - \frac{x(x^2-1)}{3x^2-1} \psi(x+1) - \frac{2x^3}{3x^2-1} \ln x \\ &\quad + \frac{x^2(x-1)^2}{2(3x^2-1)} \ln \left(1 + \frac{1}{x} \right) + \frac{x(x-1)(5x^3+3x^2+x-1)}{2(3x^2-1)(x^2+1)} =: H(x). \end{aligned}$$

By differentiating $H(x)$ and using the inequalities (2.2) and (2.3) as well as

$$\ln \left(1 + \frac{1}{x} \right) > \frac{1}{x} - \frac{1}{2x^2}, \quad x > 0,$$

we have

$$\begin{aligned} H'(x) &= \frac{6x^2(x^2-1)}{(3x^2-1)^2} [\psi(x+1) - \ln x] - \frac{x(x^2-1)}{3x^2-1} \psi'(x+1) \\ &\quad + \frac{x(x-1)(3x^3-2x+1)}{(3x^2-1)^2} \ln \left(1 + \frac{1}{x} \right) \\ &\quad + \frac{(x-1)(9x^7+17x^6+47x^5+3x^4-29x^3-13x^2-3x+1)}{2(3x^2-1)^2(x+1)(x^2+1)^2} \end{aligned}$$

$$\begin{aligned}
&> \frac{6x^2(x^2-1)}{(3x^2-1)^2} \left(\frac{1}{2x} - \frac{1}{12x^2} \right) - \frac{x(x^2-1)}{3x^2-1} \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} \right) \\
&\quad + \frac{x(x-1)(3x^3-2x+1)}{(3x^2-1)^2} \left(\frac{1}{x} - \frac{1}{2x^2} \right) \\
&\quad + \frac{(x-1)(9x^7+17x^6+47x^5+3x^4-29x^3-13x^2-3x+1)}{2(3x^2-1)^2(x+1)(x^2+1)^2} \\
&= \frac{(x-1)(27x^9+66x^8+165x^7+43x^6-43x^5-15x^4+7x^3+9x^2-4x+1)}{6x^2(x+1)(3x^2-1)^2(x^2+1)^2} \\
&> 0, \quad x > 1.
\end{aligned}$$

Here the nine-degree factor has some negative coefficients, but the substitution $x = 1 + y$ ($y > 0$) yields

$$\begin{aligned}
&27x^9 + 66x^8 + 165x^7 + 43x^6 - 43x^5 - 15x^4 + 7x^3 + 9x^2 - 4x + 1 \\
&= 256 + 1944y + 6440y^2 + 12116y^3 + 14212y^4 + 10778y^5 \\
&\quad + 5314y^6 + 1665y^7 + 309y^8 + 27y^9,
\end{aligned}$$

with all coefficients positive, so the factor is positive on $(1, \infty)$. Hence, the function $H(x)$ is strictly increasing on $(1, \infty)$, and we have

$$H(x) > H(1) = 0 \implies F'(x) > 0, \quad x > 1.$$

Therefore $F(x)$ is strictly increasing on $(1, \infty)$, and since $F(\infty) = 0$ it follows that $F(x) < 0$ for all $x > 1$, that is, $\ln f(x) < \ln(1 + 1/x^2)$, or equivalently $f(x) < 1 + 1/x^2$. In particular, $f(n) < 1 + 1/n^2$ for $n \geq 2$, which is the right-hand side of (3.1). The proof of Theorem 1 is complete. $\square$

REMARK 3.1. The constant 1 in the upper bound $1 + 1/n^2$ of (3.1) is asymptotically sharp: from the expansion (3.3) we obtain

$$n^2 \left(\frac{G_n^2}{G_{n-1}G_{n+1}} - 1 \right) = 1 + O\left(\frac{\ln n}{n}\right), \quad n \rightarrow \infty,$$

so no inequality of the form $G_n^2/(G_{n-1}G_{n+1}) \leq 1 + c/n^2$ can hold for all sufficiently large n with any $c < 1$.

REMARK 3.2. A sequence $\{a_n\}_{n \in \mathbb{N}}$ of real numbers is called strictly log-convex (log-concave), if it is positive and

$$a_{n+1}^2 < (>) a_n a_{n+2} \quad \text{for } n \in \mathbb{N}.$$

For positive sequences the arithmetic-geometric mean inequality gives that log-convexity implies convexity, and concavity implies log-concavity. From the left half of (3.1) we deduce

$$G_n G_{n+2} < G_{n+1}^2 \quad \text{for } n \in \mathbb{N}. \quad (3.4)$$

This shows that the sequence $\{G_n\}_{n \in \mathbb{N}}$ is strictly log-concave.

Turán [16] proved that, for $|x| \leq 1$ and $n \in \mathbb{N}$,

$$0 \leq P_n(x)^2 - P_{n-1}(x)P_{n+1}(x), \quad (3.5)$$

where $P_n(x)$ denotes the Legendre polynomial of degree n . The inequality (3.5) has attracted much interest from many mathematicians, various inequalities of the same type were presented for other special functions. From the left-hand side of (3.1), we obtain the following Turán-type inequality:

$$0 < G_n^2 - G_{n-1}G_{n+1} \quad \text{for } n \geq 2. \quad (3.6)$$

The Turán-type expression (3.6) is in fact of constant order, as the following Stirling computation shows. Writing

$$G_n^2 - G_{n-1}G_{n+1} = G_{n-1}G_{n+1}(f(n) - 1),$$

together with (3.3) and the Stirling expansion

$$\ln G_n = \ln n - 1 + \frac{\ln(2\pi n)}{2n} + O\left(\frac{1}{n^2}\right),$$

one finds that the $\ln(2\pi n)/n$ contributions cancel and

$$G_n^2 - G_{n-1}G_{n+1} = \frac{1}{e^2} \left(1 + \frac{3}{2n}\right) + O\left(\frac{\ln n}{n^2}\right), \quad n \rightarrow \infty. \quad (3.7)$$

The asymptotic relation (3.7), combined with numerical checks at small n , motivates the following conjecture, in which the lower constant $1/e^2$ and the leading correction $3/(2n)$ are precisely the values that would be asymptotically sharp.

CONJECTURE 3.1. For $n \geq 2$, the following Turán-type inequality holds:

$$\frac{1}{e^2} < G_n^2 - G_{n-1}G_{n+1} < \frac{1}{e^2} \left(1 + \frac{3}{2n}\right).$$

By the arithmetic-geometric mean inequality, we obtain from (3.1) that for $n \geq 2$,

$$\frac{G_n^2}{(G_{n-1} + G_{n+1})^2} < \frac{G_n^2}{(2\sqrt{G_{n-1}G_{n+1}})^2} = \frac{1}{4} \frac{G_n^2}{G_{n-1}G_{n+1}} < \frac{1}{4} \left(1 + \frac{1}{n^2}\right),$$

which yields

$$\frac{G_n}{G_{n-1} + G_{n+1}} < \frac{1}{2} \sqrt{1 + \frac{1}{n^2}}, \quad n \geq 2. \quad (3.8)$$

An analogous Stirling computation, combining (3.3) with the expansion

$$\ln G_{n+1} - \ln G_{n-1} = \frac{2}{n} + \frac{1 - \ln(2\pi n)}{n^2} + O\left(\frac{1}{n^3}\right)$$

and the identity $(G_{n-1} + G_{n+1})/(2\sqrt{G_{n-1}G_{n+1}}) = \cosh(\frac{1}{2}(\ln G_{n+1} - \ln G_{n-1}))$, gives

$$\frac{G_n}{G_{n-1} + G_{n+1}} = \frac{1}{2} + \frac{1}{8n^3} + O\left(\frac{1}{n^4}\right), \quad n \rightarrow \infty. \quad (3.9)$$

This sharpens the upper bound (3.8) from order $1/n^2$ to order $1/n^3$ and motivates the following conjecture, in which the coefficient $1/8$ in the upper bound would be asymptotically sharp.

CONJECTURE 3.2. For $n \geq 2$, the following inequality holds:

$$\frac{1}{2} < \frac{G_n}{G_{n-1} + G_{n+1}} < \frac{1}{2} + \frac{1}{8n^3}. \quad (3.10)$$

REMARK 3.3. The left-hand side of (3.10) can be written as

$$\frac{G_n + G_{n+2}}{2} < G_{n+1} \quad \text{for } n \in \mathbb{N}. \quad (3.11)$$

If Conjecture 3.2 is true, this shows that the sequence $\{G_n\}_{n \in \mathbb{N}}$ is concave. Noting that concavity implies log-concavity, we see that if the inequality (3.11) can be proved, then the inequality (3.4) is an immediate consequence of the inequality (3.11). The upper bound of (3.10) is better than the upper bound of (3.8).

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