

BOUNDEDNESS OF COMMUTATOR GENERATED BY FRACTIONAL INTEGRAL OPERATOR AND ORLICZ–BMO FUNCTION

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Abstract. For $\alpha \in (0, n)$ and a growth function $\varphi : [0, \infty) \rightarrow [0, \infty)$, it is proved that the commutator $[b, I_\alpha]$ generated by fractional integral operator I_α and Orlicz BMO function b is bounded from Orlicz-Hardy space $H_b^\varphi(\mathbb{R}^n)$ to Lebesgue space $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$, where $H_b^\varphi(\mathbb{R}^n)$ is a suitable Orlicz-Hardy space. Moreover, the authors also establish that the boundedness of commutator $[b, I_\alpha]$ from Orlicz-Hardy space $H^\varphi(\mathbb{R}^n)$ to weak Lebesgue space $L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n)$.

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