

SHARP CONSTANTS FOR A WEIGHTED GEOMETRIC-MEAN FAMILY ON BOXES

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Abstract. We determine the optimal constant in a family of weighted geometric-mean inequalities on a compact box when the right-hand side is any premean that dominates the weighted geometric mean and agrees with it on constant vectors. The proof reduces the n -variable problem to a scalar extremal problem on the box. The resulting sharp constant depends only on the box endpoints and the exponent, and not on the dimension, the weights, or the particular dominating premean. As applications, we obtain a complete phase diagram for the two-parameter weighted AM-GM family, recover the original one-parameter case as a diagonal slice, characterize equality, and derive Lambert W representations and asymptotic expansions for the boundary curves.

Mathematics subject classification (2020): 26D15, 26E60, 33E20.

Keywords and phrases: Weighted AM-GM inequality, sharp constant, box constraint, Lambert W function, power mean.

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