

TWO-EXPONENT WEIGHTED BEESACK–OPIAL INEQUALITIES ON TIME SCALES AND APPLICATIONS

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Abstract. In this paper, we establish a class of weighted Beesack–Opial inequalities with two independent exponents on arbitrary time scales. The resulting estimates extend the weighted Opial framework by allowing the powers of the function and its delta derivative to vary independently, while an auxiliary nondecreasing function contributes through both its value and its delta derivative. The main result is a product-type inequality obtained by combining a weighted Hölder estimate with the exact triangular change-of-order formula on time scales. Corresponding additive and balanced forms are also derived by means of Young’s inequality.

The general results yield several dynamic Opial-type consequences, including composite weighted inequalities, power-type estimates, and classical discrete inequalities of Wong–Lasota type. In the continuous case, the main product estimate includes the $p = q = 2$ case of the two-weight Beesack–Opial inequality of Chesneau as a special case. Discrete and quantum versions are also obtained, with the forward-jump structure preserved explicitly.

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