

BEREZIN RADIUS INEQUALITIES VIA UNIFIED CONVEXITY APPROACH: A GENERAL FRAMEWORK ENCOMPASSING CLASSICAL RESULTS

LUCA GUERRINI, V. MADHUSUDANAN* AND NIRANJAN KUMAR K A

Abstract. We present a unified approach to Berezin radius inequalities based on weighted power mean inequalities and convex analysis. Our main contribution is a general interpolation framework showing that classical inequalities, including Cauchy-Schwarz, Buzano, Kato's mixed Schwarz inequality, and their generalizations, emerge as special cases of a single master inequality. We demonstrate that the results in the recent paper by Saltan, Gürdal, Garayev, and Rajawat [22] can be understood and generalized through our unified approach. Moreover, we establish bounds with optimal parameters and derive new inequalities, including variational characterizations and dual formulations.

Mathematics subject classification (2020): 47A30, 47A63, 47B32, 26D15, 47A12.

Keywords and phrases: Berezin radius, Berezin number, reproducing kernel Hilbert space, power mean inequality, Kato's mixed Schwarz inequality, geometric convexity, operator inequality, variational method.

REFERENCES

- [1] N. ARONSZAJN, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.
- [2] P. BHUNIA, S. S. DRAGOMIR, M. S. MOSLEHIAN, AND K. PAUL, *Lectures on numerical radius inequalities*, Infosys Science Foundation Series in Mathematical Sciences, Springer, Cham, 2022.
- [3] P. BHUNIA, M. T. GARAYEV, K. PAUL, AND R. TAPDIGOGLU, *Some new applications of Berezin symbols*, Complex Anal. Oper. Theory **17** (2023), no. 5, Paper No. 96, 21 pp.
- [4] P. BHUNIA, M. GÜRDAL, K. PAUL, A. SEN, AND R. TAPDIGOGLU, *On a new norm on the space of reproducing kernel Hilbert space operators and Berezin radius inequalities*, Numer. Funct. Anal. Optim. **44** (2023), no. 9, 970–986.
- [5] M. L. BUZANO, *Generalizzazione della disuguaglianza di Cauchy-Schwarz*, Rend. Sem. Mat. Univ. Politec. Torino **31** (1971/73), 405–409 (1974).
- [6] M. T. GARAYEV, M. GÜRDAL, AND S. SALTAN, *Hardy type inequality for reproducing kernel Hilbert space operators and related problems*, Positivity **21** (2017), no. 4, 1615–1623.
- [7] M. T. GARAYEV, H. GÜEDIRI, M. GÜRDAL, AND G. M. ALSAHLI, *On some problems for operators on the reproducing kernel Hilbert space*, Linear Multilinear Algebra **69** (2021), no. 11, 2059–2077.
- [8] M. HAJMOHAMADI, R. LASHKARIPOUR, AND M. BAKHERAD, *Improvements of Berezin number inequalities*, Linear Multilinear Algebra **68** (2020), no. 6, 1218–1229.
- [9] F. HANSEN, *Operator monotone functions*, in: Operator Theory: Advances and Applications, vol. 142, Birkhäuser, Basel, 2003, pp. 1–13.
- [10] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1952.
- [11] M. HOSSEINI, R. NURAEI, AND M. S. HOSSEINI, *Generalized Cauchy-Bunyakovsky-Schwarz inequalities*, Funct. Anal. Appl. **58** (2024), no. 1, 106–120.
- [12] M. T. KARAEV, *Berezin symbol and invertibility of operators on the functional Hilbert spaces*, J. Funct. Anal. **238** (2006), no. 1, 181–192.
- [13] M. T. KARAEV, *Reproducing kernels and Berezin symbols techniques in various questions of operator theory*, Complex Anal. Oper. Theory **7** (2013), no. 4, 983–1018.

- [14] T. KATO, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.
- [15] M. KHOSRAVI, R. DRNOVŠEK, AND M. S. MOSLEHIAN, *A commutator approach to Buzano's inequality*, Filomat **26** (2012), no. 4, 827–832.
- [16] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 2, 283–293.
- [17] F. KITTANEH AND H. R. MORADI, *Cauchy–Schwarz type inequalities and applications to numerical radius inequalities*, Math. Inequal. Appl. **23** (2020), no. 3, 1117–1125.
- [18] C. P. NICULESCU AND L.-E. PERSSON, *Convex functions and their applications. A contemporary approach*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23, Springer, New York, 2006.
- [19] V. I. PAULSEN AND M. RAGHUPATHI, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics, vol. 152, Cambridge University Press, Cambridge, 2016.
- [20] J. PEČARIĆ, T. FURUTA, J. MIČIĆ HOT, AND Y. SEO, *Mond-Pečarić method in operator inequalities. Inequalities for bounded selfadjoint operators on a Hilbert space*, Monographs in Inequalities, vol. 1, Element, Zagreb, 2005.
- [21] R. T. ROCKAFELLAR, *Convex analysis*, Princeton Mathematical Series, no. 28, Princeton University Press, Princeton, N.J., 1970.
- [22] S. SALTAN, M. GÜRDAL, M. T. GARAYEV, AND R. S. RAJAWAT, *Berezin radius inequalities via classical inequalities*, J. Math. Inequal. **19** (2025), no. 3, 893–907.
- [23] X. ZHAN, *Matrix theory*, Graduate Studies in Mathematics, vol. 147, American Mathematical Society, Providence, RI, 2013.