

A NEW DETERMINANTAL FORMULA AND A NEW RECURSIVE RELATION FOR COEFFICIENTS IN THE MACLAURIN EXPANSION OF THE RECIPROCAL OF THE GAMMA FUNCTION

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Abstract. In the work, the authors establish a new determinantal formula and a new recursive relation for coefficients in the Maclaurin power series expansion of the reciprocal of the classical Euler gamma function.

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REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.
- [2] C. BERG, *Integral representation of some functions related to the gamma function*, *Mediterr. J. Math.* **1** (2004), no. 4, 433–439, <https://doi.org/10.1007/s00009-004-0022-6>.
- [3] N. BOURBAKI, *Elements of Mathematics: Functions of a Real Variable: Elementary Theory*, Translated from the 1976 French original by Philip Spain. *Elements of Mathematics* (Berlin). Springer-Verlag, Berlin, 2004, <https://doi.org/10.1007/978-3-642-59315-4>.
- [4] N. D. CAHILL, J. R. D’ERRICO, D. A. NARAYAN, AND J. Y. NARAYAN, *Fibonacci determinants*, *College Math. J.* **33** (2002), no. 3, 221–225, <https://doi.org/10.2307/1559033>.
- [5] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015, <https://doi.org/10.1016/B978-0-12-384933-5.00013-8>.
- [6] B.-N. GUO AND F. QI, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **72** (2010), no. 2, 21–30.
- [7] Q.-M. LUO, B.-N. GUO, AND F. QI, *Evaluation of a class of improper integrals of the first kind*, *Math. Gaz.* **87** (2003), no. 510, 534–539, <https://doi.org/10.1017/S0025557200173863>.
- [8] Q.-M. LUO AND F. QI, *Evaluation of a class of the first kind improper integrals* $\int_0^\infty \left(\frac{\sin(bx)}{x}\right)^n \cos(bx) dx$, *Octagon Math. Mag.* **11** (2003), no. 1, 76–81.
- [9] Q.-M. LUO AND F. QI, *Evaluation of the improper integrals* $\int_0^\infty \frac{\sin^{2m}(ax)}{x^{2n}} \cos(bx) dx$ and $\int_0^\infty \frac{\sin^{2m+1}(ax)}{x^{2n+1}} \cos(bx) dx$, *Austral. Math. Soc. Gaz.* **30** (2003), no. 2, 86–89.
- [10] W. MAGNUS, F. OBERHETTINGER, AND R. P. SONI, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third enlarged edition. *Die Grundlehren der mathematischen Wissenschaften*, Band 52. Springer-Verlag New York, Inc., Berlin–Heidelberg–GmbH, 1966, <https://doi.org/10.1007/978-3-662-11761-3>.
- [11] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht–Boston–London, 1993, <https://doi.org/10.1007/978-94-017-1043-5>.

- [12] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT, AND C. W. CLARK (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010, <http://dlmf.nist.gov/>.
- [13] F. QI, *Theory of normalized remainders in Taylor series expansions*, arXiv:2512.15797v3, <https://doi.org/10.48550/arXiv.2512.15797>. Chapter in *Contemporary Approaches to Approximation of Operators – Approximation Theory and Mathematical Methods*, edited by Vijay Gupta and Gradimir V. Milovanović, published in Birkhäuser's Trends in Mathematics series, accepted on 15 April 2026, <https://www.researchgate.net/publication/406538357>.
- [14] F. QI, *On signs of certain Toeplitz–Hessenberg determinants whose elements involve Bernoulli numbers*, *Contrib. Discrete Math.* **18** (2023), no. 2, 48–59, <https://doi.org/10.55016/ojs/cdm.v18i2.73022>.
- [15] F. QI, *Several new formulas for a class of improper integrals whose integrands contain the powers of the sine function*, *J. Korean Soc. Math. Educ. Ser. B. Theor. Math. Pedag. Math.* **33** (2026), no. 1, 77–88, <https://doi.org/10.7468/jksmeb.2026.33.1.77>.
- [16] F. QI AND C.-P. CHEN, *A complete monotonicity property of the gamma function*, *J. Math. Anal. Appl.* **296** (2004), 603–607, <https://doi.org/10.1016/j.jmaa.2004.04.026>.
- [17] F. QI, Q.-M. LUO, AND B.-N. GUO, *Evaluations of the improper integrals $\int_0^\infty \frac{\sin^{2m}(\alpha x)}{x^{2n}} dx$ and $\int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x^{2n+1}} dx$* , *J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math.* **11** (2004), no. 3, 189–196.
- [18] R. L. SCHILLING, R. SONG, AND Z. VONDRAČEK, *Bernstein Functions*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012, <https://doi.org/10.1515/9783110269338>.
- [19] Y. SHUANG, C.-Y. HE, AND F. QI, *Logarithmically absolute monotonicity of the ratio between normalized remainders for a function in an integral representation of the reciprocal of the gamma function*, *J. Math. Inequal.* **19** (2025), no. 2, 605–624, <https://doi.org/10.7153/jmi-2025-19-80>.
- [20] A. SOFO, *On the integral $\frac{(\sin x)^{p+q}}{x^d}$* , *Internat. J. Math. Ed. Sci. Tech.* **29** (1998), no. 6, 914–918.
- [21] N. M. TEMME, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996, <https://doi.org/10.1002/9781118032572>.
- [22] D. V. WIDDER, *The Laplace Transform*, Princeton Mathematical Series, vol. 6, Princeton University Press, Princeton, NJ, 1941.
- [23] J. W. WRENCH, JR., *Concerning two series for the gamma function*, *Math. Comp.* **22** (1968), 617–626, <https://doi.org/10.2307/2004538>.
- [24] J. W. WRENCH, JR., *Table Errata: “Concerning two series for the gamma function”*, *Math. Comp.* **27** (1973), 681–682, <https://www.jstor.org/stable/2005685>.
- [25] J. ZHANG AND F. QI, *Some properties of normalized remainders of the Maclaurin expansion for a function originating from an integral representation of the reciprocal of the gamma function*, *Math. Inequal. Appl.* **28** (2025), no. 2, 343–354, <https://doi.org/10.7153/mia-2025-28-23>.