

NEW INEQUALITIES INVOLVING $(n!)^{1/n}$

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Abstract. Let $G_n = (n!)^{1/n}$ denote the geometric mean of the first n positive integers. In this paper, we prove that for $n \geq 2$,

$$1 < \frac{G_n^2}{G_{n-1}G_{n+1}} < 1 + \frac{1}{n^2}.$$

Both constants are asymptotically sharp: $n^2(G_n^2/(G_{n-1}G_{n+1}) - 1) \rightarrow 1$ as $n \rightarrow \infty$.

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